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New properties of the Intermediate Point from Bonnet Theorem [1]

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Abstract

In this paper, the following result is presented: if $f,\ g : [a, b] \to \mathbb{R}$ are two continuous functions and f is monotone, then there exists a function $\overline{c} : [a, b] \to [a, b]$ continuous in a with the property:

$$\int_a^x f(t)\, g(t)\, \mathrm{d}t = f(a) \int_a^{\overline{c}(x)} g(t)\, \mathrm{d}t + f(x) \int_{\overline{c}(x)}^x g(t)\, \mathrm{d}t,$$

for all $x \in [a, b]$. Then, under specified conditions, the function $\overline{c}$ that is differentiable at point a is presented, and $\overline{c}'(a)$ is also calculated.

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1 Introduction

Mean-value theorems play an important role in mathematical analysis. One of the most significant and widely used mean-value theorems in integral calculus is presented below.

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Theorem 1 *[3] Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists an intermediate point $c \in [a, b]$ such that*

$$\int_a^b f(x)\,\mathrm{d}x = (b-a)\, f(c).$$

Properties of the intermediate point c, from this theorem, have been studied in several papers starting from 1931 ([1] – [10], [12]).

In what follows, we present two results related to the intermediate point c from Theorem 1, results that will be compared with our findings for an important mean-value theorem in integral calculus.

Theorem 2 *[3] Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$. If*

(a) the function f is derivable on $[a, b]$,

(b) the derivable f' of the function f is continuous at the point a,

(c) $f'(a) \neq 0$,

then

1^0 *There exists a continuous function $\overline{c} : [a, b] \to \mathbb{R}$ with the property*

$$\int_a^x f(t)\,\mathrm{d}t = (x-a)\, f(\overline{c}(x)), \text{ for all } x \in [a, b]. \tag{1}$$

2^0 *There exists a function $\theta : (a, b] \to [0, 1]$ with the property*

$$\int_a^x f(t)\,\mathrm{d}t = (x-a)\, f(a + (x-a)\,\theta(x)), \text{ for all } x \in (a, b].$$

3^0 *The function θ has limit at the point $x = a$ and*

$$\lim_{x \to a} \theta(x) = \frac{1}{2}.$$

4^0 *The function $\overline{c}$ is derivable at the point $x = a$ and*

$$\overline{c}'(a) = \frac{1}{2}.$$

Theorem 3 *[3] Let $a, b \in \mathbb{R}$ with $a < b$, $n \geq 2$ a natural number and $f : [a, b] \to \mathbb{R}$. If*

(a) the function f is derivable of n times on the interval $[a, b]$,

(b) the function $f^{(n)}$ is continuous at the point a,

(c) $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$,

(d) $f^{(n)}(a) \neq 0$,

then the following statements are true:

1^0 *If $\theta : (a, b] \to [0, 1]$ is a function that satisfies the relation*

$$\int_a^x f(t)\,\mathrm{d}t = (x-a)\, f(a + (x-a)\,\theta(x)), \tag{2}$$

for all $x \in (a, b]$, *then the function* θ *has limit at the point* a *and*

$$\lim_{x \to a} \theta(x) = \frac{1}{\sqrt[n]{n+1}}. \tag{3}$$

2^0 *If* $c : (a, b] \to [a, b]$ *is a function that satisfies the relation*

$$\int_a^x f(t)\,\mathrm{d}t = (x - a) f(c(x)), \text{ for all } x \in (a, b],$$

then the function $\overline{c} : [a, b] \to [a, b]$ *defined by*

$$\overline{c}(x) := \begin{cases} c(x), & \text{if } x \in (a, b] \\ a, & \text{if } x = a, \end{cases}$$

is derivable at the point a *and*

$$\overline{c}'(a) = \frac{1}{\sqrt[n]{n+1}}. \tag{4}$$

The mean-value theorem considered in this paper, known as the Bonnet Theorem or the second mean-value theorem in integral calculus, is as follows

Theorem 4 *[3] Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f, g : [a, b] \to \mathbb{R}$. *If*
(a) *the function* f *is monotone on* $[a, b]$,
(b) *the function* g *is Riemann integrable on* $[a, b]$,
then the function fg *is Riemann integrable on* $[a, b]$ *and there exists a point* $c \in [a, b]$ *such that*

$$\int_a^b f(x) g(x)\,\mathrm{d}x = f(a) \int_a^c g(x)\,\mathrm{d}x + f(b) \int_c^b g(x)\,\mathrm{d}x.$$

This paper presents several properties of the intermediate point c from Theorem 4.

