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Third-order strong differential subordinations involving fractional integral of confluent (Kummer) and Gaussian hypergeometric functions [1]

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Abstract

For the study presented in this paper, the concepts specific to strong differential subordination theory, introduced for the second-order differential subordination case, are extended for the third-order differential subordination case. The general results presented in the first theorem proved in this investigation are applied considering the prolific operators recently introduced as the fractional integral of confluent (Kummer) hypergeometric function and as the fractional integral of Gaussian hypergeometric function, respectively, adapted considering the extra parameter imposed by the strong differential subordination theory and new third-order differential subordinations are investigated. Simple examples are also constructed but it is hoped that the results given here would inspire future similar studies for applying the new third-order strong differential subordination results using other known operators adapted in order to fit this new context.

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1 Introduction

J.A. Antonino and S.S. Miller [1] extended for the context of third-order differential subordination, the differential subordination theory originally presented by S.S. Miller and P.T. Mocanu for second-order differential subordianiton [2, 3], consequently establishing a framework for further investigation on this topic. Applications of the results provided in [1] emerged promptly, and research on this subject has since continued successfully. A particular line of research focuses on identifying suitable classes of admissible functions by applying the fundamental results of third-order differential subordination. In light of this methodology, certain applications of third-order differential subordination are still obtained today. For instance, in [5] the authors obtain third-order differential subordination involving p-valent functions associated with a generalized fractional differintegral operator. The same method generates interesting third-order differential subordination results involving certain special functions in [6] and [7] and the studies involve also third-order differential superordination in [8].

The best dominant of the differential subordination is another key concept in the theory of differential subordination. Recent research has provided a different framework in third-order differential subordination theory presenting techniques to find the best dominant a third-order differential subordination in [9, 10], which also offer new ways to determine the dominant of a third-order differential subordination.

This paper focuses on showing how the classical outcomes on third-order differential subordination can be extended to the case of third-order strong differential subordination in particular. The most recent analysis [11] proposes the first results in this direction. The research done by the authors develops on the definitions of second-order strong differential subordination to include the particular context of third-order strong differential subordination. The aim is attained by employing a technique that involves selecting suitable classes of admissible functions and generating some original results. In this study, we derive specific third-order strong differential subordination results using the concept of best dominant of a third-order strong differential subordination and the operators presented as the fractional integral of confluent (Kummer) hypergeometric function and as the fractional integral of Gaussian hypergeometric function, respectively, adapted to fit the context of the strong differential subordination theory.

The basic notions of the strong differential subordination theory were initially introduced in 2009 [12], building upon the concepts established by J.A. Antonino and S. Romaguera in 1994 [13], when the concept of strong differential subordination was initially introduced within the framework of the Briot-Bouquet differential subordination particular case. The paper [12] defined the main concepts of solution, dominant and best dominant of the solutions of the strong differential subordination alongside rephrasing the three problems that serve as the foundation for the theory and adapting the concept of a class of admissible functions, which is a main tool in the analysis of strong differential subordination. The addition in 2012 [14] of specific classes of analytic functions, particularly used in strong differential subordination investigations, further enhanced the theory. Current findings applying the research results given by [14] include strong differential outcomes using various operators [15], applications of multiplier transformation and Ruscheweyh derivative in strong differential subordination theory [16], first-order strong differential subordinations [17], and q-calculus elements alongside specific operators in strong differential subordination and superordiantion studies [18].

The classes introduced in [14] and also used for this investigation, are:

$H(U\times\overline{U})$ denotes the class of analytic functions in $U\times\overline{U}$;

$$H\zeta[a,n]=$$
$$\left\{f\in H(U\times\overline{U}):f(z,\zeta)=a+a_n(\zeta)z^n+a_{n+1}(\zeta)z^{n+1}+\cdots,\ z\in U,\ \zeta\in\overline{U}\right\},$$

where $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\geq n$, $a\in\mathbb{C}$, $n\in\mathbb{N}$, the class derived from the classical:

$$H[a,n]=\{f\in H(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots,\ z\in U\};$$
$$H\zeta_U(U)=\{f\in H_\zeta[a,n]:f(\cdot,\zeta)\text{ univalent in }U\text{ for all }\zeta\in\overline{U}\};$$
$$A\zeta_n=\{f\in H(U\times\overline{U}):f(z,\zeta)=z+a_{n+1}(\zeta)z^{n+1}+\cdots,\quad z\in U,\ \zeta\in\overline{U}\},$$

with $A\zeta_1=A\zeta$ and $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k\geq n+1$, $n\in\mathbb{N}$, the class derived from the classical:

$$A_n=\{f\in H(U):f(z)=z+a_{n+1}z^{n+1}+\cdots,\quad z\in U\},\quad\text{with }A_1=A;$$

$$S^*\zeta=\{f\in A\zeta:\mathrm{Re}\frac{zf'_z(z,\zeta)}{f(z,\zeta)}>0,\quad z\in U,\ \zeta\in\overline{U}\},$$

the class of starlike functions in $U\times\overline{U}$ derived from the classical class of starlike functions:

$$S^*=\{f\in A:\mathrm{Re}\frac{zf'(z)}{f(z)}>0\};$$

$$K\zeta=\{f\in A\zeta:\mathrm{Re}\left(\frac{zf''_{z^2}(z,\zeta)}{f'_z(z,\zeta)}+1\right)>0,\quad z\in U,\ \zeta\in\overline{U}\},$$

the class of convex functions in $U\times\overline{U}$, derived from the classical class of convex functions:

$$K=\{f\in A:\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>0,\ f(0)=0,\ f'(0)\neq0,\quad z\in U\}.$$

and the class of close-to-convex functions,

$$C\zeta=\{f\in A\zeta:\exists q\in S^*\zeta,\mathrm{Re}\left(\frac{zf'_z(z,\zeta)}{g(z,\zeta)}\right)>0,\quad z\in U,\ \zeta\in\overline{U}\},$$

with function $f(z,\zeta)$ being univalent in $U\times\overline{U}$.

The following is a list of the strong differential subordination concepts that must be considered for this study.

Definition 1 *[14] Let $h(z,\zeta)$ and $f(z,\zeta)$ be analytic functions in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $h(z,\zeta)$, or $h(z,\zeta)$ is said to be strongly superordinate to $f(z,\zeta)$ if there exists a function w analytic in U with $w(0)=0$, $|w(z)|<1$ such that $f(z,\zeta)=h(w(z),\zeta)$, for all $\zeta\in\overline{U}$, $z\in U$. In such a case, we write*

$$f(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

Remark 1 *[14] a) If $f(z,\zeta)$ is analytic in $U\times\overline{U}$ and univalent in U for $\zeta\in\overline{U}$, then Definition 1 is equivalent to:*

$$f(0,\zeta)=h(0,\zeta),\ \text{for all } \zeta\in\overline{U} \text{ and } f(U\times\overline{U})\subset h(U\times\overline{U}).$$

b) If $f(z,\zeta)=f(z)$, $h(z,\zeta)=h(z)$, then the strong superordination becomes the usual superordination.

