



Dynamics of a symmetrically decoupled three-dimensional point transformation¹

Hacene Gharout, Abdel-kaddous Taha, Nourredine Akroune

Abstract

In this work, we give some results obtained on the dynamics of a symmetrically decoupled three-dimensional point transformation. We are interested, in particular, in the study of its parametric plane and in its phase space by highlighting the existence of chaotic attractors.

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1 Introduction

We will speak of a symmetrically decoupled three-dimensional point transformation, if this transformation can be put in the form:

$$(1) \quad T \begin{cases} x_{n+1} = f(y_n), \\ y_{n+1} = g(z_n), \\ z_{n+1} = h(x_n). \end{cases}$$

Such transformations are found in different fields of science, for example, in economics the duopoly of Cournot (1838) [3]; this problem has been the subject of a large number of publications T.Puu (2004) [6], J.S.Canovas (2004) [2] and F.Tramontana

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in 2011 [7]. Another case that is modilized by such recurrences and that we meet in physics is that of systems with delay, see for example the work of L.Larger and D.Fournier-Prunaret (2005) [5] and L.Larger (2010) [4]. They are also found in the case of the security of the transmissions, see for example J.Xu, D.Fournier-Prunaret, A.K.Taha and P.Chargé (2010) [8] and H.Gharout et al. [9, 10].

We will focus here on the recurrence, defined by:

$$(2) \quad T \begin{cases} x_{n+1} = y_n, \\ y_{n+1} = z_n, \\ z_{n+1} = x_n^2 + b. \end{cases}$$

2 Properties of the map T

We present in this section some definitions and properties of symmetrically decoupled point transformations, taken from A.Agliari et al. [1], necessary to the understanding of this work.

The transformation T is of the form:

$$(3) \quad T \begin{cases} x' = f(y), \\ y' = g(z), \\ z' = h(x). \end{cases}$$

With $x = x_n$, $y = y_n$ and $z = z_n$; likewise $x' = x_{n+1}$, $y' = y_{n+1}$ and $z' = z_{n+1}$.

The study of the cycles of the system defined by (2) will be done, by using the functions: $H(x) = f(g(h(x)))$, $F(y) = g(h(f(y)))$ and $G(z) = h(f(g(z)))$. Using one-dimensional transformations H , F and G it is possible to build the iterations of T . Indeed:

$$(4) \quad T^{3k}(x, y, z) = (H^k(x), F^k(y), G^k(z)),$$

$$(5) \quad T^{3k+1}(x, y, z) = (f(F^k(y)), g(G^k(z)), h(H^k(x))),$$

$$(6) \quad T^{3k+2}(x, y, z) = (f(g(G^k(z))), g(h(H^k(x))), h(f(F^k(y)))).$$

for $k \geq 0$, with H^0, F^0 and G^0 are the identity.

The cycles of the one-dimensional functions H , F and G are linked; indeed, if x is a fixed point of H , then $y = g(h(x))$ is a fixed point of F and $z = h(x)$ is a fixed point of G . Taking into account a cycle of order n of H , the cycles of the functions F and G will be called conjugate cycles, and are such that, if $X = \{x_i\}_{i=1, n}$ is a cycle of order n of H , then a conjugate cycle of F exists, and is given by $Y = g(h(X))$ and the conjugate cycle of G is given by $Z = h(X)$.

2.1 Homogeneous cycles

Definition 1. A cycle of the map T is said homogeneous if the components of its periodic points belong to conjugated cycles of H , F and G . Otherwise, it is called mixed cycle.

Proposition 1. [1] If n is not a multiple of 3, the homogeneous cycles of period n of 3D map T associated with the cycle $X = \{x_1, x_2, \dots, x_n\}$ of 1D map H of the same period have only one periodic point with first component x_1 . This periodic point is (x_1, y_{2s+1}, z_{s+1}) when $n = 3s + 1$; and (x_1, y_{s+1}, z_{2s+2}) when $n = 3s + 2$.

