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Fixed Points in Multiplicative Metric Spaces [1]

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Abstract

In this article, using only our recent concept of occasionally weakly biased mappings of type (**A**), in multiplicative metric spaces, we could find unique common fixed points.

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1 Introduction and Needed Definitions

Fixed point theory is among the active branches in mathematics. Moreover, the Banach fixed-point theorem is among the famous and useful fixed-point theorems. To ameliorate, prolong, and generalize the Banach contraction principle, the authors focused sometimes on the number of mappings and the contractive condition, and other times on the space. For this end, several authors generalized the metric space to other spaces such as, symmetric, normed, topological, hyper-convex, fuzzy, generalized, partial, weak partial, dislocated, rectangular, quasi-metric, s-metric, b-metric, b-metric-like, modular, b_2 metric, CAT(0), cone, Menger, G_b-metric, probabilistic, partially ordered, locally convex, complex valued, Hausdorff M-distance, ultra metric spaces, and so on and so forth. In particular, Bashirov et al. [6] initiated the notion of multiplicative metric spaces as alternative to the ordinary metric spaces. Many authors used this kind of spaces to search fixed points (see for instance

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[2], [1], [3], [5], [8], [9], [11], [13]-[28]). On the other hand, to get common fixed points with minimal conditions, we introduced the concept of occasionally weakly biased mappings of type (**A**) (see [7]), and we proved and illustrated that our notion is more general than the weak compatibility [12]. In this article, using our notion, we will prove the existence and uniqueness of common fixed points for quadruple mappings in multiplicative metric spaces. Of course, our theorems improve some results for instance the one of Srinivas et al. [24].

We start by giving the concept of multiplicative metric spaces.

Definition 1 *Let $\mathbb{X}$ be a nonempty set. A multiplicative metric is a mapping $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ satisfying the following conditions:*

1. $\forall\ \alpha,\ \beta \in \mathbb{X}$, $m(\alpha, \beta) \geq 1$,
2. $m(\alpha, \beta) = 1$ *if and only if* $\alpha = \beta$,
3. $\forall\ \alpha,\ \beta \in \mathbb{X}$, $m(\alpha, \beta) = m(\beta, \alpha)$,
4. *(triangle inequality)* $\forall\ \alpha,\ \beta,\ \gamma \in \mathbb{X}$, $m(\alpha, \gamma) \leq m(\alpha, \beta).m(\beta, \gamma)$.

The pair $(\mathbb{X}, m)$ *is called a multiplicative metric space.*

Example 1 *Define* $m : \mathbb{R} \times \mathbb{R} \to [1, +\infty)$ *by* $m(x, y) = \alpha^{|x-y|}$ *where* $x,\ y \in \mathbb{R}$ *and* $\alpha > 1$*. Then* $(\mathbb{R} \times \mathbb{R}, m)$ *is a multiplicative metric space.*

The following definitions were used by Srinivas et al. [24] in order to prove the existence and uniqueness of common fixed points.

Definition 2 *([24]) Mappings $\mathbb{G}$ and $\mathbb{H}$ of a multiplicative metric space $(\mathbb{X}, d)$ are said to hold E.A property if* $\lim_{n\to\infty} \mathbb{G}x_n = \lim_{n\to\infty} \mathbb{H}x_n = t$ *for some* $t \in \mathbb{X}$.

Definition 3 *([24]) Mappings $\mathbb{G}$ and $\mathbb{H}$ of a multiplicative metric space in which if $\mathbb{G}\mu = \mathbb{H}\mu$ for some μ such that $\mathbb{G}\mathbb{H}\mu = \mathbb{H}\mathbb{G}\mu$ holds then we say that $\mathbb{G}$ and $\mathbb{H}$ are weakly compatible mappings.*

In the next, we give our definition with an example.

Definition 4 *([7]) Let $\mathbb{L}$ and $\mathbb{O}$ be mappings from a multiplicative metric space $(\mathbb{X}, d)$ into itself. The pair $\{\mathbb{L}, \mathbb{O}\}$ is called occasionally weakly $\mathbb{L}$-biased (respectively $\mathbb{O}$-biased) of type $(\mathbf{A})$, if and only if, there is an element $\rho \in \mathbb{X}$ which satisfies $\mathbb{L}\rho = \mathbb{O}\rho$, this implies*

$$d(\mathbb{L}\mathbb{L}\rho, \mathbb{O}\rho) \leq d(\mathbb{O}\mathbb{L}\rho, \mathbb{L}\rho),$$
$$d(\mathbb{O}\mathbb{O}\rho, \mathbb{L}\rho) \leq d(\mathbb{L}\mathbb{O}\rho, \mathbb{O}\rho),$$

respectively.