Definition 2 *[14] We denote by Q_ζ the set of functions $q(\cdot,\zeta)$ that are analytic and injective, as function of z, on $\overline{U}\backslash E(q(z,\zeta))$ where*

$$E(q(z,\zeta))=\{\xi\in\partial U:\lim_{z\to\xi}q(z,\zeta)=\infty\}$$

and are such that $Min|q'(\xi,\zeta)|=\rho>0$ for $\xi\in\partial U\backslash E(q(z,\zeta))$, $\zeta\in\overline{U}$.

The subclass of Q_ζ for which $q(0,\zeta)=a$ is denoted by $Q_\zeta(a)$.

Definition 3 *[14] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta)\in\Omega_\zeta$ and n a positive integer. The class of admissible functions $\phi_n[\Omega_\zeta,q(\cdot,\zeta)]$ consists of those functions $\psi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t;z,\zeta)\notin\Omega_\zeta$$

whenever

$$r=q(\xi,\zeta),\ s=nq'_z(\xi,\zeta),\ Re\left(\frac{t}{s}+1\right)\geq nRe\left[\frac{\xi q''_{z^2}(\xi,\zeta)}{q'_z(\xi,\zeta)}+1\right],$$

$z\in U$, $\xi\in\partial U\backslash E(q(\cdot,\zeta)),\zeta\in\overline{U}$ and $n\geq1$. When $n=1$, we write $\phi_1[\Omega_\zeta,q(\cdot,\zeta)]$ as $\phi[\Omega_\zeta,q(\cdot,\zeta)]$.

In the special case when $h(\cdot,\zeta)$ is an analytic mapping of $U\times\overline{U}$ onto $\Omega_\zeta\neq\mathbb{C}$ we denote the class $\phi_n[h(U\times\overline{U}),q(z,\zeta)]$ by $\phi_n[h(z,\zeta),q(z,\zeta)]$.

The class of admissible functions has been extended in [11] for the case of third-order strong differential subordination as it shows the next definition and will be used as such in the present investigation.

Definition 4 *[11] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta)\in\Omega_\zeta$ and $n\geq 2$. The class of admissible functions $\phi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\psi:\mathbb{C}^4\times U\times\overline{U}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t,u;z,\zeta)\notin\Omega_\zeta, \tag{1}$$

whenever

$$r=q(\xi,\zeta),\ \ s=nq'_z(\xi,\zeta),\ \ Re\left(\frac{t}{s}+1\right)\geq nRe\left[\frac{\xi q''_{\xi^2}(\xi,\zeta)}{q'_z(\xi,\zeta)}+1\right],$$

$$Re\frac{u}{s}\geq n^2Re\frac{\xi^2 q'''_{z^3}(\xi,\zeta)}{q'_z(\xi,\zeta)},$$

$z\in U$, $\xi\in\partial U\backslash E(q(\cdot,\zeta)),\zeta\in\overline{U}$ and $n\geq 2$.

An important known result that will be used in the proof of the new results is the following lemma used in third-order differential subordination theory and given here in the form required by the theory of strong differential subordination:

Lemma 1 *([1], [19]) Let $q(z,\zeta)\in Q_\zeta(a)$ and let*

$$p(z,\zeta)=a+a_n(\zeta)z^n+a_{n+1}(\zeta)z^{n+1}+\ldots\in H\left(U\times\overline{U}\right),$$

with $p(z,\zeta)\neq a$, and $n\geq 2$.

If $p(\cdot,\zeta)$ is not subordinate to $q(\cdot,\zeta)$, then there exist points $z_0\in U$, $z_0=r_0e^{i\theta_0}$ and $\zeta_0\in\partial U\backslash E(q(\cdot,\zeta))$ for which $p\left(U\times\overline{U_{r_0}}\right)\subset q\left(U\times\overline{U}\right)$ and an $m\geq n$, such that the following conditions are satisfied:

(i) $p(z_0,\zeta)=q(\zeta_0,\zeta)$;

(ii) $Re\frac{\zeta_0 q''_{z^2}(\zeta_0,\zeta)}{q'_z(\zeta_0,\zeta)}\geq 0$ *and* $\left|\frac{zp'_z(z,\zeta)}{q'_z(\xi,\zeta)}\right|\leq m, z\in U,\xi\in\partial U\backslash E(q(\cdot,\zeta))$;

(iii) $z_0p'_z(z_0,\zeta)=m\zeta_0q'_z(\zeta_0,\zeta)$;

(iv) $Re\left(\frac{z_0p''_{z^2}(z_0,\zeta)}{p'_z(z_0,\zeta)}+1\right)\geq mRe\left(\frac{\zeta_0q''_{z^2}(\zeta_0,\zeta)}{q'_z(\zeta_0,\zeta)}+1\right)$;

(v) $Re\frac{z_0^2p'''_{z^3}(z_0,\zeta)}{p'_z(z_0,\zeta)}\geq m^2Re\frac{\zeta_0^2q'''_{z^3}(\zeta_0,\zeta)}{q'_z(\zeta_0,\zeta)}$.

Definition 5 *[11] Let $\psi:\mathbb{C}^4\times U\times\overline{U}\to\mathbb{C}$ and $h(z,\zeta)$ be univalent in $U\times\overline{U}$. If $p(z,\zeta)$ is analytic in $U\times\overline{U}$ and satisfies the third-order strong differential subordination*

$$\psi\left(p(z,\zeta),zp'_z(z,\zeta),z^2p''_{z^2}(z,\zeta),z^3p'''_{z^3}(z,\zeta);z,\zeta\right)\prec\prec h(z,\zeta), \tag{2}$$

then $p(z,\zeta)$ is called a solution of the third-order strong differential subordination (2). A univalent function $q(z,\zeta)$ is called a dominant of the solutions of the third-order strong differential subordination or more simply a dominant, if $p(z,\zeta) \prec\prec q(z,\zeta)$ for all $p(z,\zeta)$ satisfying the third-order strong differential subordination (2). A dominant $\tilde{q}(z,\zeta)$ that satisfies $\tilde{q}(z,\zeta) \prec\prec q(z,\zeta)$ for all dominants of (2) is called the best dominant of (2) and it is unique up to a rotation in $U \times \overline{U}$.

Research connecting the theory of univalent functions with the theory of special functions began to develop after the proof of Bieberbach's conjecture was published in 1985 [20], where de Branges has used a generalized hypergeometric function for his proof. One of the earliest studies analyzing the implications of using hypergeometric functions in univalent function theory was published in 1990 by Miller and Mocanu [21]. In their paper [21], Miller and Mocanu have determined conditions such that confluent (Kummer) and Gaussian hypergeometric functions to be univalent in U. The form of those functions is next presented.