Proposition 2. [1] All the different homogeneous cycles of period $3n$ of the 3D map T obtained starting from a cycle of period $n \geq 2$ of the 1D map H can be obtained by the periodic points:

$$\begin{aligned} &(x_1, y_j, z_{j+h}) \text{ with } h \leq j \leq n - 2h \text{ and } 1 \leq h \leq \lfloor \frac{n}{3} \rfloor, \\ &(x_1, y_j, z_{j+1-h}) \text{ with } 2h - 1 \leq j \leq n - h \text{ and } 1 \leq h \leq \lfloor \frac{n+1}{3} \rfloor, \end{aligned}$$

where $\lfloor x \rfloor$ is the ceiling function (the largest integer smaller than x).

2.2 Mixed cycles

Let $\{x_i\}_{i=\overline{1,n}}$, $\{a_j\}_{j=\overline{1,m}}$ and $\{\alpha_l\}_{l=\overline{1,p}}$ three cycles of H of period n , m and p (respectively), with their conjugated cycles $(y_i = g \circ h(x_i))$, $(b_j = g \circ h(a_j))$ and $(\beta_l = g \circ h(\alpha_l))$ are the periodic points of the cycles of F , with $z_i = h(x_i)$, $c_j = h(a_j)$ and $\gamma_l = h(\alpha_l)$. We have:

Proposition 3. [1] If the map H has two distinct coexisting cycles of period n and m and $s = \text{LCM}(n, m)$, then the map T has $(n + m) \frac{n \cdot m}{s}$ different mixed cycles of period $3s$, besides the homogeneous one. All the distinct mixed cycles can be obtained by the periodic points:

$$\begin{aligned} &(x_1, b_j, z_l) \text{ with } 1 \leq j \leq d \text{ and } 1 \leq l \leq n, \\ &(x_1, b_j, c_l) \text{ with } 1 \leq j \leq d \text{ and } 1 \leq l \leq m. \end{aligned}$$

Proposition 4. [1] If the map H has three coexisting cycles of period n , m and p , and $S = \text{LCM}(n, m, p)$, then map T has $2 \frac{n \cdot m \cdot p}{S}$ different mixed cycles of period $3S$, besides the homogeneous ones and those generated by any pair of the three cycles of H . All the distinct mixed cycles obtained by mixing of all the components of the three cycles can be obtained by the periodic points:

$$\begin{aligned} &(x_1, b_j, \gamma_l) \text{ with } 1 \leq j \leq d \text{ and } 1 \leq l \leq p \frac{\text{LCM}(m, n)}{S}, \\ &(x_1, \beta_l, c_j) \text{ with } 1 \leq j \leq d \text{ and } 1 \leq l \leq p \frac{\text{LCM}(m, n)}{S}, \end{aligned}$$

where $d = \text{gcf}(n, m)$.

3 Study of cycles of the 3D map T

For $k = 1$, the construction of the cycles will be done as follows:

$$\begin{aligned} T^3(x, y, z) &= (H(x), F(y), G(z)) = (x^2 + b, y^2 + b, z^2 + b). \\ T^4(x, y, z) &= (f(F(y)), g(G(z)), h(H(x))) = (y^2 + b, z^2 + b, (x^2 + b)^2 + b). \\ T^5(x, y, z) &= (f(g(G(z))), g(h(H(x))), h(f(F(y)))) \\ &= (z^2 + b, (x^2 + b)^2 + b, (y^2 + b)^2 + b). \end{aligned}$$

3.1 Fixed points of T

From the fixed points of the 1D map H and its conjugate functions F and G , we can define all homogeneous cycles of order 1 and all mixed cycles of order 3 of the 3D map T .

$$H(x) = f(g(h(x))) = x^2 + b.$$

Fixed points of T denoted X_1 and X_2 are:

$$\begin{aligned} X_1 &= (x_1, g(h(x_1)), h(x_1)) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}\right), \\ X_2 &= (x_2, g(h(x_2)), h(x_2)) = \left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} - \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} - \frac{1}{2}\sqrt{1-4b}\right); \end{aligned}$$

with $x_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}$ for $b \leq \frac{1}{4}$ is a fixed point of H , $y_1 = g(h(x_1))$ and $z_1 = h(x_1)$ are conjugate fixed points of F and G respectively. Similarly, $x_2 = \frac{1}{2} - \frac{1}{2}\sqrt{1-4b}$ for $b \leq \frac{1}{4}$ is a fixed point of H , $y_2 = g(h(x_2))$ and $z_2 = h(x_2)$ are the conjugate fixed points of F and G respectively.

