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Variational and numerical analysis of a dynamic elasto-viscoplastic piezoelectric frictional contact problem [1]

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Abstract

This work studies a mathematical model involving a dynamic contact between two elasto-viscoplastic piezoelectric bodies with damage. The contact is modeled on a combination of a normal compliance and a normal damped response law associated with friction. We derive a variational formulation of the problem and we prove the existence and uniqueness result of the weak solution. The proof is based on the classical existence and uniqueness result for parabolic inequalities, differential equations and fixed-point arguments.

This paper examines a friction problem between two piezoelectric bodies that are elastic- viscoplastic. The study uses both numerical and analytical methods. The contact surface is assumed to have a thin layer of lubricant, and a damped response contact condition is considered. The problem also takes into account electrical and frictional effects. The paper proves the existence of a unique weak solution to the problem by using variational inequalities and a fixed point argument. Additionally, a fully discrete approximation is investigated using the Euler scheme and finite element method for the spatial variable and time derivatives. Error estimates are then calculated, leading to convergence results under certain regularity conditions.

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1 Introduction

This work focuses on a model that examines the dynamic frictional contact between two elastic-viscoplastic piezoelectric bodies. Frictional contact situations are common in various industries and daily life, such as when braking pads make contact with wheels or tires touch the road. The process of frictional contact is crucial, leading to significant efforts in modeling and conducting numerical simulations. As a result, there is extensive engineering literature on this topic, though it primarily focuses on specific physical scenarios or numerical simulations.
Recent studies (refer to [1, 2], the monograph [11], and the references therein) have focused on contact problems using a viscoelastic law based on a rate-type model. This paper, which is a continuation of [5], introduces a novel approach by considering both electrical and frictional effects. The piezoelectric material is modeled using a nonlinear elasto-viscoplastic constitutive law. Piezoelectricity can be explained as follows: when a certain class of crystalline materials (such as ceramics $BaTiO_3$, $BiFeO_3$) is subjected to mechanical pressure, the crystal structure generates a voltage that is proportional to the applied pressure. On the other hand, when an electric field is applied, the structure undergoes dimensional changes, altering the material's shape. Various models have been created to explain the relationship between the electrical and mechanical fields. These models can be found in references [11, 15] and others mentioned in those references. For contact problems with elasto-piezoelectric materials, refer to reference [14]. Different models for viscoelastic piezoelectric problems have been examined in references [4, 13], while contact problems for electro-elasto-viscoplastic materials were explored in references [11, 12]. In our study, we utilize an elasto-viscoplastic piezoelectric constitutive law outlined in...

$$\boldsymbol{\sigma}^{\alpha} = \mathcal{A}^{\alpha}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{\alpha}) + \mathcal{G}^{\alpha}\boldsymbol{\varepsilon}(\boldsymbol{u}^{\alpha}) + (\mathcal{E}^{\alpha})^{*}\nabla\varphi^{\alpha} + \int_0^t \mathcal{F}^{\alpha}\Big(\boldsymbol{\sigma}^{\alpha}(s) - \mathcal{A}^{\alpha}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{\alpha}(s)) - (\mathcal{E}^{\alpha})^{*}\nabla\varphi^{\alpha},\ \boldsymbol{\varepsilon}(\boldsymbol{u}^{\alpha}(s))\Big)\, ds, \tag{1}$$

$$\boldsymbol{D}^{\alpha} = \mathcal{E}^{\alpha}\varepsilon(\boldsymbol{u}^{\alpha}) - \mathcal{B}^{\alpha}\nabla\varphi^{\alpha}, \tag{2}$$

where $\boldsymbol{D}^{\alpha}$ is a represents the electric displacement field, $\boldsymbol{u}^{\alpha}$ the displacement field, $\boldsymbol{\sigma}^{\alpha}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}^{\alpha})$ represent the stress and the linearized strain tensor, respectively. Here $\mathcal{A}^{\alpha}$ is a given nonlinear function, $\mathcal{F}^{\alpha}$ is the relaxation tensor, and $\mathcal{G}^{\alpha}$ represents the elasticity operator. $E(\varphi^{\alpha}) = -\nabla\varphi^{\alpha}$ is the electric field, $\mathcal{E}^{\alpha} = (e_{ijk})$ represents the third order piezoelectric tensor, $(\mathcal{E}^{\alpha})^{*}$ is its transposition.
We will now explain the contact in more detail. We are assuming that both bodies can change shape and that there is a thin layer of lubricant at the boundary where they make contact. In order to describe this, we use a normal damped response contact condition, as discussed in previous studies (see reference [2]). This allows us to calculate the normal stress, represented by $\sigma_{\nu}^{\alpha} = (\boldsymbol{\sigma}_{ij}^{\alpha}\boldsymbol{\nu}_{i}^{\alpha}) \cdot \boldsymbol{\nu}_{j}^{\alpha}$, on the surface where the bodies touch, using the following semilinear relationship.

$$-\sigma_{\nu} = -p_{\nu}([\dot{\boldsymbol{u}}_{\nu}])$$

The normal velocity is represented as $\dot{\boldsymbol{u}}_\nu^\alpha = \dot{\boldsymbol{u}}^\alpha \cdot \boldsymbol{\nu}^\alpha$. The prescribed normal damped response function pv is such that $p_\nu([\dot{\boldsymbol{u}}_\nu]) = 0$ when $\dot{\boldsymbol{u}}_\nu^\alpha \leq 0$, as there is no contact in such cases. A specific example can be found in references [7, 2].

$$p_\nu(r) = c_\nu r_+, \tag{3}$$

where c_ν is a constant that expresses the tendency to deform and $r_+ = \max\{0, r\}$ represents the associated friction law

$$-\sigma_\tau = -p_\tau([\dot{\boldsymbol{u}}_\tau]).$$

Here, $\dot{\boldsymbol{u}}_\tau^\alpha = \dot{\boldsymbol{u}}^\alpha - \dot{\boldsymbol{u}}_\nu^\alpha \cdot \boldsymbol{\nu}^\alpha$. is the tangential velocity, $\boldsymbol{\sigma}_\tau^\alpha = \boldsymbol{\sigma}^\alpha \boldsymbol{\nu}^\alpha - \sigma_\nu^\alpha \cdot \boldsymbol{\nu}^\alpha$. denotes the shear stresses and p_τ is a constitutive function whose properties will be described below. The following form of p_τ was considered

$$p_\tau(\boldsymbol{r}) = c_\tau \boldsymbol{r}, \quad \forall \boldsymbol{r} \in \mathbb{R}^d. \tag{4}$$

In this scenario, c_τ represents a coefficient of friction, and the amount of tangential shear is directly related to the tangential velocity. This description applies to situations where the contact surface is coated with a thin layer of a newtonian fluid. This paper examines a mathematical problem involving frictional contact between two bodies made of rate-type electro-elastic-viscoplastic materials.