Definition 6 *[21] Let a and c be complex numbers with $c \neq 0, -1, -2, \ldots$ and consider*

$$F(a,c;z) = 1 + \frac{a}{c} \cdot \frac{z}{1!} + \frac{a(a+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \ldots, \; z \in U. \tag{3}$$

This function is called confluent (Kummer) hypergeometric function, is analytic in $\mathbb{C}$ and satisfies Kummer's differential equation:

$$z \cdot w''(z) + [c-z] \cdot w'(z) - a \cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d(d+1)(d+2)\ldots(d+k-1) \text{ and } (d)_0 = 1,$$

then (3) can be written in the form

$$F(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(a)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(c+k)} \cdot \frac{z^k}{k!}. \tag{4}$$

Definition 7 *[21] Let a, b and c be complex numbers with $c \neq 0, -1, -2, \ldots$ The function*

$$F(a,b,c;z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + ..., \; z \in U, \tag{5}$$

is called Gaussian hypergeometric function, is analytic in U and satisfies the hypergeometric equation:

$$z(z-1) \cdot w''(z) + [c-(a+b+1)z] \cdot w'(z) - ab \cdot w(z) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d\,(d+1)\,(d+2)\ldots(d+k-1) \text{ and } (d)_0 = 1,$$

then (5) can be written in the form

$$F\,(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma\,(c)}{\Gamma\,(a)\,\Gamma\,(b)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma\,(a+k)\,\Gamma\,(b+k)}{\Gamma\,(c+k)} \cdot \frac{z^k}{k!}. \tag{6}$$

The fractional integral of order λ used by Owa [22] and Owa and Srivastava [23] is defined as:

Definition 8 *[22, 23] The fractional integral of order $\lambda(\lambda > 0)$ is defined for a function f by the following expression:*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{7}$$

where f is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{1-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

In the research presented in [24], the coefficients a and c complex numbers in the form (3) of the confluent (Kummer) hypergeometric function are replaced by the holomorphic functions $a(\zeta), c(\zeta)$ depending on the parameter $\zeta \in \bar{U}$. The following form is used in [24] for studies regarding second-order strong differential subordonations and will be further used in this investigation.

Definition 9 *[24]*

$$F(a(\zeta), c(\zeta); z, \zeta) = 1 + \frac{a(\zeta)}{c(\zeta)} \cdot \frac{z}{1!} + \frac{a(\zeta)[a(\zeta)+1]}{c(\zeta)[c(\zeta)+1]} \cdot \frac{z^2}{2!} + \ldots, z \in U, \tag{8}$$

where $a(\zeta) \neq 0, c(\zeta) \neq 0, -1, -2, \ldots$.

For the research presented in this paper, the fractional integral seen in (7) will be applied to the confluent (Kummer) hypergeometric function given by (8). Gaussian hypergeometric function will also be adapted to the new classes specific to strong differential subordination theory and the fractional integral (7) will be also applied to this form. Hence, two new integral operators having the forms specific to the studies regarding strong differential subordination theory will be obtained. Using those operators and the techniques specific to the strong differential subordination theory, new results concerning third-order differential subordinations are obtained as corollaries of the main theorem proved in this study.

2 Main results

The first new result presented gives the form of the Gaussian hypergeometric function, obtained by replacing the complex coefficients a, b, c seen in its definition given by (5) with certain holomorphic functions $a(\zeta), b(\zeta), c(\zeta)$ depending on the parameter $\zeta \in \overline{U}$ specific to the theory of strong differential subordination.

Definition 10 *Let $a(\zeta), b(\zeta)$ and $c(\zeta)$ be holomorphic functions with $a(\zeta) \neq 0, c(\zeta) \neq 0, -1, -2, \ldots$.*

(9)

$$F\left(a(\zeta), b(\zeta), c(\zeta); z\right) = 1 + \frac{a(\zeta)b(\zeta)}{c(\zeta)} \cdot \frac{z}{1!} + \frac{a\left(a(\zeta)+1\right)b(\zeta)\left(b(\zeta)+1\right)}{c(\zeta)\left(c(\zeta)+1\right)} \cdot \frac{z^2}{2!} + ..., \ z \in U,$$

where $c(\zeta) \neq 0, -1, -2, \ldots$.

Definition 11 *Let $a(\zeta)$ and $c(\zeta)$ be holomorphic functions with $a(\zeta) \neq 0, c(\zeta) \neq 0, -1, -2, \ldots$ and let $\lambda > 0$. The factional integral of the confluent (Kummer) hypergeometric function specific to strong differential subordination theory is the analytic function in $U \times \overline{U}$ given by:*

$$A_z^{-\lambda} F(a(\zeta), c(\zeta); z, \zeta) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{F(a(\zeta), c(\zeta); t, \zeta)}{(z-t)^{1-\lambda}} dt \tag{10}$$

$$= \frac{1}{\Gamma(\lambda)} \cdot \frac{\Gamma(c(\zeta))}{\Gamma(a(\zeta))} \cdot \sum_{k=0}^{\infty} \frac{\Gamma(a(\zeta)+k)}{\Gamma(c(\zeta)+k)\Gamma(k+1)} \int_0^z \frac{t^k}{(z-t)^{1-\lambda}} dt, z \in U, \zeta \in \overline{U}.$$

Following a straightforward computation, the confluent hypergeometric function's fractional integral assumes the following form:

$$A_z^{-\lambda} F(a(\zeta), c(\zeta); z, \zeta) = \frac{\Gamma(c(\zeta))}{\Gamma(a(\zeta))} \sum_{k=0}^{\infty} \frac{\Gamma(a(\zeta)+k)}{\Gamma(c(\zeta)+k)\Gamma(\lambda+k+1)} z^{k+\lambda} \tag{11}$$

$$= \frac{z^\lambda}{\Gamma(\lambda+1)} + \frac{a(\zeta) z^{\lambda+1}}{c(\zeta)\Gamma(\lambda+2)} + \cdots .$$

We conclude that $A_z^{-\lambda}(a(\zeta), c(\zeta); z, \zeta) \in H\zeta[0, \lambda]$.

We can also write:

$$[A_z^{-\lambda} F(a(\zeta), c(\zeta); z, \zeta)]' = \frac{\lambda}{\Gamma(\lambda+1)} \cdot z^{\lambda-1} + \frac{a(\zeta)}{c(\zeta)} \cdot \frac{\lambda+1}{\Gamma(\lambda+2)} \cdot z^\lambda + \cdots . \tag{12}$$

For $\lambda = 1$ we obtain:

$$A_z^{-1} F(a(\zeta), c(\zeta); z, \zeta) = z + \frac{a(\zeta)}{c(\zeta)} \cdot \frac{z^2}{2} + \cdots ,$$

which shows that $A_z^{-1}F(a(\zeta), c(\zeta); z, \zeta) \in A\zeta$.