Conjugate cycles have the same stability, because they have equal multipliers. The multipliers associated with the conjugated cycles are:

$$\lambda_x = (H(x_i))' = 2x_i, \lambda_y = (F(g(h(x_i))))' = 2y_i \text{ and } \lambda_z = (G(h(x_i)))' = 2z_i, i = 1, 2.$$

The multipliers associated with the conjugate cycles are equal to the eigenvalues of the Jacobian of T^3 . Indeed, the Jacobian of T^3 is:

$$(7) \quad J_{T^3} = \begin{pmatrix} 2x & 0 & 0 \\ 0 & 2y & 0 \\ 0 & 0 & 2z \end{pmatrix}$$

The characteristic polynomial $P_1(\lambda) = (2x - \lambda)(2y - \lambda)(2z - \lambda)$ vanishes for $\lambda = 2x$, $\lambda = 2y$ and $\lambda = 2z$ and the stability of the fixed points of H induces that of the conjugates of F and G , because they have the same eigenvalues. Likewise for the 3D maps T and T^3 , the stability of the fixed points of H induces that of the fixed points and third order cycles of T . Knowing that the fixed points x_1 and x_2 of H are unstable and stable respectively for $-\frac{3}{4} < b < \frac{1}{4}$, then the fixed point X_1 of T generated by x_1 is unstable and X_2 generated by x_2 is stable under the same conditions on b .

3.2 Bifurcation of fixed points of the 3D map T

Similarly, the T bifurcations are deduced from that of H .

• **Fold Bifurcation:** The Fold bifurcation of H corresponds to $H(x) = x$ and $H'(x) = 1$, then T undergoes a Fold bifurcation at $b = \frac{1}{4}$ and $x = \frac{1}{2}$, $y = g(h(x))$ and $z = h(x)$.

• **Flip Bifurcation:** The Flip bifurcation of H corresponds to $H(x) = x$ and $H'(x) = -1$, then T undergoes a Flip bifurcation at $b = -\frac{3}{4}$ and $x = y = z = -\frac{1}{2}$.

• **Transcritical bifurcation of fixed points of T :** A transcritical bifurcation of the fixed points x_1 and x_2 of H occurs when there exists a b^* such that for $b > b^*$ the two fixed points exchange their stability, with $H'(x_1) = H'(x_2) = 1$, for $b = b^*$. Here T checks this property for $b^* = \frac{1}{4}$.

3.3 Cycles of period 2

From the proposition 1, the map T admits a homogeneous cycle of period two $C_2 = \{V_1, V_2\}$, deduced from the cycle of order two $c_2 = \{V_{1x}, V_{2x}\}$ of H , where $V_{1x} = \frac{-1}{2} + \frac{1}{2}\sqrt{-3-4b}$ and $V_{2x} = \frac{-1}{2} - \frac{1}{2}\sqrt{-3-4b}$. The homogeneous cycle of period two of T , whose components are $V_1 = (V_{1x}, V_{2x}, V_{1x})$ and $V_2 = (V_{2x}, V_{1x}, V_{2x})$, is stable for $\frac{-5}{4} < b < \frac{-3}{4}$.

The Flip bifurcation of the two-period cycle of T is deduced from the Flip bifurcation of the two-period cycle of H ($H^2(x) = x$ and $(H^2)' = -1$): $b = \frac{-5}{4}$ for $x \in \{\frac{1}{2}\sqrt{2} - \frac{1}{2}, -\frac{1}{2}\sqrt{2} - \frac{1}{2}\}$.

The cycle of the period two mixed with an fixed point of 1D map H builds a cycle of period six of the 3D map T .

3.4 Construction of the period cycles greater than 2 of T

Using the cycles of the 1D map H and the conjugate cycles of F and G , the homogeneous and mixed cycles of the 3D map T are constructed.