Example 2 *Let $\mathbb{X} = [0, +\infty)$ with the multiplicative metric $d(x, y) = e^{|x-y|}$. Define $\mathbb{M}$, $\mathbb{N} : \mathbb{X} \to \mathbb{X}$ by*

$$\mathbb{M}x = \begin{cases} x+1 & \text{if } x \in [0,1] \\ \dfrac{45}{x} & \text{if } x \in (1, +\infty), \end{cases} \quad \mathbb{N}x = \begin{cases} \frac{4}{x+1} & \text{if } x \in [0,1] \\ 5x & \text{if } x \in (1, +\infty). \end{cases}$$

We have $\mathbb{M}x = \mathbb{N}x$ if and only if $x = 1$ or $x = 3$ and

$$e^8 = d(\mathbb{N}\mathbb{N}1, \mathbb{M}1) \le d(\mathbb{M}\mathbb{N}1, \mathbb{N}1) = e^{\frac{41}{2}};$$

i.e., the pair $\{\mathbb{M}, \mathbb{N}\}$ is occasionally weakly $\mathbb{N}$-biased of type $(\mathbf{A})$.

Also, we have

$$e^{12} = d(\mathbb{M}\mathbb{M}3, \mathbb{N}3) \le d(\mathbb{N}\mathbb{M}3, \mathbb{M}3) = e^{60},$$

then, $\mathbb{M}$ and $\mathbb{N}$ are occasionally weakly $\mathbb{M}$-biased of type $(\mathbf{A})$.

Note that

$$d(\mathbb{M}\mathbb{N}1, \mathbb{N}\mathbb{M}1) = e^{\left|\frac{45}{2}-10\right|} = e^{\frac{25}{2}} \neq 1,$$

and

$$d(\mathbb{M}\mathbb{N}3, \mathbb{N}\mathbb{M}3) = e^{|3-75|} = e^{72} \neq 1,$$

That is, $\mathbb{M}$ and $\mathbb{N}$ are not weakly compatible.

2 Previous Result

In 2020, in their paper [24], Srinivas et al. proved the following result:

Theorem 1 *Suppose in a complete multiplicative metric space $(\mathbb{X}, \delta)$, there are four mappings $\mathbb{G}$, $\mathbb{H}$, $\mathbb{K}$ and $\mathbb{J}$ holding the conditions*

1. $\mathbb{G}(\mathbb{X}) \subseteq \mathbb{J}(\mathbb{X})$ *and* $\mathbb{H}(\mathbb{X}) \subseteq \mathbb{K}(\mathbb{X})$

2. $$\begin{aligned} \delta(\mathbb{G}\alpha, \mathbb{H}\beta) \quad &\le \quad \max\left\{ \frac{\delta(\mathbb{G}\alpha, \mathbb{K}\alpha)\delta(\mathbb{H}\beta, \mathbb{J}\beta)}{1+\delta(\mathbb{K}\alpha, \mathbb{J}\beta)}, \frac{\delta(\mathbb{G}\alpha, \mathbb{J}\beta)\delta(\mathbb{K}\alpha, \mathbb{H}\beta)}{1+\delta(\mathbb{K}\alpha, \mathbb{J}\beta)}, \right. \\ &\left. \frac{\delta(\mathbb{G}\alpha, \mathbb{J}\beta)\delta(\mathbb{H}\beta, \mathbb{J}\beta)}{1+\delta(\mathbb{K}\alpha, \mathbb{J}\beta)}, \frac{\delta(\mathbb{G}\alpha, \mathbb{K}\alpha)\delta(\mathbb{K}\alpha, \mathbb{H}\beta)}{1+\delta(\mathbb{K}\alpha, \mathbb{J}\beta)} \right\}^{\lambda}, \end{aligned}$$

   for all α, $\beta \in \mathbb{X}$, where $\lambda \in \left(0, \frac{1}{3}\right)$,

3. *the pairs $(\mathbb{G}, \mathbb{K})$ and $(\mathbb{H}, \mathbb{J})$ are satisfying the E.A property*

4. *the pair of mappings $(\mathbb{G}, \mathbb{K})$ and $(\mathbb{H}, \mathbb{J})$ are weakly compatible.*

Then the above mappings will be having a common fixed point.