We can also write:

$$[A_z^{-1}F(a(\zeta), c(\zeta); z, \zeta)]' = 1 + \frac{a(\zeta)}{c(\zeta)} \cdot z + \cdots ,$$

and $[A_z^{-1}F(a(\zeta), c(\zeta); 0, \zeta)]' = 1 \neq 0$.

Remark 2 *By replacing $a(\zeta) = a$ and $c(\zeta) = c$ in Definition 11, the fractional integral of confluent hypergeometric function is obtained in its original form introduced in [25] and further applied in studies like [26].*

By applying Definition 8 and the form of the Gaussian hypergeometric function given in (11), the following definition describes the fractional integral of Gaussian hypergeometric function.

Definition 12 *Let $a(\zeta), b(\zeta)$ and $c(\zeta)$ be holomorphic functions with $a(\zeta) \neq 0, c(\zeta) \neq 0, -1, -2, \ldots,$ and $\lambda > 0$. For the fractional integral of Gaussian hypergeometric function, the following expression is found:*

(13)

$$D_z^{-\lambda}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right) = \frac{1}{\Gamma\left(\lambda\right)} \int_0^z \frac{\frac{\Gamma(c((\zeta)))}{\Gamma(a(\zeta))\Gamma(b(\zeta))} \sum_{k=0}^{\infty} \frac{\Gamma(a(\zeta)+k)\Gamma(b(\zeta)+k)}{\Gamma(c(\zeta)+k)} \cdot \frac{z^k}{k!}}{(z-t)^{1-\lambda}} dt$$

$$= \frac{\Gamma\left(c((\zeta))\right)}{\Gamma\left(a(\zeta)\right)\Gamma\left(b(\zeta)\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(a(\zeta)+k\right)\Gamma\left(b(\zeta)+k\right)}{\Gamma\left(c(\zeta)+k\right)} \cdot z^{k+\lambda}, z \in U, \zeta \in \overline{U}.$$

The fractional integral of Gaussian hypergeometric function can be written in the equivalent form:

$$D_z^{-\lambda}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right) = \frac{z^{\lambda}}{\Gamma(\lambda+1)} +$$

$$\frac{\Gamma\left(c((\zeta))\right)}{\Gamma\left(a(\zeta)\right)\Gamma\left(b(\zeta)\right)} \cdot \frac{\Gamma\left(a(\zeta)+1\right)\Gamma\left(b(\zeta)+1\right)}{\Gamma\left(c(\zeta)+1\right)\Gamma\left(\lambda+2\right)} \cdot z^{\lambda+1} + \cdots$$

We conclude that $D_z^{-\lambda}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right) \in H\zeta[0, \lambda]$.

We can also write:

$$[D_z^{-\lambda}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right)]' = \frac{\lambda}{\Gamma(\lambda+1)} z^{\lambda-1} + \frac{a(\zeta)b(\zeta)(\lambda+1)}{c(\zeta)\Gamma(\lambda+2)} \cdot z^{\lambda} + \cdots$$

For $\lambda = 1$,

$$D_z^{-1}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right) = z + \frac{a(\zeta)b(\zeta)}{c(\zeta)} \cdot \frac{z^2}{2} + \cdots,$$

which shows that $D_z^{-1}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right) \in A\zeta$,

and

$$[D_z^{-1}F\left(a(\zeta), b(\zeta), c(\zeta); z, \zeta\right)]' = 1 + \frac{a(\zeta)b(\zeta)z}{c(\zeta)} \cdot z^{\lambda} + \cdots .$$

Remark 3 *If $a(\zeta) = a, b(\zeta) = b, c(\zeta) = c$, the fractional integral of Gaussian hypergeometric function the form is obtained in its original form introduced in [27].*

In the following theorem, the best dominant for a third-order strong differential subordination is obtained. Using the outcome of this theorem, two results will be given as applications involving the fractional integral confluent (Kummer) hypergeometric function and the fractional integral of Gaussian hypergeometric function, respectively, considered in their forms adapted to the strong differential subordination theory.

Theorem 1 *Consider the function $q(z,\zeta) \in K\zeta$, the functions θ and $\phi \in \mathcal{H}\left(U \times \overline{U}\right)$, where the domanin $D \supset q\left(U \times \overline{U}\right)$ and $\phi(w) \neq 0$, $w \in q\left(U \times \overline{U}\right)$.*

Let

$$Q(z,\zeta) = zq'_z(z,\zeta)\,\phi\left(q(z,\zeta), z^2 q''_z(z,\zeta), z^3 q'''_z(z,\zeta)\right)$$

and

$$\begin{aligned} h(z,\zeta) &= \theta(q(z,\zeta)) + zq'_z(z,\zeta)\,\phi\left(q(z,\zeta), z^2 q''_z(z,\zeta), z^3 q'''_z(z,\zeta)\right) \\ &= \theta(q(z,\zeta)) + Q(z,\zeta). \end{aligned}$$

Suppose that:

(i) Re$\frac{\zeta q''_z(z,\zeta)}{q'_z(z,\zeta)} \geq 0$ and $\left|\frac{zp'_z(z,\zeta)}{q'_z(\xi,\zeta)}\right| \leq u$, $z \in U$, $\zeta \in \overline{U}$, $\xi \in \partial U - E(q(z,\zeta))$,

(ii) Functions Q and ϕ are starlike in $U \times \overline{U}$,

(iii) Re$\frac{\theta'_z(q(z,\zeta))}{\phi(q(z,\zeta), z^2 q''_z(z,\zeta), z^3 q'''_z(z,\zeta))} > 0$, $z \in U$, $\zeta \in \overline{U}$.

If the function $p(z,\zeta) \in \mathcal{H}\left(U \times \overline{U}\right)$ with $p(0,\zeta) = q(0,\zeta)$ and $p\left(U \times \overline{U}\right) \subset D$, then

$$\theta(p(z,\zeta)) + zp'_z(z,\zeta)\,\phi\left(p(z,\zeta), z^2 p''_z(z,\zeta), z^3 p'''_z(z,\zeta)\right) \prec\prec h(z,\zeta) \tag{14}$$

implies

$$p(z,\zeta) \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}$$

and $q(z,\zeta)$ is the best dominant of the strong differential subordination (14).