2 Preliminaries

From Theorem 4, follows that, if the functions $f, g : [a, b] \to \mathbb{R}$ are continuous and f is monotone on $[a, b]$, then, for each $x \in (a, b]$ there exists a point $c_x \in [a, x]$ such that:

$$\int_a^x f(t) g(t)\,\mathrm{d}t = f(a) \int_a^{c_x} g(t)\,\mathrm{d}t + f(x) \int_{c_x}^x g(t)\,\mathrm{d}t. \tag{5}$$

If for each $x \in (a, b]$ there exists a unique point $c_x \in [a, x]$ such that (5) holds, we can define the function $c : (a, b] \to [a, b]$ by

$$c(x) = c_x, \text{ for all } x \in (a, b]. \tag{6}$$

The function c has the property that

$$\int_a^x f(t)\,g(t)\,\mathrm{d}t = f(a)\int_a^{c(x)} g(t)\,\mathrm{d}t + f(x)\int_{c(x)}^x g(t)\,\mathrm{d}t, \tag{7}$$

for all $x \in (a,b]$.

If for some $x \in (a,b]$, there exist more points $c_x \in [a,x]$ such that (5) is true, then for each $x \in (a,b]$ choose $c_x \in [a,x]$ which satisfies (7). It follows that we can also define the function $c : (a,b] \to [a,b]$ by formula (6). This function c satisfies (7), too.

Consequently, the following statement is true.

Theorem 5 *Let $a, b \in \mathbb{R}$ with $a < b$. If the functions $f, g : [a,b] \to \mathbb{R}$ are continuous and f is monotone on $[a,b]$, then there exists a function $c : (a,b] \to [a,b]$ such that (7) is true.*

If $x \in (a,b]$ tends to a, because $|c(x) - a| \leq |x - a|$, we have

$$\lim_{x \to a} c(x) = a.$$

Then the function $\overline{c} : [a,b] \to [a,b]$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in (a,b] \\ a, & \text{if } x = a, \end{cases}$$

is continuous at $x = a$.

The purpose of this paper is to establish under which circumstances the function $\overline{c}$ is differentiable at the point $x = a$ and to compute its derivatives. Do the derivatives depend upon the functions f and g? If there exist several functions $\overline{c}$ which satisfy (7), do the derivatives of the function $\overline{c}$ at $x = a$ depend upon the function $\overline{c}$ we choose?

Since for $x \in (a,b]$,

$$\frac{\overline{c}(x) - \overline{c}(a)}{x - a} = \frac{c(x) - a}{x - a},$$

if we denote by

$$\theta(x) = \frac{c(x) - a}{x - a},$$

then

$$\theta(x) \in [0,1]$$

and

$$c(x) = a + (x - a)\,\theta(x) \tag{8}$$

and hence

$$\int_a^x f(t)\,g(t)\,\mathrm{d}t = \tag{9}$$

$$= f(a)\int_a^{a+(x-a)\theta(x)} g(t)\,\mathrm{d}t + f(x)\int_{a+(x-a)\theta(x)}^{x} g(t)\,\mathrm{d}t,$$

for all $x \in (a,b]$.

Consequently, the following statement is true.

Theorem 6 *Let $a,b \in \mathbb{R}$ with $a < b$. If the functions $f,\, g : [a,b] \to \mathbb{R}$ are continuous and f is monotone on $[a,b]$, then there exists a function $\theta : (a,b] \to [0,1]$ such that (9) is true.*

Another purpose of this paper is to establish in which conditions the function $\theta : (a,b] \to [0,1]$ has limit at the point $x = a$.

3 Main results

Note that H is an antiderivative of the function fg and G is an antiderivative of the function g, i.e.

$$H'(x) = f(x)\,g(x)\,,\quad G'(x) = g(x)\,,\ \text{for all } x \in [a,b]\,,$$

then, from the Leibniz-Newton Theorem, the equality (8) becomes

$$\begin{aligned}(10)\qquad & H(x) - H(a) = \\ & = f(a)[G(a+(x-a)\theta(x)) - G(a)] + f(x)[G(x) - G(a+(x-a)\theta(x))],\end{aligned}$$

for all $x \in (a,b]$, or, equivalently

$$\begin{aligned}(11)\qquad & \frac{f(x)-f(a)}{x-a}\left[G(a+(x-a)\,\theta(x)) - G(a)\right] = \\ & = \frac{G(x)-G(a)}{x-a} f(x) - \frac{H(x)-H(a)}{x-a},\end{aligned}$$

for all $x \in (a,b]$.