In this paper, we will improve the above result.

3 Main Results

Theorem 2 *Let $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be self-mappings of a multiplicative metric space $(\mathbb{M}, d)$ such that*

1. *mappings $\mathbb{W}$ and $\mathbb{Y}$ (respectively $\mathbb{X}$ and $\mathbb{Z}$) are occasionally weakly $\mathbb{Y}$-biased (respectively $\mathbb{Z}$-biased) of type $(\mathbf{A})$,*

2. *$\forall\ x,\ y \in \mathbb{M}$*

$$\begin{aligned} d(\mathbb{W}x, \mathbb{X}y) \quad &\leq \quad \max\left\{\frac{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{X}y, \mathbb{Z}y)}{1+d(\mathbb{Y}x, \mathbb{Z}y)}, \frac{d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{Y}x, \mathbb{X}y)}{1+d(\mathbb{Y}x, \mathbb{Z}y)},\right. \\ &\left.\frac{d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{X}y, \mathbb{Z}y)}{1+d(\mathbb{Y}x, \mathbb{Z}y)}, \frac{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{Y}x, \mathbb{X}y)}{1+d(\mathbb{Y}x, \mathbb{Z}y)}\right\}^{\lambda}, \end{aligned}$$

where $\lambda \in \left(0, \frac{1}{2}\right)$, then $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ have a unique common fixed point.

Proof. By the virtue of the first condition, there exist two elements ρ and μ in $\mathbb{M}$ such that $\mathbb{W}\rho = \mathbb{Y}\rho$ implies $d(\mathbb{Y}\mathbb{Y}\rho, \mathbb{W}\rho) \leq d(\mathbb{W}\mathbb{Y}\rho, \mathbb{Y}\rho)$ and $\mathbb{X}\mu = \mathbb{Z}\mu$ implies $d(\mathbb{Z}\mathbb{Z}\mu, \mathbb{X}\mu) \leq d(\mathbb{X}\mathbb{Z}\mu, \mathbb{Z}\mu)$. We need four steps for proving the existence and uniqueness of the common fixed point. Indeed

First step: We claim that $d(\mathbb{W}\rho, \mathbb{X}\mu) = 1$; i.e., $\mathbb{W}\rho = \mathbb{X}\mu$. Suppose that we have the contrary; i.e., $d(\mathbb{W}\rho, \mathbb{X}\mu) > 1$, then, we get

$$\begin{aligned} d(\mathbb{W}\rho, \mathbb{X}\mu) \quad &\leq \quad \max\left\{\frac{d(\mathbb{W}\rho, \mathbb{Y}\rho)d(\mathbb{X}\mu, \mathbb{Z}\mu)}{1+d(\mathbb{Y}\rho, \mathbb{Z}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{Z}\mu)d(\mathbb{Y}\rho, \mathbb{X}\mu)}{1+d(\mathbb{Y}\rho, \mathbb{Z}\mu)},\right. \\ &\left.\frac{d(\mathbb{W}\rho, \mathbb{Z}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mu)}{1+d(\mathbb{Y}\rho, \mathbb{Z}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{Y}\rho)d(\mathbb{Y}\rho, \mathbb{X}\mu)}{1+d(\mathbb{Y}\rho, \mathbb{Z}\mu)}\right\}^{\lambda} \\ &= \quad \max\left\{\frac{1}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}, \frac{d^2(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)},\right. \\ &\left.\frac{d(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}\right\}^{\lambda} \\ &= \quad \max\left\{\frac{1}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}, \frac{d^2(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}\right\}^{\lambda} \\ &= \quad \left[\frac{d^2(\mathbb{W}\rho, \mathbb{X}\mu)}{1+d(\mathbb{W}\rho, \mathbb{X}\mu)}\right]^{\lambda} \\ &< \quad d(\mathbb{W}\rho, \mathbb{X}\mu), \end{aligned}$$

a contradiction, which implies that $\mathbb{W}\rho = \mathbb{X}\mu$.