The paper is structured as follows: In Section 2, we present the mathematical models for the frictional contact problem. In Section 3, we outline the assumptions on the data and derive the variational formulation of the problem. In Section 4, we present our main result, which is Theorem 1, regarding the existence and uniqueness of the solution. In Section 5, we present a method that uses finite elements to solve the problem. The spatial approximation is done using the finite element method, while the time derivatives are discretized using the Euler scheme. We provide a proof of the main error estimates result, Theorem 3, which allows us to deduce the linear convergence of the algorithm under certain regularity conditions.

2 Problem Statement

We consider the following physical setting. Let us consider two electro-elastic- viscoplastics bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2, 3)$. For each domain Ω^α, the boundary Γ^α is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^α, Γ_2^α and Γ_3^α, on one hand, and on two measurable parts Γ_a^α and Γ_b^α, on the other hand, such that $meas\Gamma_1^\alpha > 0$, $meas\Gamma_a^\alpha > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The Ω^α body is submitted to $\boldsymbol{f}_0^\alpha$ forces and volume electric charges of density q_0^α. The bodies are assumed to be clamped on $\Gamma_1^\alpha \times (0, T)$. The surface tractions $\boldsymbol{f}_2^\alpha$ act on $\Gamma_2^\alpha \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a^\alpha \times (0, T)$ and a surface electric charge of density q_2^α is prescribed on $\Gamma_b^\alpha \times (0, T)$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. with a lubricant layer assumed between them. With these assumptions, the classical formulation of frictional contact

problem between two electro-elastic-viscoplastic bodies is the following .

Problem P. For $\alpha = 1, 2$, find a displacement field $\boldsymbol{u}^\alpha : \Omega^\alpha \times (0,T) \longrightarrow \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^\alpha : \Omega^\alpha \times (0,T) \longrightarrow \mathbb{S}^d$, an electric potential field $\varphi^\alpha : \Omega^\alpha \times (0,T) \longrightarrow \mathbb{R}$ and a electric displacement field $\boldsymbol{D}^\alpha : \Omega^\alpha \times (0,T) \longrightarrow \mathbb{R}^d$ such that

$$\begin{aligned} \boldsymbol{\sigma}^\alpha &= \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + (\mathcal{E}^\alpha)^* \nabla \varphi^\alpha + \\ &\int_0^t \mathcal{F}^\alpha \Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \varphi^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s)) \Big)\, ds \end{aligned} \quad \text{in } \Omega^\alpha \times (0,T), \tag{5}$$

$$\boldsymbol{D}^\alpha = \mathcal{E}^\alpha \varepsilon(\boldsymbol{u}^\alpha) - \mathcal{B}^\alpha \nabla \varphi^\alpha \quad \text{in } \Omega^\alpha \times (0,T), \tag{6}$$

$$\rho^\alpha \ddot{\boldsymbol{u}}^\alpha = \operatorname{Div} \boldsymbol{\sigma}^\alpha + \boldsymbol{f}_0^\alpha \quad \text{in } \Omega^\alpha \times (0,T), \tag{7}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\alpha + \boldsymbol{f}_0^\alpha = 0 \quad \text{in } \Omega^\alpha \times (0,T), \tag{8}$$

$$\operatorname{div} \boldsymbol{D}^\alpha - q_0^\alpha = 0 \quad \text{in } \Omega^\alpha \times (0,T), \tag{9}$$

$$\boldsymbol{u}^\alpha = 0 \quad \text{on } \Gamma_1^\alpha \times (0,T), \tag{10}$$

$$\boldsymbol{\sigma}^\alpha \boldsymbol{\nu}^\alpha = \boldsymbol{f}_2^\alpha \quad \text{on } \Gamma_2^\alpha \times (0,T), \tag{11}$$

$$\begin{cases} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ -\sigma_\nu = -p_\nu([\dot{\boldsymbol{u}}_\nu]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{12}$$

$$\begin{cases} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ -\boldsymbol{\sigma}_\tau = p_\tau([\dot{\boldsymbol{u}}_\tau]) \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{13}$$

$$\varphi^\alpha = 0 \quad \text{on } \Gamma_a^\alpha \times (0,T), \tag{14}$$

$$\boldsymbol{D}^\alpha . \boldsymbol{\nu}^\alpha = q_2^\alpha \quad \text{on } \Gamma_b^\alpha \times (0,T), \tag{15}$$

$$\boldsymbol{u}^\alpha(0) = \boldsymbol{u}_0^\alpha, \quad \dot{\boldsymbol{u}}^\alpha(0) = \boldsymbol{v}_0^\alpha \quad \text{in } \Omega^\alpha. \tag{16}$$

Initially, equations (5) and (6) depict the constitutive law of the material, incorporating the electro-elastic-viscoplastic nature. In equation (5), the symbol $\varepsilon(\boldsymbol{u}^\alpha)$ represents the linearized strain tensor, while in equation (6), $E(\varphi^\alpha) = -\nabla \varphi^\alpha$ represents the electric field, with φ^α denoting the electric potential. The operators $\mathcal{A}^\alpha$

and $\mathcal{G}^\alpha$ correspond to non-linear functions describing the purely viscous and elastic properties of the material, respectively. $\mathcal{F}^\alpha$ is a non-linear function that describes the viscoplastic behavior of the material. $\mathcal{E}^\alpha$ represents the piezoelectric operator, while $(\mathcal{E}^\alpha)^*$ is its transpose. B represents the electric permittivity operator, and $\boldsymbol{D}^\alpha = (D_1^\alpha, ..., D_d^\alpha)$ is the electric displacement vector. Equations (8) and (9) are used to describe the balance of stress and electric displacement fields, respectively. The terms "Div" and "div" refer to the divergence operator for tensor and vector valued functions, respectively. Equations (10) and (11) represent the displacement and traction boundary conditions. Equation (16) represents the initial displacement field and initial velocity. Finally, equations (14) and (15) represent the electric boundary conditions.