Next, we present some results related to the intermediate point functions $\overline{c}$ and θ.

Theorem 7 *Let $a,b \in \mathbb{R}$ with $a < b$ and $f,g : [a,b] \to \mathbb{R}$ be twice differentiable functions on $[a,b]$. If*

(a) the function f is monotone on $[a,b]$,

(b) $f'(a)g(a) \neq 0$,

then

1° The function $\theta : (a,b] \to [0,1]$ has limit at the point $x = a$ and

$$\lim_{x\to a}\ \theta(x) = \frac{1}{2}.$$

2° The function $\overline{c}$ is differentiable at $x = a$ and

$$\overline{c}'(a) = \frac{1}{2}.$$

Proof. 1^0 The relation (11) is equivalent to

$$\frac{f\left(x\right)-f\left(a\right)}{x-a}\frac{G(a+(x-a)\theta(x))-G(a)}{\left(x-a\right)\theta\left(x\right)}\theta\left(x\right)=$$

$$=\frac{\frac{G(x)-G(a)}{x-a}-G'\left(a\right)}{x-a}f\left(x\right)+\frac{G'\left(a\right)f\left(x\right)-H'\left(a\right)}{x-a}-\frac{\frac{H(x)-H(a)}{x-a}-H'\left(a\right)}{x-a},$$

for all $x\in(a,b]$.

By letting $x\to a$, we obtain

$$f'\left(a\right)G'\left(a\right)\lim_{x\to a}\theta\left(x\right)=\frac{1}{2}G''\left(a\right)f\left(a\right)+G'\left(a\right)f'\left(a\right)-\frac{1}{2}H''\left(a\right).$$

Since

$$G'\left(a\right)=g\left(a\right),\ G''\left(a\right)=g'\left(a\right),$$

and

$$H'\left(a\right)=f\left(a\right)g\left(a\right),\ H''\left(a\right)=f'\left(a\right)g\left(a\right)+f\left(a\right)g'\left(a\right),$$

it follows that

$$f'\left(a\right)g\left(a\right)\lim_{x\to a}\theta\left(x\right)=$$

$$=\frac{1}{2}f\left(a\right)g'\left(a\right)+f'\left(a\right)g\left(a\right)-\frac{1}{2}\left[f'\left(a\right)g\left(a\right)+f\left(a\right)g'\left(a\right)\right].$$

Hence

$$\lim_{x\to a}\theta\left(x\right)=\frac{1}{2}.$$

2^0 Follows from (8) and 1^0.

In what follows we analyze the calculus of $\overline{c}'\left(a\right)$ and of the limit $\lim\limits_{x\to a}\theta\left(x\right)$ when $f'\left(a\right)=0$ and $g\left(a\right)\neq 0$.

The following statement is true.

Theorem 8 *Let $a,b\in\mathbb{R}$ with $a<b$ and $f,g:[a,b]\to\mathbb{R}$ be thrice differentiable functions on $[a,b]$. If*

(a) the function f is monotone on $[a,b]$,

(b) $f'(a)=0,\ f''\left(a\right)\neq 0,\ g\left(a\right)\neq 0,$

then

1° The function $\theta:(a,b]\to[0,1]$ has limit at the point $x=a$ and

$$\lim_{x\to a}\ \theta(x)=\frac{2}{3}.$$

2° The function $\overline{c}$ is differentiable at $x=a$ and

$$\overline{c}'(a)=\frac{2}{3}.$$

Proof. 1^0 By Taylor's Theorem, for each $x \in (a,b]$, there exist $c_f, c_G, c_H \in [0,1]$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(c_f)(x-a)^3,$$

$$G(x) = G(a) + \frac{G'(a)}{1!}(x-a) + \frac{G''(a)}{2!}(x-a)^2 + \frac{1}{3!}G'''(c_G)(x-a)^3,$$

$$H(x) = H(a) + \frac{H'(a)}{1!}(x-a) + \frac{H''(a)}{2!}(x-a)^2 + \frac{1}{3!}H'''(c_H)(x-a)^3.$$