3.4.1 Cycles of periods 4 and 5

Under the proposition 1, the transformation T has a unique homogeneous cycle of order four C_4 , and its components are derived from the cycle of the period four of H , $c_4 = \{a_1, a_2, a_3, a_4\}$ (points noted c_4 in the figure 1(a)), using the periodic point (a_1, a_4, a_3) . The cycle of period four of T is stable for $-1.38 < b < \frac{-5}{4}$ (figure 1(b)). Similarly, there is a single homogeneous cycle C_5 of period five of and its components are deduced from the cycle of period five of H , $c_5 = \{b_1, b_2, b_3, b_4, b_5\}$. The cycle components of five C_5 periods of T are constructed using the periodic point (b_1, b_3, b_5) .

3.4.2 Cycles of period 3

By virtue of the proposition 1, the point transformation T admits only two mixed cycles of period 3, deduced from the fixed points of H , x_1 and x_2 . The cycles of period three of T , are given by:

- $C_{3_1} = \{V_{3_1}, V_{3_2}, V_{3_3}\}$:

$$\begin{aligned} V_{3_1} &= (x_2, x_1, x_2) = [\frac{1}{2} - \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} - \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}], \\ V_{3_2} &= T(V_{3_1}) \text{ and } V_{3_3} = T(V_{3_2}). \end{aligned}$$

- $C3_2 = \{V3_4, V3_5, V3_6\}$:

$$V3_4 = (x_2, x_1, x_1) = \left[\frac{1}{2} - \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-4b}\right],$$

$$V3_5 = T(V3_4) \text{ and } V3_6 = T(V3_5).$$

The Flip bifurcation of the three-period cycle of T , occurs for $b = -1.768529152$; and the Fold bifurcation of a three-period cycle of T occurs for $b = \frac{-7}{4}$.

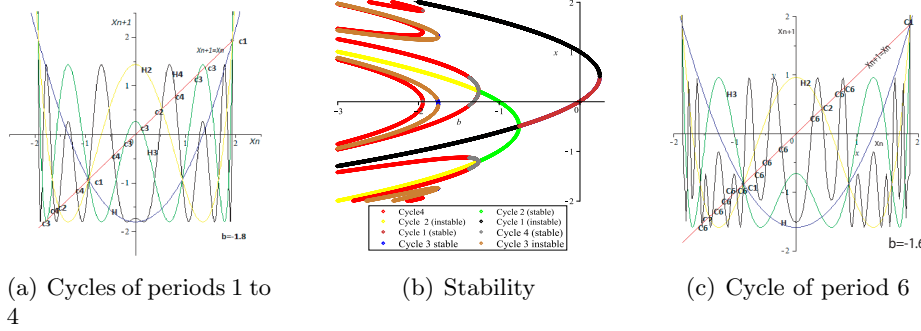


Figure 1: A few cycles of H and their stability.

3.4.3 Cycles of period 6

Under the stated propositions, there is a total of nine cycles of period six for the map T ; indeed:

1. From the proposition (2), a six-period homogeneous cycle of T deduced from the two-period cycle $c2 = \{V_{1x}, V_{2x}\}$ of H , is obtained using the periodic point $(\alpha_1, \alpha_1, \alpha_2)$, with $\alpha_1 = V_{1x}$ and $\alpha_2 = V_{2x}$; stable for $\frac{-5}{4} < b < \frac{-3}{4}$.
2. From the proposition (4), we have the existence of eight mixed cycles of period six of T :
 - (a) There are two mixed cycles of six period deduced from the coexistence of the two fixed points, $\{x_1\}$ and $\{x_2\}$ and the period cycle two $c2$ of H (here we have $n = 1$, $m = 1$ and $p = 2$, then $S = p.p.c.m(n, m, p) = 2$). The first component of each of the two cycles of period six is (α_1, x_2, x_1) and (α_1, x_1, x_2) respectively.
 - (b) From the coexistence of the cycles $\{x_1\}$ and $c2$ of H , three periodic cycles of period six are obtained from the periodic points: $(\alpha_1, x_1, \alpha_1)$, $(\alpha_1, x_1, \alpha_2)$ and (α_1, x_1, x_1) .
 - (c) And from the coexistence of the cycles $\{x_2\}$ and $c2$ de H , three other periodic cycles of period six are obtained from the periodic points: $(\alpha_1, x_2, \alpha_1)$, $(\alpha_1, x_2, \alpha_2)$ and (α_1, x_2, x_2) .