Proof. Let the function $h(z,\zeta)$ be differentiated with respect to z:

$$h'_z(z,\zeta) = q'_z(z,\zeta) \cdot \theta'(q(z,\zeta)) + Q'(z,\zeta). \tag{15}$$

We assess:

$$\mathrm{Re}\frac{zh'_z(z,\zeta)}{Q(z,\zeta)} = \tag{16}$$

$$\mathrm{Re}\left[\frac{zq'_z(z,\zeta) \cdot \theta'(q(z,\zeta))}{zq'_z(z,\zeta)\,\phi(q(z,\zeta), z^2 q''_z(z,\zeta), z^3 q'''_z(z,\zeta))} + \frac{zQ'(z,\zeta)}{Q(z,\zeta)}\right] =$$

$$\mathrm{Re}\frac{\theta'(q(z,\zeta))}{\phi(q(z,\zeta), z^2 q''_z(z,\zeta), z^3 q'''_z(z,\zeta))} + \mathrm{Re}\frac{zQ'(z,\zeta)}{Q(z,\zeta)}.$$

Since $Q(z,\zeta)$ is a starlike function, using (iii), relation (16) becomes:

$$\operatorname{Re}\frac{zh'_z(z,\zeta)}{Q(z,\zeta)} \geq 0. \tag{17}$$

Using relation (17), according to the definition of the strong close-to-convex functions, we conclude that $h(z,\zeta)$ is a univalent function in $U \times \overline{U}$.

Since $h(z,\zeta)$ is univalent, strong differential subordination (14) is equivalent to:

$$\left\{\theta\left(p(z,\zeta)\right) + zp'_z(z,\zeta)\,\phi\left(p(z,\zeta), z^2p''_z(z,\zeta), z^3p'''_z(z,\zeta)\right)\right\} \subset h\left(U \times \overline{U}\right). \tag{18}$$

For $z = z_0 \in U$, relation (18) becomes

$$\left\{\theta\left(p(z_0,\zeta)\right) + z_0p'_z(z_0,\zeta)\,\phi\left(p(z_0,\zeta), z_0^2p''_z(z_0,\zeta), z_0^3p'''_z(z_0,\zeta)\right)\right\} \in h\left(U \times \overline{U}\right). \tag{19}$$

For completing the proof, Lemma 1 will be applied. In order to do that, consider the function $\psi : \mathbb{C}^4 \to \mathbb{C}$ given by:

$$\psi(r,s,t,u) = \theta(r) + s\phi(r,t,u). \tag{20}$$

For $r = p(z,\zeta)$, $s = zp'_z(z,\zeta)$, $t = z^2p''_z(z,\zeta)$, $u = z^3p'''_z(z,\zeta)$, relation (20) becomes

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_z(z,\zeta), z^3p'''_z(z,\zeta)\right) = \tag{21}$$

$$\theta\left(p(z,\zeta)\right) + zp'_z(z,\zeta)\,\phi\left(p(z,\zeta), z^2p''_z(z,\zeta), z^3p'''_z(z,\zeta)\right).$$

Using (21), strong differential subordination (14) becomes:

$$\psi\left(p(z,\zeta), zp'_z(z,\zeta), z^2p''_z(z,\zeta), z^3p'''_z(z,\zeta)\right) \prec\prec h(z,\zeta). \tag{22}$$

We assume that $p(z,\zeta) \prec\not\prec q(z,\zeta)$. Then, according to Lemma 1 there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$, such that:

$$p(z_0,\zeta) = q(\zeta_0,\zeta),\quad z_0p'_z(z_0,\zeta) = m\zeta_0q'(\zeta_0,\zeta),\quad t = z_0^2p''_z(z_0,\zeta),\quad u = z_0^3p'''_z(z_0,\zeta), \tag{23}$$

satisfying conditions $(i - v)$ of Lemma 1.

By replacing in the admissibility condition $r = q(\zeta_0,\zeta)$, $s = m\zeta_0q'(\zeta_0,\zeta)$, t, u, given by relation (23), we have:

$$\psi\left(q(\zeta_0,\zeta), m\zeta_0q'(\zeta_0,\zeta), t, u\right) \notin h\left(U \times \overline{U}\right). \tag{24}$$

Using the equalities given by (23), we write:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_z(z_0,\zeta), z_0^3p'''_z(z_0,\zeta)\right) \notin h\left(U \times \overline{U}\right). \tag{25}$$

If in relation (21), we take $z = z_0$, we obtain:

$$\psi\left(p(z_0,\zeta), z_0p'_z(z_0,\zeta), z_0^2p''_z(z_0,\zeta), z_0^3p'''_z(z_0,\zeta)\right) = \tag{26}$$

$$\theta\left(p\left(z_0,\zeta\right)\right)+z_0\phi\left(p\left(z_0,\zeta\right),z_0^2p_z''\left(z_0,\zeta\right),z_0^3p_z'''\left(z_0,\zeta\right)\right).$$

Using (26) in (25), we write:

$$\theta\left(p\left(z_0,\zeta\right)\right)+z_0\phi\left(p\left(z_0,\zeta\right),z_0^2p_z''\left(z_0,\zeta\right),z_0^3p_z'''\left(z_0,\zeta\right)\right)\notin h\left(U\times\overline{U}\right). \tag{27}$$

Since (27) contradicts (19), we conclude that the assumption made is not true, hence:

$$p\left(z,\zeta\right)\prec\prec q\left(z,\zeta\right),\quad z\in U,\ \zeta\in\overline{U}.$$

Since function $q\left(z,\zeta\right)$ is convex, we know that it is univalent, and since it verifies the relation

$$h\left(z,\zeta\right)=\theta\left(q\left(z,\zeta\right)\right)+zq_z'\left(z,\zeta\right)\cdot\phi\left(q\left(z,\zeta\right),z^2q_z''\left(z,\zeta\right),z^3q_z'''\left(z,\zeta\right)\right),$$

we have that it is the best dominant of the strong differential subordination (14).

Remark 4 *For $p\left(z,\zeta\right)=A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)=z+\frac{a(\zeta)}{c(\zeta)}\frac{z^2}{2!}+\frac{a(\zeta)[a(\zeta)+1]}{c(\zeta)[c(\zeta)+1]}\frac{z^3}{3!}+...$ with $p\left(0,\zeta\right)=0$ and $p'\left(0,\zeta\right)=1\neq0$, $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}$ cu $q\left(0,\zeta\right)=0$, $q'\left(0,\zeta\right)=1$, a convex function in $U\times\overline{U}$, using Theorem 1 we write the following corollary:*

Corollary 1 *Consider the function $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}$, with $q\left(0,\zeta\right)=0$, $q'\left(0,\zeta\right)=1$, the functions θ si $\phi\in\mathcal{H}\left(D\right)$, $D\subset U\times\overline{U}$, where the domain $q\left(U\times\overline{U}\right)\subset D$, $\phi\left(w\right)\neq0$, $w\in q\left(U\times\overline{U}\right)$ and let $p\left(z,\zeta\right)=A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)$ be analytic in $U\times\overline{U}$, $p\left(0,\zeta\right)=q\left(0,\zeta\right)=0$ and $p\left(U\times\overline{U}\right)\subset D$.*