3 The Formulation of Variations and Necessary Initial Steps

In this part, we outline the presumptions made about the data and establish a variational approach to tackling the contact problem. To do so, we must introduce certain symbols and initial information. In this and subsequent sections, $\mathbb{S}^d$ refers to the set of second-order symmetric tensors in the space of $\mathbb{R}^d$. Now, in order to continue with the variational formulation, we require the following function spaces:

$$H^\alpha = \{\boldsymbol{w}^\alpha = (w_i^\alpha);\ w_i^\alpha \in L^2(\Omega^\alpha)\},\ H_1^\alpha = \{\boldsymbol{w}^\alpha = (w_i^\alpha);\ w_i^\alpha \in H^1(\Omega^\alpha)\},$$
$$Q^\alpha = \{\boldsymbol{\tau}^\alpha = (\tau_{ij}^\alpha);\ \tau_{ij}^\alpha = \tau_{ji}^\alpha \in L^2(\Omega^\alpha)\},\ Q_1^\alpha = \{\boldsymbol{\tau}^\alpha = (\tau_{ij}^\alpha) \in Q^\alpha;\ \mathrm{div}\boldsymbol{\tau}^\alpha \in H^\alpha\}.$$

The spaces H^α, H_1^α, Q^α and Q_1^α are Hilbert spaces that are real and have the canonical inner products defined by.

$$(\boldsymbol{u}^\alpha, \boldsymbol{w}^\alpha)_{H^\alpha} = \int_{\Omega^\alpha} \boldsymbol{u}^\alpha.\boldsymbol{w}^\alpha dx, \quad (\boldsymbol{u}^\alpha, \boldsymbol{w}^\alpha)_{H_1^\alpha} = \int_{\Omega^\alpha} \boldsymbol{u}^\alpha.\boldsymbol{w}^\alpha dx + \int_{\Omega^\alpha} \nabla\boldsymbol{u}^\alpha.\nabla\boldsymbol{w}^\alpha dx,$$
$$(\boldsymbol{\sigma}^\alpha, \boldsymbol{\tau}^\alpha)_{Q^\alpha} = \int_{\Omega^\alpha} \boldsymbol{\sigma}^\alpha.\boldsymbol{\tau}^\alpha dx, \quad (\boldsymbol{\sigma}^\alpha, \boldsymbol{\tau}^\alpha)_{Q_1^\alpha} = \int_{\Omega^\alpha} \boldsymbol{\sigma}^\alpha.\boldsymbol{\tau}^\alpha dx + \int_{\Omega^\alpha} \mathrm{div}\,\boldsymbol{\sigma}^\alpha.\,\mathrm{Div}\,\boldsymbol{\tau}^\alpha dx$$

and the associated norms $\|.\|_{H^\alpha}$, $\|.\|_{H_1^\alpha}$, $\|.\|_{Q^\alpha}$, and $\|.\|_{Q_1^\alpha}$ respectively. When $\boldsymbol{\sigma}^\alpha$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{\sigma}^\alpha, \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + (\mathrm{Div}\,\boldsymbol{\sigma}^\alpha, \boldsymbol{w}^\alpha)_{H^\alpha} = \int_{\Gamma^\alpha} \boldsymbol{\sigma}^\alpha\boldsymbol{\nu}^\alpha.\boldsymbol{w}^\alpha da \quad \forall\boldsymbol{w}^\alpha \in H_1^\alpha.$$

Next, we define the space

$$V^\alpha = \{\boldsymbol{w}^\alpha \in H_1^\alpha;\ \boldsymbol{w}^\alpha = 0 \text{ on } \Gamma_1^\alpha\}.$$

Since $meas\Gamma_1^\alpha > 0$, the following Korn's inequality holds :

$$\|\varepsilon(\boldsymbol{w}^\alpha)\|_{Q^\alpha} \geq c_K\|\boldsymbol{w}^\alpha\|_{H_1^\alpha} \quad \forall\boldsymbol{w}^\alpha \in V^\alpha, \tag{17}$$

where the constant c_K denotes a positive constant which may depends only on Ω^α, Γ_1^α (see [9]). Over the space V^α we consider the inner product given by

$$(\boldsymbol{u}^\alpha, \boldsymbol{w}^\alpha)_{V^\alpha} = (\varepsilon(\boldsymbol{u}^\alpha), \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha}, \quad \forall \boldsymbol{u}^\alpha, \boldsymbol{w}^\alpha \in V^\alpha, \tag{18}$$

and let $\|.\|_{V^\alpha}$ be the associated norm. It follows from Korn's inequality (17) that the norms $\|.\|_{H_1^\alpha}$ and $\|.\|_{V^\alpha}$ are equivalent on V^α. Then $(V^\alpha, \|.\|_{V^\alpha})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (18), there exists a constant $c_0 > 0$, depending only on Ω^α, Γ_1^α and Γ_3 such that

$$\|\boldsymbol{w}^\alpha\|_{L^2(\Gamma_3)^d} \leq c_0 \|\boldsymbol{w}^\alpha\|_{V^\alpha} \quad \forall \boldsymbol{w}^\alpha \in V^\alpha. \tag{19}$$

We also introduce the spaces

$$W^\alpha = \left\{ \zeta^\alpha \in H^1(\Omega^\alpha);\ \zeta^\alpha = 0 \text{ on } \Gamma_a^\alpha \right\},$$

$$\mathcal{W}^\alpha = \left\{ \boldsymbol{D}^\alpha = (D_i^\alpha);\ D_i^\alpha \in L^2(\Omega^\alpha),\ \operatorname{div} \boldsymbol{D}^\alpha \in L^2(\Omega^\alpha) \right\}.$$

Since $meas\Gamma_a^\alpha > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla \zeta^\alpha\|_{L^2(\Omega^\alpha)^d} \geq c_F \|\zeta^\alpha\|_{H^1(\Omega^\alpha)} \quad \forall \zeta^\alpha \in W^\alpha, \tag{20}$$

where $c_F > 0$ is a constant which depends only on Ω^α, Γ_a^α.
Over the space W^α, we consider the inner product given by

$$(\varphi^\alpha, \zeta^\alpha)_{W^\alpha} = \int_{\Omega^\alpha} \nabla \varphi^\alpha . \nabla \zeta^\alpha dx,$$

and let $\|.\|_{W^\alpha}$ be the associated norm. It follows from (20) that $\|.\|_{H^1(\Omega^\alpha)}$ and $\|.\|_{W^\alpha}$ are equivalent norms on W^α and therefore $(W^\alpha, \|.\|_{W^\alpha})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$, depending only on Ω^α, Γ_a^α and Γ_3, such that

$$\|\phi^\alpha\|_{L^2(\Omega^\alpha)} \leq \mathbf{c}_0 \|\phi^\alpha\|_{W^\alpha} \quad \forall \phi^\alpha \in W^\alpha. \tag{21}$$

The space $\mathcal{W}^\alpha$ is real Hilbert space with the inner product

$$(\boldsymbol{D}^\alpha, \boldsymbol{E}^\alpha)_{\mathcal{W}^\alpha} = \int_{\Omega^\alpha} \boldsymbol{D}^\alpha . \boldsymbol{E}^\alpha dx + \int_{\Omega^\alpha} \operatorname{div} \boldsymbol{D}^\alpha . \operatorname{div} \boldsymbol{E}^\alpha dx,$$

where $\operatorname{div} \boldsymbol{D}^\alpha = (\boldsymbol{D}_{i,i}^\alpha)$, and the associated norm $\|.\|_{\mathcal{W}^\alpha}$.
In order to simplify the notations, we define the product spaces

$$\begin{gathered} \boldsymbol{V} = V^1 \times V^2, \quad H = H^1 \times H^2, \quad H_1 = H_1^1 \times H_1^2, \quad Q = Q^1 \times Q^2, \\ Q_1 = Q_1^1 \times Q_1^2, \quad W = W^1 \times W^2, \quad \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2. \end{gathered} \tag{22}$$

The spaces $\boldsymbol{V}$, W and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $(.,.)_{\boldsymbol{V}}$, $(.,.)_W$, and $(.,.)_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\boldsymbol{V}}$, $\|.\|_W$, and $\|.\|_{\mathcal{W}}$, respectively.