Second step: Suppose that $d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)>1$, using condition 2, we obtain

$$\begin{aligned}
d(\mathbb{W}\mathbb{W}\rho,\mathbb{X}\mu) &\leq \max\left\{\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)d(\mathbb{X}\mu,\mathbb{Z}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Z}\mu)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{X}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\right.\\
&\left.\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Z}\mu)d(\mathbb{X}\mu,\mathbb{Z}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{X}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)}\right\}^{\lambda}\\
&= \max\left\{\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Z}\mu)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{X}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\right.\\
&\left.\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Z}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{X}\mu)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{Z}\mu)}\right\}^{\lambda};
\end{aligned}$$

that is,

$$\begin{aligned}
d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho) &\leq \max\left\{\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)},\right.\\
&\left.\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{W}\rho)d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{W}\rho,\mathbb{W}\rho)}\right\}^{\lambda}\\
&= \max\left\{\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{Y}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\right.\\
&\left.\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{Y}\mathbb{Y}\rho)d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}\right\}^{\lambda}\\
&\leq \max\left\{\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d(\mathbb{W}\rho,\mathbb{Y}\mathbb{Y}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\right.\\
&\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)},\\
&\left.\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d^2(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}\right\}^{\lambda}\\
&= \left[\frac{d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d^2(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}\right]^{\lambda}.
\end{aligned}$$

As we have

$$\frac{d^2(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}{1+d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)}\leq d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho),$$

we get

$$d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho) \leq [d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho)d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)]^{\lambda}$$

and as

$$d(\mathbb{Y}\mathbb{Y}\rho,\mathbb{W}\rho)\leq d(\mathbb{W}\mathbb{Y}\rho,\mathbb{Y}\rho)=d(\mathbb{W}\mathbb{W}\rho,\mathbb{W}\rho),$$

we obtain

$$\begin{aligned} d(\mathbb{W}\mathbb{W}\rho, \mathbb{W}\rho) &\leq d^{2\lambda}(\mathbb{W}\mathbb{W}\rho, \mathbb{W}\rho) \\ &< d(\mathbb{W}\mathbb{W}\rho, \mathbb{W}\rho), \end{aligned}$$

which is a contradiction, hence, $\mathbb{W}\mathbb{W}\rho = \mathbb{W}\rho$ and consequently $\mathbb{Y}\mathbb{W}\rho = \mathbb{W}\rho$.

Third step: Now, assume that $d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu) > 1$, then

$$\begin{aligned} d(\mathbb{W}\rho, \mathbb{X}\mathbb{X}\mu) \leq \max\Bigg\{ & \frac{d(\mathbb{W}\rho, \mathbb{Y}\rho)d(\mathbb{X}\mathbb{X}\mu, \mathbb{Z}\mathbb{X}\mu)}{1 + d(\mathbb{Y}\rho, \mathbb{Z}\mathbb{X}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{Z}\mathbb{X}\mu)d(\mathbb{Y}\rho, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{Y}\rho, \mathbb{Z}\mathbb{X}\mu)}, \\ & \frac{d(\mathbb{W}\rho, \mathbb{Z}\mathbb{X}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{Z}\mathbb{X}\mu)}{1 + d(\mathbb{Y}\rho, \mathbb{Z}\mathbb{X}\mu)}, \frac{d(\mathbb{W}\rho, \mathbb{Y}\rho)d(\mathbb{Y}\rho, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{Y}\rho, \mathbb{Z}\mathbb{X}\mu)} \Bigg\}^{\lambda}; \end{aligned}$$

that is,

$$\begin{aligned} d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu) &\leq \max\Bigg\{ \frac{d(\mathbb{X}\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \\ &\qquad \frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \frac{d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)} \Bigg\}^{\lambda} \\ &\leq \max\Bigg\{ \frac{d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \\ &\qquad \frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \frac{d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)} \Bigg\}^{\lambda} \\ &= \max\Bigg\{ \frac{d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \frac{d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}, \\ &\qquad \frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)} \Bigg\}^{\lambda} \\ &= \left[\frac{d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)} \right]^{\lambda} \\ &= \left[\frac{d^2(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)}{1 + d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)} \right]^{\lambda} \\ &\leq [d(\mathbb{X}\mu, \mathbb{Z}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)]^{\lambda} \\ &\leq [d(\mathbb{Z}\mu, \mathbb{X}\mathbb{Z}\mu)d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu)]^{\lambda} \\ &= d^{2\lambda}(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu) \\ &< d(\mathbb{X}\mu, \mathbb{X}\mathbb{X}\mu), \end{aligned}$$