Let

$$\begin{aligned}Q\left(z,\zeta\right)&=zq_z'\left(z,\zeta\right)\phi\left(q\left(z,\zeta\right),z^2q_z''\left(z,\zeta\right),z^3q_z'''\left(z,\zeta\right)\right)\\&=\left(z-z^2\zeta\right)\phi\left(z-z^2\frac{\zeta}{2},-z^2\zeta,0\right)\end{aligned}$$

and

$$h\left(z,\zeta\right)=\theta\left(q\left(z,\zeta\right)\right)+Q\left(z,\zeta\right)=$$
$$\theta\left(z-\frac{z^2\zeta}{2}\right)+\left(z-z^2\zeta\right)\phi\left(z-\frac{z^2\zeta}{2},-z^2\zeta,0\right).$$

Suppose that:

(i) $Re\frac{\zeta q_z''(z,\zeta)}{q_z'(z,\zeta)}=Re\frac{-\zeta\xi}{1-\zeta\xi}>0$ and $\left|\frac{z\left(A_z^{-1}F(a(\zeta),c(\zeta);z,\zeta)\right)_z'}{q_z'(xi,\zeta)}\right|\leq u$, $z\in U$, $\zeta\in\overline{U}$, $\xi\in\partial U\backslash E\left(q\left(\cdot,\zeta\right)\right)$, $u\geq2$.

(ii) Functions $Q\left(z,\zeta\right)$ and $\phi\left(z,\zeta\right)$ are starlike $U\times\overline{U}$,

(iii) $Re\frac{\theta'\left(z-\zeta\frac{z^2}{2}\right)}{\phi\left(z-\frac{z^2\zeta}{2},-z^2\zeta,0\right)}>0$, $z\in U$, $\zeta\in\overline{U}$.

If function $p\left(z,\zeta\right)=A_z^{-1}F_z'\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)$ is analytic in $U\times\overline{U}$ cu $p\left(0,\zeta\right)=q\left(0,\zeta\right)$, then

$$\theta\left(A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\right)+z\left(A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\right)_z'\cdot$$

(28)

$$\phi\left(A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right),z^2\left(A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\right)''_z,z^3\left(A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\right)'''_z\right) \prec\prec h\left(z,\zeta\right),$$

implies

a) $A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\prec\prec z-\zeta\frac{z^2}{2}$,

b) $ReA_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)>Re\left(z-\zeta\frac{z^2}{2}\right)$, $z\in U$, $\zeta\in\overline{U}$ *and* $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}$ *is the best dominant of the strong differential subordination considered.*

Proof. We prove that function $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}, q'(0,\zeta)=1$, is convex in $U\times\overline{U}$. After a simple calculation, we write:

$$\mathrm{Re}\left(\frac{zq''_z\left(z,\zeta\right)}{q'_z\left(z,\zeta\right)}+1\right)=2-\mathrm{Re}\frac{1}{1-z\zeta}=$$

$$2-\mathrm{Re}\frac{1}{1-\left(\rho\cos\alpha+i\rho\sin\alpha\right)\left(\rho\cos\beta+i\rho\sin\beta\right)}=$$

$$\frac{1+2\rho^4-3\rho^2\cos\left(\alpha+\beta\right)}{1-2\rho^2\cos\left(\alpha+\beta\right)+\rho^4}.$$

For $\rho\to 1$ we obtain:

$$\mathrm{Re}\left(\frac{zq''_z\left(z,\zeta\right)}{q'_z\left(z,\zeta\right)}+1\right)=\lim_{\rho\to 1}\frac{1+2\rho^4-3\rho^2\cos\left(\alpha+\beta\right)}{1-2\rho^2\cos\left(\alpha+\beta\right)+\rho^4}=$$

$$\frac{1+2-3\cos\left(\alpha+\beta\right)}{1-2\cos\left(\alpha+\beta\right)+1}=\frac{3\left[1-\cos\left(\alpha+\beta\right)\right]}{2\left[1-\cos\left(\alpha+\beta\right)\right]}=\frac{3}{2}\geq 0,$$

and we conclude that $q\left(z,\zeta\right)$ is convex in $U\times\overline{U}$.

We assess now:

$$\mathrm{Re}\frac{\zeta q''_z\left(z,\zeta\right)}{q'_z\left(z,\zeta\right)}=\mathrm{Re}\frac{-\zeta\zeta}{1-\zeta\zeta}=1-\mathrm{Re}\frac{\zeta\zeta}{1-\zeta\zeta}=\frac{\rho^4-\rho^2\cos\left(\alpha+\beta\right)}{1-2\rho^2\cos\left(\alpha+\beta\right)+\rho^4}.$$

For $\rho\to 1$ we get:

$$\mathrm{Re}\frac{\zeta q''_z\left(z,\zeta\right)}{q'_z\left(z,\zeta\right)}=\lim_{\rho\to 1}\frac{\rho^4-\rho^2\cos\left(\alpha+\beta\right)}{1-2\rho^2\cos\left(\alpha+\beta\right)+\rho^4}=\frac{1-\cos\left(\alpha+\beta\right)}{2\left(1-\cos\left(\alpha+\beta\right)\right)}=\frac{1}{2}\geq 0.$$

Having fulfilled the conditions of Theorem 1, for $p\left(z,\zeta\right)=A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)$ and $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}$, we have:

(29) $$A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)\prec\prec z-\zeta\frac{z^2}{2},\ \ z\in U,\zeta\in\overline{U}.$$

Since $q\left(z,\zeta\right)=z-\zeta\frac{z^2}{2}$ is convexa in $U\times\overline{U}$ and $A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);0,\zeta\right)=q\left(0,\zeta\right)=0$, the strong differential subordination (29) is equivalent to:

$$\mathrm{Re}A_z^{-1}F\left(a\left(\zeta\right),c\left(\zeta\right);z,\zeta\right)>\mathrm{Re}\left(z-\zeta\frac{z^2}{2}\right),\quad z\in U,\ \zeta\in\overline{U}.$$

Remark 5 *For $p(z,\zeta)$ taken as the function:*

$$D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)=z+\frac{a(\zeta)\,b(\zeta)}{c(\zeta)}\frac{z^2}{2!}+\frac{a(\zeta)\,[a(\zeta)+1]\,b(\zeta)\,[b(\zeta)+1]}{c(\zeta)\,[c(\zeta)+1]}\frac{z^3}{3!}+\cdots$$

and $q(z,\zeta)=z+\zeta\frac{z^2}{2}$, $z\in U$, $\zeta\in\overline{U}$, using Theorem 1 we write the following corollary.