In the study of the Problem **P**, we consider the following assumptions:
We assume that the *viscosity operator* $\mathcal{A}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(23)
$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\alpha} > 0 \text{ such that} \\ \quad |\mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_2)| \leq L_{\mathcal{A}^\alpha} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|, \ \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(b) There exists } m_{\mathcal{A}^\alpha} > 0 \text{ such that} \\ \quad (\mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \geq m_{\mathcal{A}^\alpha} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^2, \ \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\alpha, \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^\alpha(\boldsymbol{x}, \mathbf{0}) \text{ is continuous on } \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \end{cases}$$

The *elasticity operator* $\mathcal{G}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(24)
$$\begin{cases} \text{(a) There exists } L_{\mathcal{G}^\alpha} > 0 \text{ such that} \\ \quad |\mathcal{G}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_1) - \mathcal{G}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}_2)| \leq L_{\mathcal{G}^\alpha} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|, \ \forall \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\alpha(\boldsymbol{x}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\alpha, \text{ for any } \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}^\alpha(\boldsymbol{x}, \mathbf{0}) \text{ belongs to } Q^\alpha. \end{cases}$$

The *viscoplasticity operator* $\mathcal{F}^\alpha : \Omega^\alpha \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

(25)
$$\begin{cases} \text{(a) There exists } L_{\mathcal{F}^\alpha} > 0 \text{ such that} \\ \quad |\mathcal{F}^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1) - \mathcal{F}^\alpha(\boldsymbol{x}, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2)| \leq L_{\mathcal{F}^\alpha} \big(|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|\big) \\ \quad \forall \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\alpha(\boldsymbol{x}, \boldsymbol{\eta}, \boldsymbol{\xi}) \text{ is Lebesgue measurable on } \Omega^\alpha, \\ \quad \text{for any } \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{F}^\alpha(\boldsymbol{x}, \mathbf{0}, \mathbf{0}) \text{ belongs to } \mathrm{Q}^\alpha. \end{cases}$$

The *piezoelectric tensor* $\mathcal{E}^\alpha : \Omega^\alpha \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies:

(26)
$$\begin{cases} \text{(a) } \mathcal{E}^\alpha(\boldsymbol{x}, \tau) = (e^\alpha_{ijk}(\boldsymbol{x})\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d \ \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(b) } e^\alpha_{ijk} = e^\alpha_{ikj} \in L^\infty(\Omega^\alpha), \ 1 \leq i, j, k \leq d. \end{cases}$$

Recall also that the transposed operator $(\mathcal{E}^\alpha)^*$ is given by $(\mathcal{E}^\alpha)^* = (e^{\alpha,*}_{ijk})$ where $e^{\alpha,*}_{ijk} = e^\alpha_{kij}$ and the following equality holds

$$\mathcal{E}^\alpha \sigma . \boldsymbol{w} = \sigma . (\mathcal{E}^\alpha)^* \boldsymbol{w} \quad \forall \sigma \in \mathbb{S}^d, \ \ \forall \boldsymbol{w} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^\alpha = (b^\alpha_{ij}) : \Omega^\alpha \times \mathbb{R}^d \to \mathbb{R}^d$ verifies:

(27)
$$\begin{cases} \text{(a) } \mathcal{B}^\alpha(\boldsymbol{x}, \mathbf{E}) = (b^\alpha_{ij}(\boldsymbol{x}) E_j) \quad \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \ \ \text{a.e. } \boldsymbol{x} \in \Omega^\alpha. \\ \text{(b) } b^\alpha_{ij} = b^\alpha_{ji}, \ b^\alpha_{ij} \in L^\infty(\Omega^\alpha), \quad 1 \leq i, j \leq d. \\ \text{(c) There exists } m_{\mathcal{B}^\alpha} > 0 \text{ such that } \mathcal{B}^\alpha \mathbf{E} . \mathbf{E} \geq m_{\mathcal{B}^\alpha} |\mathbf{E}|^2, \ \forall \mathbf{E} = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega^\alpha. \end{cases}$$

The *contact functions* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ and $p_\tau : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies:

(28)
$$\begin{cases} \text{(a) } \exists\, L_\nu > 0 \,\text{such that } |p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2)| \le L_\nu |r_1 - r_2|,\ \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\boldsymbol{x}, r_1) - p_\nu(\boldsymbol{x}, r_2))(r_1 - r_2) \ge 0,\ \forall\, r_1, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\ \forall r \in \mathbb{R}. \\ \text{(d) The mapping } \ r \in \mathbb{R} \to p_\nu(\boldsymbol{x}, r) \in \mathbb{R} \text{ is continuous in } \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

(29)
$$\begin{cases} \text{(a) } \exists\, L_\tau > 0 \ \text{ such that } |p_\tau(\boldsymbol{x}, \boldsymbol{r}_1) - p_\tau(\boldsymbol{x}, \boldsymbol{r}_2)| \le L_\tau |\boldsymbol{r}_1 - \boldsymbol{r}_2| \ , \forall\, \boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) } (p_\tau(\boldsymbol{x}, \boldsymbol{r}_1) - p_\tau(\boldsymbol{x}, \boldsymbol{r}_2))(\boldsymbol{r}_1 - \boldsymbol{r}_2) \ge 0,\ \forall\, \boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, \boldsymbol{r}), \text{ is Lebesgue measurable on } \Gamma_3,\ \forall \boldsymbol{r} \in \mathbb{R}^d. \\ \text{(d) The mapping } \ \boldsymbol{r} \in \mathbb{R}^d \to p_\tau \nu(\boldsymbol{x}, \boldsymbol{r}) \in \mathbb{R}^d \text{ is continuous in } \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(e) } p_\tau(\boldsymbol{x}, \boldsymbol{r}) \cdot \boldsymbol{\nu}(x) = 0\ \forall \boldsymbol{r} \in \mathbb{R}^d \text{ such that } \boldsymbol{r} \cdot \boldsymbol{\nu}(x) = 0 \,\text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