which is a contradiction, hence, $d(\mathbb{X}\mathbb{X}\mu, \mathbb{X}\mu) = 1 \Rightarrow \mathbb{X}\mathbb{X}\mu = \mathbb{X}\mu$, consequently, $d(\mathbb{Z}\mathbb{X}\mu, \mathbb{X}\mu) = 1$ which implies that $\mathbb{Z}\mathbb{X}\mu = \mathbb{X}\mu$. Putting $\mathbb{W}\rho = \mathbb{X}\mu = z$, then, z is a common fixed point of the four mappings.

Fourth step: Finally, suppose there exists another common fixed point of mappings $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ (say z'), using inequality 2, we obtain

$$\begin{aligned} d(\mathbb{W}z, \mathbb{X}z') &\leq \max\left\{\frac{d(\mathbb{W}z, \mathbb{Y}z)d(\mathbb{X}z', \mathbb{Z}z')}{1 + d(\mathbb{Y}z, \mathbb{Z}z')}, \frac{d(\mathbb{W}z, \mathbb{Z}z')d(\mathbb{Y}z, \mathbb{X}z')}{1 + d(\mathbb{Y}z, \mathbb{Z}z')},\right. \\ &\qquad \left.\frac{d(\mathbb{W}z, \mathbb{Z}z')d(\mathbb{X}z', \mathbb{Z}z')}{1 + d(\mathbb{Y}z, \mathbb{Z}z')}, \frac{d(\mathbb{W}z, \mathbb{Y}z)d(\mathbb{Y}z, \mathbb{X}z')}{1 + d(\mathbb{Y}z, \mathbb{Z}z')}\right\}^{\lambda}; \end{aligned}$$

that is,

$$\begin{aligned} d(z, z') &\leq \max\left\{\frac{d(z, z)d(z', z')}{1 + d(z, z')}, \frac{d(z, z')d(z, z')}{1 + d(z, z')},\right. \\ &\qquad \left.\frac{d(z, z')d(z', z')}{1 + d(z, z')}, \frac{d(z, z)d(z, z')}{1 + d(z, z')}\right\}^{\lambda} \\ &= \max\left\{\frac{1}{1 + d(z, z')}, \frac{d^2(z, z')}{1 + d(z, z')}, \frac{d(z, z')}{1 + d(z, z')}\right\}^{\lambda} \\ &= \left[\frac{d^2(z, z')}{1 + d(z, z')}\right]^{\lambda} \\ &\leq d^{\lambda}(z, z') \\ &< d(z, z'), \end{aligned}$$

which is a contradiction, hence, $z' = z$, this completes the proof.

Remark 1 *We have*

$$1 + d(\mathbb{Y}x, \mathbb{Z}y) \geq 2$$

this implies that

$$\frac{1}{1 + d(\mathbb{Y}x, \mathbb{Z}y)} \leq \frac{1}{2}$$

and hence

$$\begin{aligned} d(\mathbb{W}x, \mathbb{X}y) &\leq \max\left\{\frac{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{X}y, \mathbb{Z}y)}{1 + d(\mathbb{Y}x, \mathbb{Z}y)}, \frac{d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{Y}x, \mathbb{X}y)}{1 + d(\mathbb{Y}x, \mathbb{Z}y)},\right. \\ &\qquad \left.\frac{d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{X}y, \mathbb{Z}y)}{1 + d(\mathbb{Y}x, \mathbb{Z}y)}, \frac{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{Y}x, \mathbb{X}y)}{1 + d(\mathbb{Y}x, \mathbb{Z}y)}\right\}^{\lambda} \\ &\leq \frac{1}{2^{\lambda}} \max\{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{X}y, \mathbb{Z}y), d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{Y}x, \mathbb{X}y), \\ &\qquad d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{X}y, \mathbb{Z}y), d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{Y}x, \mathbb{X}y)\}^{\lambda}. \end{aligned}$$