Corollary 2 *Consider the function $q(z,\zeta)=z+\zeta\frac{z^2}{2}, q'(0,\zeta)=1$, convex in $U\times\overline{U}$, the functions θ, $\phi\in\mathcal{H}(D)$, where the domain $q(U\times\overline{U})\subset D$, $\phi(w)\neq 0$, $w\in q(U\times\overline{U})$, and let $p(z,\zeta)=D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)$ with $p(0,\zeta)=D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);0,\zeta)=0$ with $p(U\times\overline{U})\subset D$.*

Let

$$\begin{aligned}Q(z,\zeta)&=zq_z'(z,\zeta)\,\phi\left(q(z,\zeta),z^2q_z''(z,\zeta),z^3q_z'''(z,\zeta)\right)\\&=\left(z+z^2\zeta\right)\phi\left(z+z^2\frac{\zeta}{2},z^2\zeta,0\right),\end{aligned}$$

and

$$h(z,\zeta)=\theta\left(z+\zeta\frac{z^2}{2}\right)+\left(z+z^2\zeta\right)\phi\left(z+\frac{z^2}{2}\zeta,z^2\zeta,0\right).$$

Suppose that:

(i) $Re\frac{\zeta q_z''(z,\zeta)}{q_z'(z,\zeta)}\geq 0$ and $\left|\frac{z\left(D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\right)_z'}{q_z'(\zeta,\zeta)}\right|\leq u$, $z\in U$, $\zeta\in\overline{U}$, $u\geq 2$, $\zeta\in\partial U$.

(ii) Functions θ and ϕ are starlike in $U\times\overline{U}$,

(iii) $Re\frac{\theta'\left(z+\frac{z^2\zeta}{2}\right)}{\phi\left(z+\frac{z^2\zeta}{2},z^2\zeta,0\right)}>0$, $z\in U$, $\zeta\in\overline{U}$.

If the function $p(z,\zeta)=D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\in\mathcal{H}(U\times\overline{U})$ with $p(0,\zeta)=q(0,\zeta)$, $p(U\times\overline{U})\subset D$, then:

$$\begin{aligned}(30)\qquad&\theta\left(D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\right)+z\left(D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\right)_z'\cdot\\&\phi\left(p(z,\zeta),z^2p(z,\zeta)_z'',z^3\left(p(z,\zeta)\right)_z'''\right)\prec\prec h(z,\zeta),\quad z\in U,\zeta\in\overline{U},\end{aligned}$$

implies

a) $D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\prec\prec z+\frac{z^2\zeta}{2}$, $z\in U,\zeta\in\overline{U}$;

b) $ReD_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)>Re\left(z+\frac{z^2\zeta}{2}\right)$, $z\in U$, $\zeta\in\overline{U}$.

and $q(z,\zeta)=z+\frac{z^2\zeta}{2}$ is the best dominant of the strong differential subordination considered.

Proof. We prove that the function $q(z,\zeta)=z+\frac{z^2}{2}\zeta$ is convex in $U\times\overline{U}$. After a simple calculation, we write:

$$\mathrm{Re}\left(\frac{zq_z''(z,\zeta)}{q_z'(z,\zeta)}+1\right)=\mathrm{Re}\left(\frac{z\zeta}{1+z\zeta}\right)=2-\mathrm{Re}\frac{1}{1+z\zeta}=$$

$$2-\mathrm{Re}\frac{1}{1+\rho^2\cos(\alpha+\beta)+i\rho^2\sin(\alpha+\beta)}=$$
$$\frac{2\rho^4+1+3\rho^2\cos(\alpha+\beta)}{1+2\rho^2\cos(\alpha+\beta)+\rho^4}.$$

For $\rho\to 1$ we obtain:

$$\mathrm{Re}\left(\frac{zq''_z(z,\zeta)}{q'_z(z,\zeta)}+1\right)=\lim_{\rho\to 1}\frac{2\rho^4+1+3\rho^2\cos(\alpha+\beta)}{1+2\rho^2\cos(\alpha+\beta)+\rho^4}=$$
$$\frac{3[1+\cos(\alpha+\beta)]}{2[1+\cos(\alpha+\beta)]}=\frac{3}{2}\geq 0,$$

and we conclude that $q(z,\zeta)=z+\frac{z^2}{2}\zeta$ is a convex function in $U\times\overline{U}$.

We assess:

$$\mathrm{Re}\frac{\zeta q''_z(\zeta,\zeta)}{q'_z(\zeta,\zeta)}=\mathrm{Re}\frac{z\zeta}{1+z\zeta}=1-\mathrm{Re}\frac{1}{1+z\zeta}=\frac{\rho^4+\rho^2\cos(\alpha+\beta)}{1+2\rho^2\cos(\alpha+\beta)+\rho^4}.$$

For $\rho\to 1$ we obtain:

$$\mathrm{Re}\frac{\zeta q''_z(\zeta,\zeta)}{q'_z(\zeta,\zeta)}=\lim_{\rho\to 1}\frac{\rho^4+\rho^2\cos(\alpha+\beta)}{1+2\rho^2\cos(\alpha+\beta)+\rho^4}=\frac{1+\cos(\alpha+\beta)}{2(1+\cos(\alpha+\beta))}=\frac{1}{2}\geq 0.$$

Having fulfilled the conditions of Theorem 1 pentru

$$p(z,\zeta)=D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)$$

si

$$q(z,\zeta)=z+\frac{z^2\zeta}{2},$$

we have:

a) $D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\prec\prec z+\frac{z^2\zeta}{2},\ \ z\in U,\ \zeta\in\overline{U}$.

Since $q(z,\zeta)=z+\frac{z^2\zeta}{2}$ is convex in $U\times\overline{U}$, and $D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);0,\zeta)=q(0,\zeta)$, the domain $q\left(U\times\overline{U}\right)$ is convex, and subordination a) is equivalent to:

$$\text{b) }\mathrm{Re}D_z^{-1}F(a(\zeta),b(\zeta),c(\zeta);z,\zeta)\geq\mathrm{Re}\left(z+\frac{z^2\zeta}{2}\right),\quad z\in U,\ \zeta\in\overline{U}.$$

Example 1 *For $a(\zeta)=-2$, $b(\zeta)=\zeta+i$, $c(\zeta)=\zeta-i$, we have:*

$$D_z^{-1}F(-2,\zeta+i,\zeta-i;z,\zeta)=z-\frac{\zeta^2-1+2\zeta i}{1+\zeta^2}\frac{z^2}{2}+\frac{\left(\zeta^2-1+2\zeta i\right)\left[\zeta^2+2\zeta+2i(1+\zeta)\right]}{(\zeta^2+1)(2+2\zeta+\zeta^2)}\frac{z^3}{6}.$$