If we replace these relations in (11) we get

$$\left[f'(a) + \frac{1}{2!}f''(a)(x-a) + \frac{1}{3!}f'''(c_f)(x-a)^2\right] \times$$

$$\times \left[G'(a)\theta(x)(x-a) + \frac{1}{2!}G''(a)\theta^2(x)(x-a)^2 + \frac{1}{3!}G'''(c_G)\theta^3(x)(x-a)^3\right] =$$

$$= \left[G'(a) + \frac{1}{2!}G''(a)(x-a) + \frac{1}{3!}G'''(c_G)(x-a)^2\right] \times$$

$$\times \left[f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(c_f)(x-a)^3\right] -$$

$$-H'(a) - \frac{1}{2!}H''(a)(x-a) - \frac{1}{3!}H'''(c_H)(x-a)^2.$$

Since

$$G' = g,\ G'' = g',\ G''' = g'',$$

$$H' = fg,\ H'' = f'g + fg',\ H''' = f''g + 2f'g' + fg'',$$

the last relation becomes

$$f'(a)g(a)\theta(x)(x-a) + \frac{1}{2!}f'(a)g'(a)\theta^2(x)(x-a)^2 +$$

$$+\frac{1}{3!}f'(a)g''(c_G)\theta^3(x)(x-a)^3 +$$

$$+\frac{1}{2!}f''(a)g(a)\theta(x)(x-a)^2 + \frac{1}{2!}\frac{1}{2!}f''(a)g'(a)\theta^2(x)(x-a)^3 +$$

$$+\frac{1}{3!}\frac{1}{2!}f''(a)g''(c_G)\theta^3(x)(x-a)^4 +$$

$$+\frac{1}{3!}f'''(c_f)g(a)\theta(x)(x-a)^3 + \frac{1}{2!}\frac{1}{3!}f'''(c_f)g'(a)\theta^2(x)(x-a)^4 +$$

$$+\frac{1}{3!}\frac{1}{3!}f'''(c_f)g''(c_G)\theta^3(x)(x-a)^5 =$$

$$= f(a)\, g(a) + \frac{1}{2!} f(a)\, g'(a)\,(x-a) +$$

$$+\frac{1}{3!} f(a)\, g''(c_G)\,(x-a)^2 + f'(a)\, g(a)\,(x-a) +$$

$$+\frac{1}{2!} f'(a)\, g'(a)\,(x-a)^2 + \frac{1}{3!} f'(a)\, g''(c_G)\,(x-a)^3 +$$

$$+\frac{1}{2!} f''(a)\, g(a)\,(x-a)^2 + \frac{1}{2!}\frac{1}{2!} f''(a)\, g'(a)\,(x-a)^3 +$$

$$+\frac{1}{3!}\frac{1}{2!} f''(a)\, g''(c_G)\,(x-a)^4 + \frac{1}{3!} f'''(c_f)\, g(a)\,(x-a)^3 +$$

$$+\frac{1}{2!}\frac{1}{3!} f'''(c_f)\, g'(a)\,(x-a)^4 + \frac{1}{3!}\frac{1}{3!} f'''(c_f)\, g''(c_G)\,(x-a)^5 -$$

$$-f(a)\, g(a) - \frac{1}{2!} f'(a)\, g(a)\,(x-a) - \frac{1}{2!} f(a)\, g'(a)\,(x-a) -$$

$$-\frac{1}{3!} f''(c_H)\, g(c_H)\,(x-a)^2 - \frac{2}{3!} f'(c_H)\, g'(c_H)\,(x-a)^2 -$$

$$-\frac{1}{3!} f(c_H)\, g''(c_H)\,(x-a)^2 .$$

But $f'(a) = 0$, and hence we have

$$\frac{1}{2!} f''(a)\, g(a)\,\theta(x) + \frac{1}{2!}\frac{1}{2!} f''(a)\, g'(a)\,\theta^2(x)\,(x-a) +$$

$$+\frac{1}{3!}\frac{1}{2!} f''(a)\, g''(c_G)\,\theta^3(x)\,(x-a)^2 + \frac{1}{3!} f'''(c_f)\, g(a)\,\theta(x)\,(x-a) +$$

$$+\frac{1}{2!}\frac{1}{3!} f'''(c_f)\, g'(a)\,\theta^2(x)\,(x-a)^2 + \frac{1}{3!}\frac{1}{3!} f'''(c_f)\, g''(c_G)\,\theta^3(x)\,(x-a)^3 =$$

$$= \frac{1}{3!} f(a)\, g''(c_G) + \frac{1}{2!} f''(a)\, g(a) +$$

$$+\frac{1}{2!}\frac{1}{2!} f''(a)\, g'(a)\,(x-a) + \frac{1}{3!}\frac{1}{2!} f''(a)\, g''(c_G)\,(x-a)^2 +$$