We note that the functions defined in (3) and (4) satisfy assumptions (12) and (13), respectively.
We suppose that the mass density satisfies

$$\rho^\alpha \in L^\infty(\Omega^\alpha) \text{ and } \ \exists \rho_0 > 0 \text{ such that } \ \rho^\alpha(x) \ge \rho_0 \ \text{ a.e. } x \in \Omega^\alpha,\ \alpha = 1, 2. \tag{30}$$

The following regularity is assumed on the density of volume forces, traction, volume electric charges and surface electric charges:

$$\begin{aligned} \mathbf{f}_0^\alpha \in L^2(0, T; L^2(\Omega^\alpha)^d), \quad \mathbf{f}_2^\alpha \in L^2(0, T; L^2(\Gamma_2^\alpha)^d), \\ q_0^\alpha \in C(0, T; L^2(\Omega^\alpha)), \quad q_2^\alpha \in C(0, T; L^2(\Gamma_b^\alpha)), \end{aligned} \tag{31}$$

$$q_2^\alpha(t) = 0 \quad \text{on } \ \Gamma_3 \ \ \forall t \in [0, T]. \tag{32}$$

finally, the initial data satisfy

$$\boldsymbol{u}_0 \in \boldsymbol{V}, \quad \boldsymbol{v}_0 \in H. \tag{33}$$

We will use a modified inner product on H, given by

$$(\!(\boldsymbol{u}, \boldsymbol{w})\!)_H = \sum_{\alpha=1}^{2} (\rho^\alpha \boldsymbol{u}^\alpha, \boldsymbol{w}^\alpha)_{H^\alpha}, \quad \forall \boldsymbol{u}, \boldsymbol{w} \in H,$$

that is, it is weighted with ρ^α, and we let $|\!|\!|.|\!|\!|_H$ be the associated norm, i.e.,

$$|\!|\!|\boldsymbol{w}|\!|\!|_H = (\!(\boldsymbol{w}, \boldsymbol{w})\!)_H^{\frac{1}{2}}, \quad \forall w \in H.$$

It follows from assumption (30) that $|\!|\!|.|\!|\!|_H$ and $\|.\|_H$ are equivalent norms on H, and the inclusion mapping of $(\boldsymbol{V}, \|.\|_{\boldsymbol{V}})$ into $(H, |\!|\!|.|\!|\!|_H)$ is continuous and dense. We

denote by $\boldsymbol{V}'$ the dual of $\boldsymbol{V}$. Identifying H with its own dual, we can write the Gelfand triple

$$\boldsymbol{V} \subset H \subset \boldsymbol{V}'.$$

Using the notation $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ to represent the duality pairing between $\boldsymbol{V}'$ and $\boldsymbol{V}$ we have

$$(\boldsymbol{u},\boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} = (\!(\boldsymbol{u},\boldsymbol{w})\!)_H, \quad \forall \boldsymbol{u} \in H, \forall \boldsymbol{w} \in \boldsymbol{V}.$$

Next, we define the four mappings $j : \boldsymbol{V}\times\boldsymbol{V} \longrightarrow \mathbb{R}$, $\mathbf{f} : [0,T] \to \boldsymbol{V}'$ and $q : [0,T] \to W$ respectively, by

$$(\mathbf{f}(t),\boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2}\int_{\Omega^\alpha} \mathbf{f}_0^\alpha(t)\cdot\boldsymbol{w}^\alpha\,dx + \sum_{\alpha=1}^{2}\int_{\Gamma_2} \mathbf{f}_2^\alpha(t)\cdot\boldsymbol{w}^\alpha\,da \quad \forall \boldsymbol{w}\in\boldsymbol{V}, \tag{34}$$

$$(q(t),\phi)_W = \sum_{\alpha=1}^{2}\int_{\Omega^\alpha} q_0^\alpha(t)\phi^\alpha\,dx - \sum_{\alpha=1}^{2}\int_{\Gamma_b^\alpha} q_2^\alpha(t)\phi^\alpha\,da \quad \forall \phi\in W, \tag{35}$$

$$j(\boldsymbol{u},\boldsymbol{w}) = \int_{\Gamma_3} p_\nu([\boldsymbol{u}_\nu])[\boldsymbol{w}_\nu]da + \int_{\Gamma_3} p_\tau([\boldsymbol{u}_\tau])[\boldsymbol{w}_\tau]da \quad \forall \boldsymbol{w}\in\boldsymbol{V}. \tag{36}$$

Finally, by a standard procedure based on Green's formula, we derive the following variational formulation of the mechanical (5).

Problem PV. Find a displacement field $\boldsymbol{u} : [0,T] \to \boldsymbol{V}$, a stress field $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$, an electric potential field $\varphi : [0,T] \to W$ and a electric displacement field $\boldsymbol{D} : [0,T] \to \mathcal{W}$ such that

$$\begin{aligned}\boldsymbol{\sigma}^\alpha &= \mathcal{A}^\alpha\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha) + \mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha) + (\mathcal{E}^\alpha)^*\nabla\varphi^\alpha + \\ &\int_0^t \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha(s)) - (\mathcal{E}^\alpha)^*\nabla\varphi^\alpha, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s))\Big)\,ds \qquad \text{in } \Omega^\alpha\times(0,T),\end{aligned} \tag{37}$$

$$\boldsymbol{D}^\alpha = \mathcal{E}^\alpha\varepsilon(\boldsymbol{u}^\alpha) - \mathcal{B}^\alpha\nabla\varphi^\alpha \quad \text{in } \Omega^\alpha\times(0,T), \tag{38}$$

$$\begin{aligned}(\ddot{u},w)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\alpha=1}^{2}(\boldsymbol{\sigma}^\alpha,\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + j(\dot{\boldsymbol{u}}(t),\boldsymbol{w}) &= (\mathbf{f}(t),\boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} \\ \forall \boldsymbol{w}\in\boldsymbol{V}, t\in(0,T).&\end{aligned} \tag{39}$$

$$\begin{aligned}\sum_{\alpha=1}^{2}(\mathcal{B}^\alpha\nabla\varphi^\alpha(t),\nabla\phi^\alpha)_{H^\alpha} - \sum_{\alpha=1}^{2}(\mathcal{E}^\alpha\varepsilon(\boldsymbol{u}^\alpha(t)),\nabla\phi^\alpha)_{H^\alpha} &= (q(t),\phi)_W \\ \forall\phi\in W,\ t\in(0,T),&\end{aligned} \tag{40}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0. \tag{41}$$

We notice that the variational Problem **PV** is formulated in terms of a displacement field, a stress field, an electrical potential field, a bonding field and a electric displacement field. The existence of the unique solution to Problem **PV** is stated and proved in the next section.