Theorem 3 *Let* $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ *and* $\mathbb{Z}$ *be self-mappings of a multiplicative metric space* $(\mathbb{M}, d)$ *such that*

1. *mappings* $\mathbb{W}$ *and* $\mathbb{Y}$ *(respectively* $\mathbb{X}$ *and* $\mathbb{Z}$*) are occasionally weakly* $\mathbb{Y}$*-biased (respectively* $\mathbb{Z}$*-biased) of type* $(\mathbf{A})$,

2. $\forall\ x,\ y \in \mathbb{M}$

$$\begin{aligned} d(\mathbb{W}x, \mathbb{X}y) \ \leq\ & \frac{1}{2^\lambda} \max\{d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{X}y, \mathbb{Z}y), d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{Y}x, \mathbb{X}y), \\ & d(\mathbb{W}x, \mathbb{Z}y)d(\mathbb{X}y, \mathbb{Z}y), d(\mathbb{W}x, \mathbb{Y}x)d(\mathbb{Y}x, \mathbb{X}y)\}^\lambda \end{aligned}$$

where $\lambda \in \left(0, \frac{1}{3}\right)$, *then* $\mathbb{W}$, $\mathbb{X}$, $\mathbb{Y}$ *and* $\mathbb{Z}$ *have a unique common fixed point.*

Proof. As in the first theorem, the use of the first condition gives the existence of two elements say a and b in $\mathbb{M}$ such that $\mathbb{W}a = \mathbb{Y}a$ implies $d(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a) \leq d(\mathbb{W}\mathbb{Y}a, \mathbb{Y}a)$ and $\mathbb{X}b = \mathbb{Z}b$ implies $d(\mathbb{Z}\mathbb{Z}b, \mathbb{X}b) \leq d(\mathbb{X}\mathbb{Z}b, \mathbb{Z}b)$. Four cases are needed to prove the existence and uniqueness of the common fixed point. Indeed

Case one: Suppose that $d(\mathbb{W}a, \mathbb{X}b) > 1$, then, using the second condition, we get

$$\begin{aligned} d(\mathbb{W}a, \mathbb{X}b) \ &\leq\ \frac{1}{2^\lambda} \max\left\{d(\mathbb{W}a, \mathbb{Y}a)d(\mathbb{X}b, \mathbb{Z}b), d(\mathbb{W}a, \mathbb{Z}b)d(\mathbb{Y}a, \mathbb{X}b),\right. \\ & \quad \left. d(\mathbb{W}a, \mathbb{Z}b)d(\mathbb{X}b, \mathbb{Z}b), d(\mathbb{W}a, \mathbb{Y}a)d(\mathbb{Y}a, \mathbb{X}b)\right\}^\lambda \\ &=\ \frac{1}{2^\lambda} \max\left\{d(\mathbb{W}a, \mathbb{W}a)d(\mathbb{X}b, \mathbb{X}b), d(\mathbb{W}a, \mathbb{X}b)d(\mathbb{W}a, \mathbb{X}b),\right. \\ & \quad \left. d(\mathbb{W}a, \mathbb{X}b)d(\mathbb{X}b, \mathbb{X}b), d(\mathbb{W}a, \mathbb{W}a)d(\mathbb{W}a, \mathbb{X}b)\right\}^\lambda \\ &=\ \frac{1}{2^\lambda} \max\left\{1, d^2(\mathbb{W}a, \mathbb{X}b), d(\mathbb{W}a, \mathbb{X}b)\right\}^\lambda \\ &=\ \frac{1}{2^\lambda} d^{2\lambda}(\mathbb{W}a, \mathbb{X}b) \\ &<\ d(\mathbb{W}a, \mathbb{X}b), \end{aligned}$$

a contradiction, hence, $d(\mathbb{W}a, \mathbb{X}b) = 1$; that is, $\mathbb{W}a = \mathbb{X}b$.