$$+\frac{1}{3!} f'''(c_f)\, g(a)\,(x-a) + \frac{1}{2!}\frac{1}{3!} f'''(c_f)\, g'(a)\,(x-a)^2 +$$

$$+\frac{1}{3!}\frac{1}{3!} f'''(c_f)\, g''(c_G)\,(x-a)^3 -$$

$$-\frac{1}{3!} f''(c_H)\, g(c_H) - \frac{2}{3!} f'(c_H)\, g'(c_H) - \frac{1}{3!} f(c_H)\, g''(c_H) .$$

If now $x \in (a, b]$ tends to a,

$$|c_f - a| \leq |x-a|, \;\; |c_G - a| \leq |x-a|, |c_H - a| \leq |x-a|,$$

we have that c_f, c_G, $c_H \to a$, following that

$$\frac{1}{2!}f''(a)\,g(a)\lim_{x\to a}\theta(x) =$$

$$= \frac{1}{3!}f(a)\,g''(a) + \frac{1}{2!}f''(a)\,g(a) - \frac{1}{3!}f''(a)\,g(a) - \frac{1}{3!}f(a)\,g''(a),$$

or

$$\frac{1}{2!}f''(a)\,g(a)\lim_{x\to a}\theta(x) = \frac{2}{3!}f''(a)\,g(a)$$

and, consequently,

$$\lim_{x\to a}\theta(x) = \frac{2}{3}.$$

2^0 Follows from 1^0 by considering (8).

Next, we analyze the calculus of $\overline{c}'(a)$ and of the limit $\lim\limits_{x\to a}\theta(x)$ when

$$f'(a) = f''(a) = ... = f^{(n-1)}(a) = 0,\ \ f^{(n)}(a) \neq 0,\ \ g(a) \neq 0.$$

Theorem 9 *Let $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \to \mathbb{R}$ be $n+1$ times differentiable functions on $[a, b]$. If*

(a) $f^{(n+1)}$ and $g^{(n+1)}$ are continuous on $[a, b]$,

(b) the function f is monotone on $[a, b]$,

(c) $f'(a) = f''(a) = ... = f^{(n-1)}(a)$, $f^{(n)}(a) \neq 0$,

(d) $g(a) \neq 0$,

then

$1°$ The function $\theta : (a, b] \to [0, 1]$ has limit at the point $x = a$ and

$$\lim_{x\to a}\ \theta(x) = \frac{n}{n+1}.$$

$2°$ The function $\overline{c}$ is differentiable at $x = a$ and

$$\overline{c}'(a) = \frac{n}{n+1}.$$

Proof. $1°$ By Taylor's Theorem, for each $x \in (a, b]$, there exist $c_f, c_G, c_H \in [0, 1]$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n +$$
$$+\frac{1}{(n+1)!}f^{(n+1)}(c_f)(x-a)^{n+1},$$

$$G(x) = G(a) + \frac{G'(a)}{1!}(x-a) + ... + \frac{G^{(n)}(a)}{n!}(x-a)^n +$$
$$+\frac{1}{(n+1)!}G^{(n+1)}(c_G)(x-a)^{n+1},$$

$$H(x) = H(a) + \frac{H'(a)}{1!}(x-a) + ... + \frac{H^{(n)}(a)}{n!}(x-a)^n +$$

$$+\frac{1}{(n+1)!}H^{(n+1)}(c_H)(x-a)^{n+1}.$$

But

$$f'(a)=...=f^{(n-1)}(a)=0,\ f^{(n)}(a)\neq 0,$$

and then

$$f(x)=f(a)+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(c_f)}{(n+1)!}(x-a)^{n+1},$$

$$G(x)=\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^k+\frac{g^{(n)}(c_G)}{(n+1)!}(x-a)^{n+1},$$

$$H(x)=\sum_{k=1}^{n}\frac{h^{(k-1)}(a)}{k!}(x-a)^k+\frac{h^{(n)}(c_H)}{(n+1)!}(x-a)^{n+1}=$$

$$=\sum_{k=1}^{n}\frac{1}{k!}\sum_{j=0}^{k-1}C_{k-1}^{j}f^{(k-1-j)}(a)\,g^{(j)}(a)\,(x-a)^k+\frac{h^{(n)}(c_H)}{(n+1)!}(x-a)^{n+1}=$$

$$=f(a)\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^k+\frac{h^{(n)}(c_H)}{(n+1)!}(x-a)^{n+1}.$$