4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result

Theorem 1 *Assume that* (24)–(33) *hold. Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}\}$ *to Problem* $\boldsymbol{PV}$*, Moreover, the solution satisfies*

$$\boldsymbol{u} \in H^1(0,T;\boldsymbol{V}) \cap C^1(0,T;H), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'), \tag{42}$$

$$\varphi \in C(0,T;W). \tag{43}$$

The weak solution to the contact Problem **P** is defined by the functions $\boldsymbol{u}, \varphi, \boldsymbol{\sigma}$ and $\boldsymbol{D}$, which fulfill (37)-(41) . It is established that, when assumptions (23)– (33) hold, the mechanical problem(5) has a singular weak solution that meets the criteria in (42). The regularity of the weak solution is determined by (42) and can be expressed in terms of stresses.

$$\boldsymbol{\sigma} \in L^2(0,T;Q), \tag{44}$$

$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{45}$$

Indeed, it follows from (39) and (40) that $\rho^\alpha \ddot{\boldsymbol{u}}^\alpha = \mathrm{Div}\,\boldsymbol{\sigma}^\alpha(t) + \mathbf{f}_0^\alpha(t)$, $\mathrm{div}\,\boldsymbol{D}^\alpha(t) - q_0^\alpha(t) = 0$ for all $t \in [0,T]$ and therefore the regularity (42) and (43) of $\boldsymbol{u}$ and φ, combined with (30)–(32) implies (44)–(45).
We will now provide the proof for Theorem 1 in multiple steps. In this section, we assume that the assumptions of Theorem 1 are satisfied, and we also consider a generic positive constant C that depends on Ω^α, Γ_1^α, Γ_3, p_ν, p_τ, and can vary depending on the context. Let $\hat{I}$ be an element of the $\eta \in L^2(0,T;\boldsymbol{V}')$ space. The first step involves examining the following variational problem.

Problem $\mathbf{PV}_\eta^u$**.** Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ such that

$$\begin{aligned}(\ddot{u}_\eta(t), w)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\dot{\boldsymbol{u}}^\alpha(t)),\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + j(\dot{\boldsymbol{u}}(t), \boldsymbol{w}) \\ +(\eta(t), w)_{\boldsymbol{V}'\times\boldsymbol{V}} = (\mathbf{f}(t), \boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall \boldsymbol{w} \in \boldsymbol{V}, a.e.\ t \in (0,T),\end{aligned} \tag{46}$$

$$\boldsymbol{u}^\alpha(0) = \boldsymbol{u}_0^\alpha, \quad \dot{\boldsymbol{u}}^\alpha(0) = \boldsymbol{v}_0^\alpha \quad \text{in } \Omega^\alpha, \tag{47}$$

To solve Problem $\mathbf{PV}_\eta^u$, we apply an abstract existence and uniqueness result which we recall now, for the convenience of the reader. Let $\boldsymbol{V}$ and H denote real Hilbert spaces such that $\boldsymbol{V}$ is dense in H and the inclusion map is continuous, H is identified with its dual and with a subspace of the dual $\boldsymbol{V}'$ of $\boldsymbol{V}$, i.e., $\boldsymbol{V} \subset H \subset \boldsymbol{V}'$, and we say that the inclusions above define a Gelfand triple. The notations $\|.\|_{\boldsymbol{V}}$, $\|.\|_{\boldsymbol{V}'}$ and $(.,.)_{\boldsymbol{V}'\times\boldsymbol{V}}$ represent the norms on $\boldsymbol{V}$ and on $\boldsymbol{V}'$ and the duality pairing between $\boldsymbol{V}'$ them, respectively. The following abstract result may be found in [16, p.48].

Theorem 2 *Let $\boldsymbol{V}$, H be as above, and let $A : \boldsymbol{V} \to \boldsymbol{V}'$ be a hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{w}, \boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq a\|\boldsymbol{w}\|_{\boldsymbol{V}}^2 + \lambda \quad \forall \boldsymbol{w} \in \boldsymbol{V}, \tag{48}$$

$$\|A\boldsymbol{w}\|_{\boldsymbol{V}'} \leq C(\|\boldsymbol{w}\|_{\boldsymbol{V}} + 1) \quad \forall \boldsymbol{w} \in \boldsymbol{V}, \tag{49}$$

for some constants $a > 0$, $C > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{u}_0 \in H$ and $f \in L^2(0,T;\boldsymbol{V}')$, there exists a unique function $\boldsymbol{u}$ which satisfies

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap C^1(0,T;H), \quad \dot{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{V}'),$$
$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \mathbf{f}(t) \text{ a.e. } t \in (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$

We have the following result for the problem.

Lemma 1 *There exists a unique solution to Problem $\boldsymbol{PV}_\eta^u$ and it has its regularity expressed in* (42).

Proof. We define the operator $A : \boldsymbol{V} \to \boldsymbol{V}'$ by

$$(A\boldsymbol{u}, \boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\boldsymbol{u}^\alpha),\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + j(\boldsymbol{u}, \boldsymbol{w}) \quad \forall \boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{V}. \tag{50}$$

Let $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V}$, using (50) and (36), we find

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\boldsymbol{u}_1^\alpha) - \mathcal{A}^\alpha \varepsilon(\boldsymbol{u}_2^\alpha),\ \varepsilon(\boldsymbol{u}_1^\alpha - \boldsymbol{u}_2^\alpha))_{Q^\alpha} +$$
$$\int_{\Gamma_3} (p_\nu([\boldsymbol{u}_{1\nu}]) - p_\nu([\boldsymbol{u}_{2\nu}]))([\boldsymbol{u}_{1\nu} - \boldsymbol{u}_{2\nu}])da + \int_{\Gamma_3} (p_\tau([\boldsymbol{u}_{1\tau}]) - p_\tau([\boldsymbol{u}_{2\tau}]))([\boldsymbol{u}_{1\tau} - \boldsymbol{u}_{2\tau}])da,$$

and keeping in mind (23), (28) and (29) we obtain

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_1)_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{V}}^2 \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_1 \in \boldsymbol{V}. \tag{51}$$