Using Corollary 2, the function

$$p(z,\zeta)=z-\frac{\zeta^2-1+2\zeta i}{1+\zeta^2}\frac{z^2}{2}+\frac{\left(\zeta^2-1+2\zeta i\right)\left[\zeta^2+2\zeta+2i(1+\zeta)\right]}{(\zeta^2+1)(2+2\zeta+\zeta^2)}\frac{z^3}{6},$$

and $q(z,\zeta) = z + \frac{\zeta}{2}z^2$, *we can write:*

$$z - \frac{\zeta^2 - 1 + 2\zeta i}{1+\zeta^2}\frac{z^2}{2} + \frac{\left(\zeta^2 - 1 + 2\zeta i\right)\left[\zeta^2 + 2\zeta + 2i\left(1+\zeta\right)\right]}{\left(\zeta^2+1\right)\left(2+2\zeta+\zeta^2\right)}\frac{z^3}{6} \prec\prec z + \frac{\zeta}{2}z^2,$$

and

$$Re\left(z - \frac{\zeta^2 - 1 + 2\zeta i}{1+\zeta^2}\frac{z^2}{2} + \frac{\left(\zeta^2 - 1 + 2\zeta i\right)\left[\zeta^2 + 2\zeta + 2i\left(1+\zeta\right)\right]}{\left(\zeta^2+1\right)\left(2+2\zeta+\zeta^2\right)}\frac{z^3}{6}\right) \geq Re\left(z + \frac{\zeta}{2}z^2\right),$$

$z \in U$, $\zeta \in \overline{U}$.

References

[1] J. A. Antonino, S.S. Miller, *Third-order differential inequalities and subordinations in the complex plane*, Complex Var. Elliptic Equ., vol. 56, no. 5, 2011, 439-454. https://doi.org/10.1080/17476931003728404

[2] S. S. Miller, P. T. Mocanu, *Second order-differential inequalities in the complex plane*, J. Math. Anal. Appl., vol. 65, 1978, 298-305.

[3] S. S. Miller, P. T. Mocanu, *Differential subordinations and univalent functions*, Michig. Math. J., vol. 28, 1981, 157-171.

[4] V. Gupta, T. M. Rassias, P. N. Agrawal, A. M. Acu, *Recent Advances in Constructive Approximation Theory*, Cham, Springer, 2018.

[5] H. M. Zayed, T. Bulboacă, *Applications of differential subordinations involving a generalized fractional differintegral operator*, J. Inequal. Appl., 2019, 242. https://doi.org/10.1186/s13660-019-2198-0

[6] H. Al-Janaby, F. Ghanim, M. Darus, *On The Third-Order Complex Differential Inequalities of ζ-Generalized-Hurwitz–Lerch Zeta Functions*, Mathematics, vol. 8, 2020, 845. https://doi.org/10.3390/math8050845

[7] A. A. Attiya, T. M. Seoudy, A. Albaid, *Third-Order Differential Subordination for Meromorphic Functions Associated with Generalized Mittag-Leffler Function*, Fractal Fract., vol. 7, 2023, 175. https://doi.org/10.3390/fractalfract7020175

[8] W. G. Atshan, R. A. Hiress, S. Altınkaya, *On Third-Order Differential Subordination and Superordination Properties of Analytic Functions Defined by a Generalized Operator*, Symmetry, vol. 14, 2022, 418. https://doi.org/10.3390/sym14020418

[9] G. I. Oros, G. Oros, L. F. Preluca, *Third-Order Differential Subordinations Using Fractional Integral of Gaussian Hypergeometric Function*, Axioms 2023, 12, 133. https://doi.org/10.3390/axioms12020133

[10] G. I. Oros, G. Oros, L. F. Preluca, *New Applications of Gaussian Hypergeometric Function for Developments on Third-Order Differential Subordinations*, Symmetry, vol. 15, 2023, 1306. https://doi.org/10.3390/sym15071306

[11] M. M. Soren, A. K. Wanas, L. -I. Cotîrlă, *Results of Third-Order Strong Differential Subordinations*, Axioms, vol. 13, 2024, 42. https://doi.org/10.3390/axioms13010042

[12] G. I. Oros, G. Oros, *Strong differential subordination*, Turk. J. Math., vol. 33, 2009, 249-257.

[13] J. A. Antonino, S. Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, J. Differ. Equ., vol. 114, 1994, 101-105.

[14] G. I. Oros, *On a new strong differential subordination*, Acta Univ. Apulensis, vol. 32, 2012, 243-250.

[15] P. Arjomandinia, R. Aghalary, *Strong subordination and superordination with sandwich-type theorems using integral operators*, Stud. Univ. Babeş-Bolyai Math., vol. 66, 2021, 667-675.

[16] A. Alb Lupaş, *Applications of a Multiplier Transformation and Ruscheweyh Derivative for Obtaining New Strong Differential Subordinations*, Symmetry, vol. 13, 2021, 1312. https://doi.org/10.3390/sym13081312

[17] R. Aghalary, P. Arjomandinia, *On a first order strong differential subordination and application to univalent functions*, Commun. Korean Math. Soc., vol. 37, 2022, 445-454.

[18] A. Alb Lupaş, F. Ghanim, *Strong Differential Subordination and Superordination Results for Extended q-Analogue of Multiplier Transformation*, Symmetry, vol. 15, 2023, 713. https://doi.org/10.3390/sym15030713

[19] H. Tang, H. M. Srivastava, S.-H. Li, L. Ma, *Third-Order Differential Subordination and Superordination Results for Meromorphically Multivalent Functions Associated with the Liu-Srivastava Operator*, Abstr. Appl. Anal., 2014, 1-11.

[20] L. De Branges, *A proof of the Bieberbach conjecture.* Acta Math., vol. 154, 1985, 137-152. http://doi.org/10.1007/BF02392821

[21] S. S. Miller, P. T. Mocanu, *Univalence of Gaussian and confluent hypergeometric Functions*, Proc. Am. Math. Soc., vol. 110, 1990, 333-342. http://dx.doi.org/10.1090/S0002-9939-1990-1017006-8

[22] S. Owa, *On the distortion theorems I*, Kyungpook Math. J., vol. 18, 1978, 53-59.

[23] S. Owa, H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Can. J. Math., vol. 39, 1987, 1057-1077.

[24] G. I. Oros, G. Oros, A. M. Rus, *Applications of confluent hypergeometric function in strong superordination theory*, Axioms, vol. 11, no. 5, 2022, 209. https://doi.org/10.3390/axioms11050209

[25] A. Alb Lupaş, G. I. Oros, *Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function*, Symmetry, vol. 13, 2021, 327. https://doi.org/10.3390/sym13020327

[26] A. Alb Lupaş, G. I. Oros, *Fractional Integral of a Confluent Hypergeometric Function Applied to Defining a New Class of Analytic Functions*, Symmetry, vol. 14, 2022, 427. https://doi.org/10.3390/sym14020427

[27] G. I. Oros, S. Dzitac, *Applications of Subordination Chains and Fractional Integral in Fuzzy Differential Subordinations*, Mathematics, vol. 10, 2022, 1690. https://doi.org/10.3390/math10101690

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