Then the relation (11) becomes

$$\left\{\frac{f^{(n)}(a)}{n!}(x-a)^{n-1}+\frac{f^{(n+1)}(c_f)}{(n+1)!}(x-a)^n\right\}\times$$

$$\times\left\{\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}\theta^k(x)(x-a)^k+\frac{g^{(n)}(c_G)}{(n+1)!}\theta^{n+1}(x)(x-a)^{n+1}\right\}=$$

$$=\left\{\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^{k-1}+\frac{g^{(n)}(c_G)}{(n+1)!}(x-a)^n\right\}\times$$

$$\times\left\{f(a)+\frac{f^{(n)}(a)}{n!}(x-a)^n+\frac{f^{(n+1)}(c_f)}{(n+1)!}(x-a)^{n+1}\right\}-$$

$$-f(a)\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^{k-1}-\frac{h^{(n)}(c_H)}{(n+1)!}(x-a)^n,$$

or, after opening the brackets, reducing the same terms and dividing by $(x-a)^n$, we obtain

$$\left(\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}\theta^k(x)(x-a)^{k-1}\right)\frac{f^{(n)}(a)}{n!}+$$

$$+\left(\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}\theta^k(x)(x-a)^k\right)\frac{f^{(n+1)}(c_f)}{(n+1)!}+$$

$$+\frac{f^{(n)}(a)}{n!}\frac{g^{(n)}(c_G)}{(n+1)!}\theta^{n+1}(x)(x-a)^n+$$

$$+\frac{f^{(n+1)}(c_f)}{(n+1)!}\frac{g^{(n)}(c_G)}{(n+1)!}\theta^{n+1}(x)(x-a)^{n+1}=$$

$$=f(a)\frac{g^{(n)}(c_G)}{(n+1)!}+$$

$$+\frac{f^{(n)}(a)}{n!}\left(\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^{k-1}\right)+\frac{f^{(n)}(a)}{n!}\frac{g^{(n)}(c_G)}{(n+1)!}(x-a)^n+$$

$$+\frac{f^{(n+1)}(c_f)}{(n+1)!}\left(\sum_{k=1}^{n}\frac{g^{(k-1)}(a)}{k!}(x-a)^{k}\right)+\frac{f^{(n+1)}(c_f)}{(n+1)!}\frac{g^{(n)}(c_G)}{(n+1)!}(x-a)^{n+1}-$$

$$-\frac{h^{(n)}(c_H)}{(n+1)!}.$$

If now $x\in(a,b]$ tends to a,

$$|c_f-a|\leq|x-a|\,,\ \ |c_G-a|\leq|x-a|\,,|c_H-a|\leq|x-a|\,,$$

we have that c_f, c_G, $c_H\to a$, following that

$$\frac{f^{(n)}(a)}{n!}\frac{g(a)}{1!}\lim_{x\to a}\theta(x)=f(a)\frac{g^{(n)}(a)}{(n+1)!}+\frac{f^{(n)}(a)}{n!}\frac{g(a)}{1!}-\frac{h^{(n)}(a)}{(n+1)!}. \tag{12}$$

Because

$$h^{(n)}(a)=(fg)^{(n)}(a)=\sum_{k=0}^{n}C_n^k f^{(n-k)}(a)\,g^{(k)}(a)=$$

$$=f^{(n)}(a)\,g(a)+f(a)\,g^{(n)}(a)\,,$$

relation (12) becomes

$$\frac{f^{(n)}(a)}{n!}\frac{g(a)}{1!}\theta(a)=$$

$$=f(a)\frac{g^{(n)}(a)}{(n+1)!}+\frac{f^{(n)}(a)}{n!}\frac{g(a)}{1!}-\frac{f^{(n)}(a)\,g(a)}{(n+1)!}-\frac{f(a)\,g^{(n)}(a)}{(n+1)!},$$

and, consequently

$$\lim_{x\to a}\theta(x)=\frac{n}{n+1}.$$

2° The statement 2^0 follows from the statement 1^0.

4 Conclusions and further challenges

In this paper we have established conditions for the functions f and g such under which the intermediate point function $\overline{c}$ is differentiable at the point a and we have provided the derivative $\overline{c}'(a)$.

In the future, we aim to investigate the conditions under which the functions $\overline{c}$ and θ are differentiable of order n at the point a and to compute the corresponding derivatives.

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