Use again (50) and (36), it follows that

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^{2} (\mathcal{A}^\alpha \varepsilon(\boldsymbol{u}_1^\alpha) - \mathcal{A}^\alpha \varepsilon(\boldsymbol{u}_2^\alpha),\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} +$$
$$\int_{\Gamma_3} (p_\nu([\boldsymbol{u}_{1\nu}]) - p_\nu([\boldsymbol{u}_{2\nu}]))([\boldsymbol{w}_\nu])da + \int_{\Gamma_3} (p_\tau([\boldsymbol{u}_{1\tau}]) - p_\tau([\boldsymbol{u}_{2\tau}]))([\boldsymbol{w}_\tau])da \quad \forall \boldsymbol{w} \in \boldsymbol{V},$$

and by (23) and (19), we deduce that

$$|A\boldsymbol{u}_1 - A\boldsymbol{u}_2|_{\boldsymbol{V}'} \leq L_{\mathcal{A}^\alpha}|\boldsymbol{u}_1 - \boldsymbol{u}_2|_{\boldsymbol{V}} + c_0|p_\nu([\boldsymbol{u}_{1\nu}]) - p_\nu([\boldsymbol{u}_{2\nu}])|_{L^2(\Gamma_3)}$$
$$+c_0|p_\tau([\boldsymbol{u}_{1\tau}]) - p_\tau([\boldsymbol{u}_{2\tau}])|_{L^2(\Gamma_3)^d}, \quad \forall \boldsymbol{u}_1, \boldsymbol{u}_2 \in \boldsymbol{V},$$

and keeping in mind the Krasnoselski Theorem (see [3, p.60]), we deduce that $A : \boldsymbol{V} \to \boldsymbol{V}'$ is a continuous operator. Now, by (50), (18) and (23), we find where the positive constant $m = \min\{m_{\mathcal{A}^1}, m_{\mathcal{A}^2}\}$. Choosing $\boldsymbol{u}_2 = 0_{\boldsymbol{V}}$ in (52) we obtain

$$(A\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{V}'\times\boldsymbol{V}} \geq m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \|Ao\|^2_{\boldsymbol{V}'}\|\boldsymbol{u}\|_{\boldsymbol{V}} \tag{52}$$

$$\geq \frac{1}{2}m\|\boldsymbol{u}\|^2_{\boldsymbol{V}} - \frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'} \quad \forall \boldsymbol{u} \in \boldsymbol{V}, \tag{53}$$

which implies that A satisfies condition (48) with $\omega = \frac{m}{2}$ and $\lambda = -\frac{1}{2m}\|Ao\|^2_{\boldsymbol{V}'}$. Moreover, by (50) and (23) we find

$$\|A\boldsymbol{u}\|_{\boldsymbol{V}'} \leq C^1\|u\|_{\boldsymbol{V}} + C^2 \quad \forall \boldsymbol{u} \in \boldsymbol{V},$$

where $C^1 = \max\{C^1_{\mathcal{A}^1}, C^1_{\mathcal{A}^2}\}$ and $C^2 = \max\{C^2_{\mathcal{A}^1}, C^2_{\mathcal{A}^2}\}$. This inequality and (18) imply that A satisfies condition (49). Finally, we recall that by (31) and (33) we have $\mathbf{f} - \eta \in L^2(0,T;\boldsymbol{V}')$ and $\boldsymbol{v}_0 \in H$.
It follows now from Theorem 2 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;\boldsymbol{V}) \cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;\boldsymbol{V}'), \tag{54}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \eta(t) = \mathbf{f}(t), \quad a.e.\ t \in [0,T] \tag{55}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{56}$$

Let $\boldsymbol{u}_\eta : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0,T]. \tag{57}$$

It follows from (50) and (54)–(57) that $\boldsymbol{u}_\eta$ is a unique solution to the variational problem $\mathbf{PV}^u_\eta$ and it satisfies the regularity expressed in (42).
In the second, let $\eta \in L^2(0,T;\boldsymbol{V}')$, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 1 and we consider the following variational problem.

Problem $\mathbf{PV}^\varphi_\eta$. Find the electric potential field $\varphi_\eta : [0,T] \to W$ such that

$$\sum_{\alpha=1}^{2}(\mathcal{B}^\alpha\nabla\varphi^\alpha_\eta(t), \nabla\phi^\alpha)_{H^\alpha} - \sum_{\alpha=1}^{2}(\mathcal{E}^\alpha\varepsilon(\boldsymbol{u}^\alpha_\eta(t)), \nabla\phi^\alpha)_{H^\alpha} = (q(t), \phi)_W \tag{58}$$

$$\forall \phi \in W,\ a.e.\ t \in (0,T).$$

We have the following result.

Lemma 2 *Problem $\boldsymbol{PV}^\varphi_\eta$ has a unique solution φ_η which satisfies the regularity* (43).

Proof. We define a bilinear form: $b(.,.) : W \times W \to \mathbb{R}$ such that

$$b(\varphi, \phi) = \sum_{\alpha=1}^{2}(\mathcal{B}^\alpha\nabla\varphi^\alpha, \nabla\phi^\alpha)_{H^\alpha} \quad \forall \varphi, \phi \in W. \tag{59}$$

We use (59), (20) and (27) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on W. Moreover, using the Riesz Representation Theorem we may define an element $q_\eta : [0,T] \to W$ such that

$$(q_\eta(t),\phi)_W = \sum_{\alpha=1}^{2}(q_\eta^\alpha(t),\phi^\alpha)_{W^\alpha} + \sum_{\alpha=1}^{2}(\mathcal{E}^\alpha\varepsilon(\boldsymbol{u}_\eta^\alpha(t)),\nabla\phi^\alpha)_{H^\alpha} \quad \forall\phi\in W, t\in(0,T).$$

We apply the Lax-Milgram Theorem to deduce that there exists a unique element $\varphi_\eta(t)\in W$ such that

$$b(\varphi_\eta(t),\phi) = (q_\eta(t),\phi)_W \quad \forall\phi\in W. \tag{60}$$

We conclude that $\varphi_\eta(t)$ is a solution to Problem $\mathbf{P}_\eta^\varphi$. Let $t_1,t_2\in[0,T]$, it follows from (58) that

$$\|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W \le C\big(\|\boldsymbol{u}_\eta(t_1)-\boldsymbol{u}_\eta(t_2)\|_{\boldsymbol{V}} + \|q(t_1)-q(t_2)\|_W\big),$$

and the previous inequality, the regularity of $\boldsymbol{u}_\eta$ and q imply that $\varphi_\eta\in C(0,T;W)$. In the third step, we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 1 and φ_η obtained in Lemma 2 to construct the following Cauchy problem for the stress field.

Problem $\mathbf{PV}_\eta^\sigma$. *Find a stress field $\boldsymbol{\sigma}_\eta = (\boldsymbol{\sigma}_\eta^1,\boldsymbol{\sigma}_\eta^2) : [0,T]\to Q$ such that*

$$\boldsymbol{\sigma}_\eta^\alpha(t) = \mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\alpha(t)) + \int_0^t \mathcal{F}^\alpha(\boldsymbol{\sigma}_\eta^\alpha(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\alpha(s)))\,ds, \quad \alpha=1,2, \tag{61}$$

for all $t\in[0,T]$.
In the study of Problem $\mathbf{PV}_\eta^\sigma$ we have the following result.