Case two: Assume that $d(\mathbb{W}\mathbb{W}a, \mathbb{W}a) > 1$, then, the use of condition two furnishes

$$\begin{aligned} d(\mathbb{W}\mathbb{W}a, \mathbb{X}b) \ \leq\ & \frac{1}{2^\lambda} \max\left\{d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a)d(\mathbb{X}b, \mathbb{Z}b), d(\mathbb{W}\mathbb{W}a, \mathbb{Z}b)d(\mathbb{Y}\mathbb{W}a, \mathbb{X}b),\right. \\ & \left. d(\mathbb{W}\mathbb{W}a, \mathbb{Z}b)d(\mathbb{X}b, \mathbb{Z}b), d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a)d(\mathbb{Y}\mathbb{W}a, \mathbb{X}b)\right\}^\lambda ; \end{aligned}$$

that is,

$$\begin{aligned} d(\mathbb{W}\mathbb{W}a, \mathbb{W}a) &\leq \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a)d(\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{Y}\mathbb{W}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a)d(\mathbb{Y}\mathbb{W}a, \mathbb{W}a)\}^\lambda \\ &= \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{Y}\mathbb{W}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{W}a)d(\mathbb{Y}\mathbb{W}a, \mathbb{W}a)\}^\lambda \\ &= \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{Y}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{Y}\mathbb{Y}a)d(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a)\}^\lambda \\ &\leq \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{W}a, \mathbb{Y}\mathbb{Y}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{W}a, \mathbb{Y}\mathbb{Y}a)d(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a)\}^\lambda \\ &= \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{W}a, \mathbb{Y}\mathbb{Y}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d^2(\mathbb{Y}\mathbb{Y}a, \mathbb{W}a)\}^\lambda, \end{aligned}$$

since mappings $\mathbb{W}$ and $\mathbb{Y}$ are occasionally weakly $\mathbb{Y}$-biased of type (**A**), we get

$$\begin{aligned} d(\mathbb{W}\mathbb{W}a, \mathbb{W}a) &\leq \frac{1}{2^\lambda} \max\{d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d(\mathbb{Y}a, \mathbb{W}\mathbb{Y}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), \\ &\quad d(\mathbb{W}\mathbb{W}a, \mathbb{W}a)d^2(\mathbb{W}\mathbb{Y}a, \mathbb{Y}a)\}^\lambda \\ &= \frac{1}{2^\lambda} \max\left\{d^2(\mathbb{W}\mathbb{W}a, \mathbb{W}a), d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), d^3(\mathbb{W}\mathbb{W}a, \mathbb{W}a)\right\}^\lambda \\ &= \frac{1}{2^\lambda} d^{3\lambda}(\mathbb{W}\mathbb{W}a, \mathbb{W}a) \\ &< d(\mathbb{W}\mathbb{W}a, \mathbb{W}a), \end{aligned}$$

a contradiction, hence, $d(\mathbb{W}\mathbb{W}a, \mathbb{W}a) = 1$, which implies that $\mathbb{W}\mathbb{W}a = \mathbb{W}a$ and consequently, $\mathbb{Y}\mathbb{W}a = \mathbb{W}a$.

Case three: Now, assume that $d(\mathbb{X}b, \mathbb{X}\mathbb{X}b > 1)$, then, by second condition, we find

$$\begin{aligned} d(\mathbb{W}a, \mathbb{X}\mathbb{X}b) &\leq \frac{1}{2^\lambda} \max\{d(\mathbb{W}a, \mathbb{Y}a)d(\mathbb{X}\mathbb{X}b, \mathbb{Z}\mathbb{X}b), d(\mathbb{W}a, \mathbb{Z}\mathbb{X}b)d(\mathbb{Y}a, \mathbb{X}\mathbb{X}b), \\ &\quad d(\mathbb{W}a, \mathbb{Z}\mathbb{X}b)d(\mathbb{X}\mathbb{X}b, \mathbb{Z}\mathbb{X}b), d(\mathbb{W}a, \mathbb{Y}a)d(\mathbb{Y}a, \mathbb{X}\mathbb{X}b)\}^\lambda; \end{aligned}$$