Note that it is still possible for us to construct mixed cycles with a period greater than six by using the proposition (4).

3.5 Bifurcation diagramme of the 3D map T

The figures 2(a) and 2(b) give an example of a Feigenbaum bifurcation diagram of the transformation T defined in the space of the variables x , y and b , and its projection in the plane $[b, x]$. As shown in the figure 1(b), when the parameter b varies, we can observe the bifurcation in an attractive fixed point in an attractive cycle of order 2, and then in an attractive cycle order 4, etc.

In the figure 2(a), we can see the projection of the bifurcation diagram in the plane $[x, y]$, which represents the set of attractors of T (we chose here the initial value $X_0 = (0, -0.5, 0)$). Notice that the point $(-0.5, -0.5, -0.5)$ for $b = \frac{-3}{4}$, corresponds to a point in the doubly period bifurcation. The coexistence of the fixed points and the cycle of order two of H gives rise to a cycle of order six of the 3D map T . From the bifurcation diagram (figure 2(b)), we can find the observations: evolution towards a fixed point for $b = -0.4$, a cycle of order two for $-0.8 < b < -0.75$, a cycle order four for $b = -1.25$. For $b = -1.6$, we no longer distinguish cycles; the system presents a chaotic character.

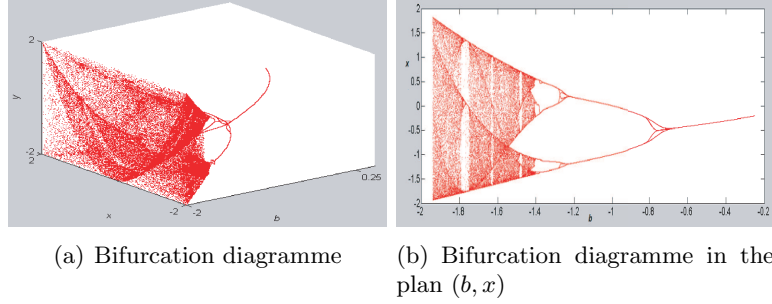


Figure 2: Bifurcation diagramme of T ($b \in [-1.99, -0.3]$).

4 Phase space of the 3D map T

In this section we will give some attractors of the three-dimensional map T , delimited by critical manifolds and an example of the chaotic dynamics of T by varying b .

4.1 Critical manifolds

We notice in the case of this map, that the critical manifolds are plans, denoted PC , dividing the space of the phases into zones denoted Z_i , i integer and each Z_i is the set of points in the phase space that has i antecedents of rank one. It is the generalization to dimension three of the notions of critical point and critical line defined in dimension one and two. A critical plan of rank $k + 1$ noted PC_k is the

consequent plan of rank k of PC_0 , $k = 1, 2, \dots$. PC_{-1} is the antecedent of rank one of PC_0 . These critical planes delimit the chaotic attractors. The equation of the critical manifold PC_{-1} of T satisfies $|J_1(x, y, z)| = 0$, where $J(x, y, z)$ is the Jacobian of T at the point (x, y, z) .

$$(8) \quad J(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 0 & 0 \end{pmatrix}$$

This critical variety is the plane $PC_{-1} = \{(0, y, z), y, z \in \mathbb{R}\}$.

The critical plane $PC = PC_0$ is therefore $PC_0 = T(PC_{-1}) = \{(y, z, b), y, z \in \mathbb{R}\}$, it is a plane parallel to the plane $[x, y]$ which cuts the axis of z into b ; PC separates the phase space into two regions. A region Z_2 verifying $z - b > 0$ and consisting of all points that have two rank 1 antecedents; and a region denoted Z_0 such that $z - b < 0$ and whose points do not have antecedents. We denote by R_1 and R_2 the regions verifying respectively $x > 0$ and $x < 0$, located on both sides of PC_{-1} (figure 3). The two inverse determinations corresponding to the two antecedents of rank 1 of the points of the region Z_2 are:

$$\begin{aligned} T_1^{-1} : Z_2 &\rightarrow R_1 & T_2^{-1} : Z_2 &\rightarrow R_2 \\ T_1^{-1} : \begin{cases} x = \sqrt{z' - b}, \\ y = x', \\ z = y'. \end{cases} & & T_2^{-1} : \begin{cases} x = -\sqrt{z' - b}, \\ y = x', \\ z = y'. \end{cases} \end{aligned}$$

Critical plans of order $n + 1$, are defined by $PC_{n+1} = T(PC_n)$ for all $n \geq 0$.