that is,

$$\begin{aligned}
d(\mathbb{X}b,\mathbb{X}\mathbb{X}b) &\leq \frac{1}{2^\lambda}\max\{d(\mathbb{X}b,\mathbb{X}b)d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{X}b), d(\mathbb{X}b,\mathbb{Z}\mathbb{X}b)d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d(\mathbb{X}b,\mathbb{Z}\mathbb{X}b)d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{X}b), d(\mathbb{X}b,\mathbb{X}b)d(\mathbb{X}b,\mathbb{X}\mathbb{X}b)\}^\lambda\\
&= \frac{1}{2^\lambda}\max\{d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{X}b), d(\mathbb{X}b,\mathbb{Z}\mathbb{X}b)d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d(\mathbb{X}b,\mathbb{Z}\mathbb{X}b)d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{X}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b)\}^\lambda\\
&= \frac{1}{2^\lambda}\max\{d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b)d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b)d(\mathbb{X}\mathbb{X}b,\mathbb{Z}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b)\}^\lambda\\
&\leq \frac{1}{2^\lambda}\max\{d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b)d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b)d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b)\}^\lambda\\
&= \frac{1}{2^\lambda}\max\{d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)d(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d^2(\mathbb{X}b,\mathbb{Z}\mathbb{Z}b)d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)\Big\}^\lambda,
\end{aligned}$$

since mappings $\mathbb{X}$ and $\mathbb{Z}$ are occasionally weakly $\mathbb{Z}$-biased of type $(\mathbf{A})$, we get

$$\begin{aligned}
d(\mathbb{X}b,\mathbb{X}\mathbb{X}b) &\leq \frac{1}{2^\lambda}\max\{d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)d(\mathbb{Z}b,\mathbb{X}\mathbb{Z}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),\\
&\quad d^2(\mathbb{Z}b,\mathbb{X}\mathbb{Z}b)d(\mathbb{X}\mathbb{X}b,\mathbb{X}b)\Big\}^\lambda\\
&= \frac{1}{2^\lambda}\max\left\{d^2(\mathbb{X}\mathbb{X}b,\mathbb{X}b), d(\mathbb{X}b,\mathbb{X}\mathbb{X}b), d^3(\mathbb{X}\mathbb{X}b,\mathbb{X}b)\right\}^\lambda\\
&= \frac{1}{2^\lambda}d^{3\lambda}(\mathbb{X}\mathbb{X}b,\mathbb{X}b)\\
&< d(\mathbb{X}b,\mathbb{X}\mathbb{X}b),
\end{aligned}$$

a contradiction, hence, $d(\mathbb{X}b,\mathbb{X}\mathbb{X}b) = 1$ which implies that $\mathbb{X}b = \mathbb{X}\mathbb{X}b$ and in consequence $\mathbb{X}b = \mathbb{Z}\mathbb{X}b$, therefore, $\mathbb{W}a = \mathbb{X}b = c$ is a common fixed point for the four mappings.

Case four: Imagine the existence of a second common fixed point (say c^*), then

$$\begin{aligned}
d(\mathbb{W}c,\mathbb{X}c^*) &\leq \frac{1}{2^\lambda}\max\{d(\mathbb{W}c,\mathbb{Y}c)d(\mathbb{X}c^*,\mathbb{Z}c^*), d(\mathbb{W}c,\mathbb{Z}c^*)d(\mathbb{Y}c,\mathbb{X}c^*),\\
&\quad d(\mathbb{W}c,\mathbb{Z}c^*)d(\mathbb{X}c^*,\mathbb{Z}c^*), d(\mathbb{W}c,\mathbb{Y}c)d(\mathbb{Y}c,\mathbb{X}c^*)\}^\lambda;
\end{aligned}$$

that is,

$$\begin{aligned} d(c,c^*) &\leq \frac{1}{2^\lambda}\max\{d(c,c)d(c^*,c^*), d(c,c^*)d(c,c^*), \\ &\quad d(c,c^*)d(c^*,c^*), d(c,c)d(c,c^*)\}^\lambda \\ &= \frac{1}{2^\lambda}\max\left\{1, d^2(c,c^*), d(c,c^*)\right\}^\lambda \\ &= \frac{1}{2^\lambda}d^{2\lambda}(c,c^*) \\ &< d(c,c^*), \end{aligned}$$

a contradiction, hence, $c^* = c$; i.e., the common fixed point is unique.

4 Conclusion

We have found unique common fixed points in multiplicative metric spaces under minimal conditions; that is, we have only used our notion of occasionally weakly biased mappings of type $(\mathbf{A})$. In other words, we have removed some required conditions such as the property $(E.A)$, the completeness of the space and the inclusions between the range spaces, using our Definition 4 which is more general than the weakly compatibility (see [7]). Our theorems improve some results such as Theorem 3.1 of [24] and probably some other results existing in multiplicative metric spaces.

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