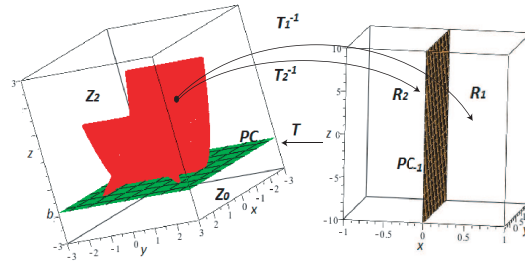
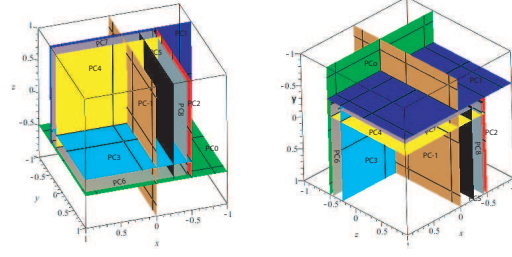


Figure 3: The regions R_1 and R_2 after applying inverse determinations of T .

Equivalently for an order n , $n \geq 0$, are defined by $PC_n = T^{n+1}(PC_{-1})$. All critical plans for T , depend on the b parameter, except PC_{-1} . The critical planes of T are all parallel to one of the plane of the three-dimensional space (x, y, z) , as shown in figure 4. The distance between two parallels critical planes depend for the value of b . For $b = 0$, we can have critical plans confused, in this case, PC_{-1} , PC_2 , PC_5 and PC_8 .

Figure 4: A few critical plans of T at $b = -0.5$.

4.2 Chaotic Attractors

By choosing the initial value $X_0 = (0, 0, -0.5)$, and $b = -\frac{3}{4}$, we have the appearance of an attractor formed of six branches that intersect at the point $(-0.5, -0.5, -0.5)$. At $b = -0.8$, the six branches split into two groups of three branches each, which forms an attractor of order six. By decreasing b up to $b = -1.45$, we have the appearance of six surfaces forming an attractor of order six. When b reaches the value of -1.864 , the attractor of order 6 becomes an attractor of order 3, consisting of a surface and two secant planes. For $b = -2$, the attractor is now composed of three line segments.

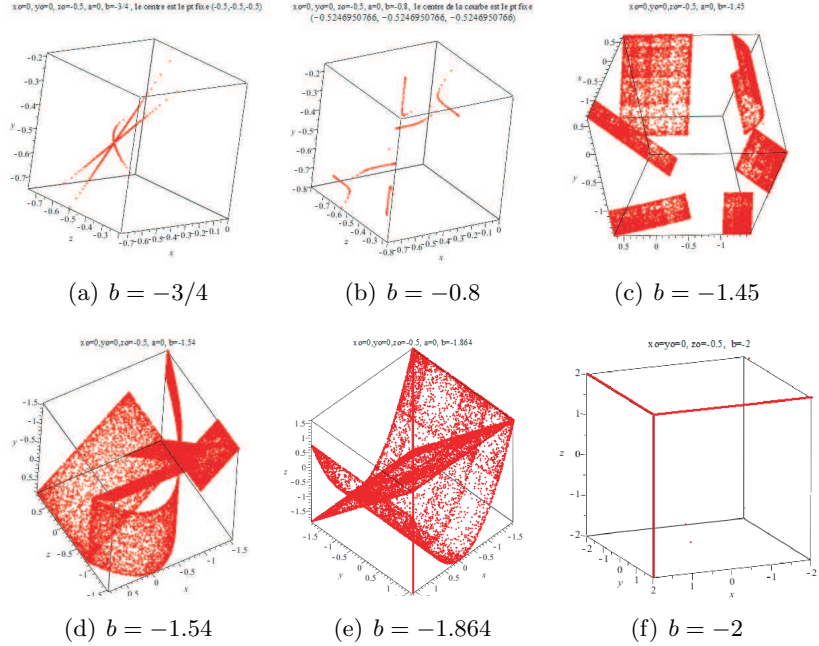


Figure 5: Evolution of an attractor of order six to a attractor of order three.

The exponents of Lyapounov allowed us to conclude, for $b = -1.864$ (figure 6)

and $b = -2$ (figure 7), that the attractors we obtained are chaotic attractors. Indeed, for $b = -1.864$, the Lyapounov exponents, calculated numerically, have the following values: $l_1 = 0.1535$, $l_2 = 0.1532$ and $l_3 = 0.1532$, who are all positive. Similarly for $b = -2$, Lyapounov's exponents are: $l_1 = 0.231048$, $l_2 = 0.231048$ and $l_3 = 0.231046$, are all positive, which makes us allows to consider it chaotic.

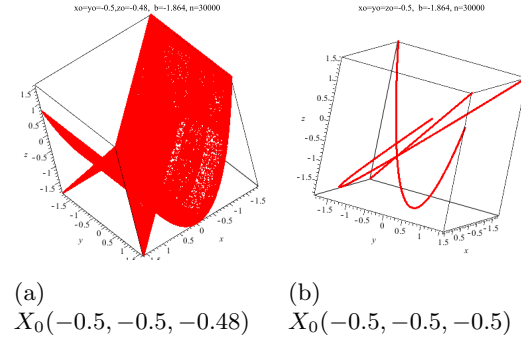


Figure 6: The attractors of order six at $b = -1.864$.

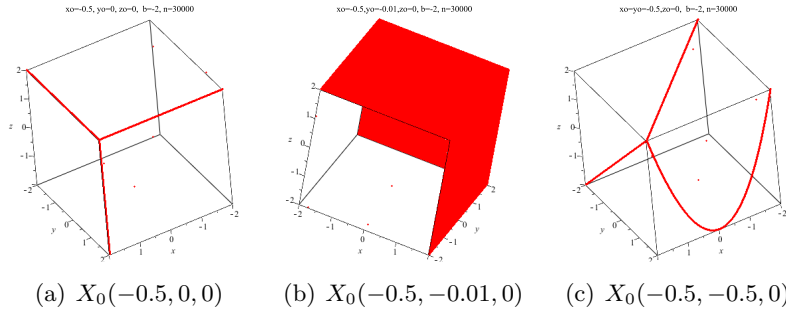
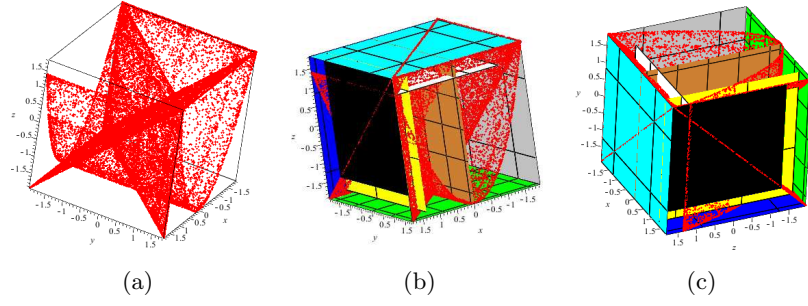
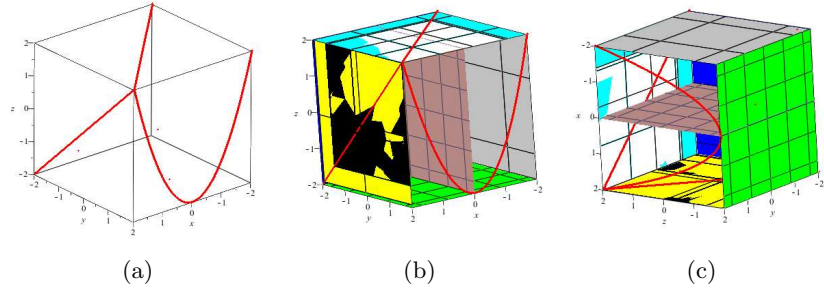


Figure 7: Third order attractors for different values of X_0 at $b = -2$.

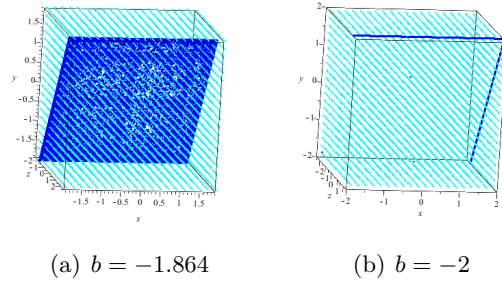
The figures 8 and 9 represent two orbits of 30000 points, as well as some critical planes which delimit the chaotic attractors defined for $b = -1.864$ and $b = -2$ respectively, taking $X_0 = (0, -0.5, 0.5)$ as the initial value. Critical plans are represented in different colors: PC_{-1} in brown, PC in green, PC_1 in blue, PC_2 in gray, PC_3 in cyan, PC_5 in black, PC_6 in white and PC_8 in yellow.

The chaotic attractors defined for these conditions correspond to unstable chaos. Chaos is said to be unstable when there is existence of a strange transient due to the presence of an infinity of unstable periodic solutions; we then speak of a chaotic repeller, such a set can be associated with the existence of an attractor at infinity (divergence for the initial conditions chosen) or the existence of a fuzzy boundary between the basins of two attractors. For $b = -1.864$, all fixed points are unstable

Figure 8: Chaotic attractor bounded by critical planes at $b = -1.864$.Figure 9: Chaotic attractor bounded by critical planes at $b = -2$.

nodes or unstable focus nodes.

In the same way we consider the case of stable chaos, for which we have the exis-

Figure 10: Attraction Basins with $X_0(0, -0.5, 0.5)$.

tence of a neighborhood ensuring the convergence of the trajectories of the iterates. In other words, it is necessary to determine the initial values of the attractor basin allowing the convergence of the trajectories towards the attractor. The graphical representation of the attraction ponds of two dense orbits in the chaotic attractors

defined for the two values of b ($b = -1.864$ and $b = -2$) with $X_0 = (0, -0.5, 0.5)$, is given in the figure 10.

We note that the two orbits touch the boundaries of their basins of attraction. For the same values of b , we have the existence of several chaotic attractors as shown in the figure 11, whose basins of attraction of coexisting attractors are represented with different colors.

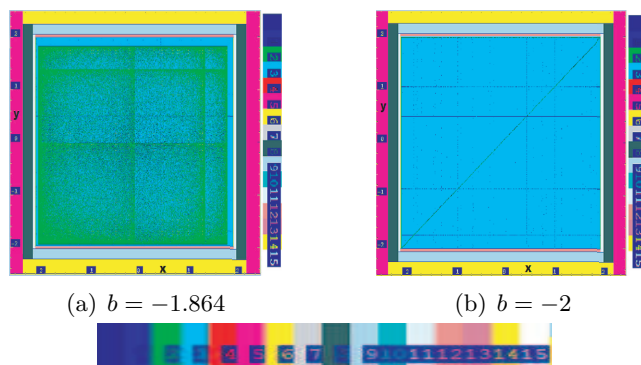


Figure 11: Basins of chaotic attractors of T , for two values of b ($X_0(0, -0.5, 0.5)$).

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Hacene Gharout

Laboratoire des Mathématiques Appliquées
 Faculté des Sciences Exactes and Faculté des Sciences de la Nature et de la Vie
 Université de Bejaia, 06000 Bejaia, ALGERIA
 professional e-mail: hacene-gharout@univ-bejaia.dz
 personal e-mail: gharouthacene@gmail.com

Abdel-kaddous Taha

INSA, University of Toulouse
 135 Avenue de Rangueil, 31077 Toulouse, FRANCE
 e-mail: taha@insa-toulouse.fr

Nourredine Akroune

Laboratoire des Mathématiques Appliquées
 Faculté des Sciences Exactes
 Université de Bejaia, 06000 Bejaia, ALGERIA
 e-mail: akroune_n@yahoo.fr