Lemma 3 *There exists a unique solution of Problem $\boldsymbol{PV}_\eta^\sigma$ and it satisfies $\boldsymbol{\sigma}_\eta\in W^{1,2}(0,T;Q)$. Moreover, if $\boldsymbol{\sigma}_i$ and $\boldsymbol{u}_i$ represent the solutions of problem $\boldsymbol{PV}_\eta^\sigma$ and $\boldsymbol{PV}_\eta^u$, respectively, for $\boldsymbol{\eta}_i\in L^2(0,T;\boldsymbol{V}')$, $i=1,2$, then there exists $c>0$ such that*

$$\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_Q \le c\Big(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V + \int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V\,ds\Big), \tag{62}$$

for all $t\in[0,T]$.

Proof. Let $\Lambda_\eta = (\Lambda_\eta^1,\Lambda_\eta^2) : L^2(0,T;Q)\to L^2(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta^\alpha\boldsymbol{\sigma}(t) = \mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha{}_\eta(t)) + \int_0^t \mathcal{F}^\alpha(\boldsymbol{\sigma}^\alpha(s),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta^\alpha(s)))\,ds, \quad \alpha=1,2 \tag{63}$$

for all $\boldsymbol{\sigma}=(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2)\in L^2(0,T;Q)$ and $t\in[0,T]$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2\in L^2(0,T;Q)$ we use (63) and (25) to obtain

$$\|\Lambda_\eta\,\boldsymbol{\sigma}_1(t)-\Lambda_\eta\,\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \le \max(L_{\mathcal{F}^1},L_{\mathcal{F}^2})\int_0^t \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q\,ds,$$

for all $t \in [0,T]$. It follows from this inequality that for p large enough, a power Λ_η^p of the operator Λ_η is a contraction on the Banach space $L^2(0,T;\mathbf{V})$ and, therefore, there exists a unique element $\boldsymbol{\sigma}_\eta \in L^2(0,T;Q)$ such that $\Lambda_\eta \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution of Problem $\mathbf{PV}_\eta^\sigma$ and, using (61), the regrularity of $\boldsymbol{u}_\eta$ and the properties of the operators $\mathcal{G}^\alpha$ and $\mathcal{F}^\alpha$, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;Q)$. Consider now $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0,T;\boldsymbol{V}')$ and, for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$. We have

$$\boldsymbol{\sigma}_i^\alpha(t) = \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_i^\alpha(t)) + \int_0^t \mathcal{F}^\alpha(\boldsymbol{\sigma}_i^\alpha(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i^\alpha(s)))\, ds, \quad \alpha = 1,2 \ \ \forall t \in [0,T],$$

and, using the properties (24) and (25) of $\mathcal{F}^\alpha$, and $\mathcal{G}^\alpha$ we find

$$\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q \le c\Big(\|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|_{\mathbf{V}} + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q\, ds + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V\, ds\Big) \quad \forall t \in [0,T].$$

Using now a Gronwall argument in the previous inequality we deduce (62), which concludes the proof.

Finally as a consequence of these results and using the properties of the operator $\mathcal{G}^\alpha$, the operator $\mathcal{E}^\alpha$, we consider the operator

$$\Lambda : L^2(0,T;\boldsymbol{V}') \longrightarrow L^2(0,T;\boldsymbol{V}')$$

which maps every element $\eta \in L^2(0,T;\boldsymbol{V}')$ to the element $\Lambda(\eta) \in L^2(0,T;\boldsymbol{V}')$ defined by

$$(\Lambda(\eta)(t), \boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^2 \big(\mathcal{G}^\alpha \varepsilon(\boldsymbol{u}_\eta^\alpha(t)),\ \varepsilon(\boldsymbol{w}^\alpha)\big)_{Q^\alpha} + \sum_{\alpha=1}^2 \big((\mathcal{E}^\alpha)^* \nabla \varphi_\eta^\alpha\ \varepsilon(\boldsymbol{w}^\alpha)\big)_{Q^\alpha} + \sum_{\alpha=1}^2 \Big(\int_0^t \mathcal{F}^\alpha\big(\boldsymbol{\sigma}_\eta^\alpha, \varepsilon(\boldsymbol{u}_\eta^\alpha(s))\big)\, ds,\ \varepsilon(\boldsymbol{w}^\alpha)\Big)_{Q^\alpha}, \quad \forall \boldsymbol{w} \in \boldsymbol{V}. \tag{64}$$

Here, for every $\eta \in L^2(0,T;\boldsymbol{V}')$, $\boldsymbol{u}_\eta$, $\boldsymbol{\sigma}_\eta$ and φ_η represent the displacement field, the stress field, the the potential electric field and bonding field obtained in Lemmas 1, 3 and 2 respectively. We have the following result.

Lemma 4 *The operator Λ has a unique fixed point $\eta^* \in L^2(0,T;\boldsymbol{V}')$.*

Proof. We show that, for a positive integer n, the mapping Λ^n is a contraction on $L^2(0,T;\boldsymbol{V}')$. To this end, we suppose that η_1 and η_2 are two functions in $L^2(0,T;\boldsymbol{V}')$ and denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\varphi_{\eta_i} = \varphi_i$ for $i = 1,2$. We use (24), (26), (28), (29) and (25), we have

$$\|\Lambda(\eta_1)(t) - \Lambda(\eta_2)(t)\|_{\boldsymbol{V}'}^2 \le \sum_{\alpha=1}^2 \int_0^t \big\|\mathcal{F}^\alpha\big(\boldsymbol{\sigma}_1^\alpha(s), \varepsilon(\boldsymbol{u}_1^\alpha(s))\big) - \mathcal{F}^\alpha\big(\boldsymbol{\sigma}_2^\alpha(s), \varepsilon(\boldsymbol{u}_2^\alpha(s))\big)\big\|_{Q^\alpha}^2\, ds + \sum_{\alpha=1}^2 \|\mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_1^\alpha(t)) - \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_2^\alpha(t))\|_{Q^\alpha}^2 + \sum_{\alpha=1}^2 \|(\mathcal{E}^\alpha)^* \nabla \varphi_1^\alpha(t) - (\mathcal{E}^\alpha)^* \nabla \varphi_2^\alpha(t)\|_{Q^\alpha}^2.$$

Therefore,

$$\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|^2_{\boldsymbol{V}'} \le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \int_0^t \|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s))\|^2_{Q}\,ds+\|\varphi_1(t)-\varphi_2(t)\|^2_{W}\bigg). \tag{65}$$

We use estimate (62) to obtain

$$\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|^2_{\boldsymbol{V}'} \le C\bigg(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds + \|\varphi_1(t)-\varphi_2(t)\|^2_{W}\bigg). \tag{66}$$

Moreover, from (46) we obtain

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_1^\alpha)-\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_2^\alpha),\varepsilon(\boldsymbol{v}_1^\alpha-\boldsymbol{v}_2^\alpha))_{Q^\alpha}$$

$$+j(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)+(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}=0. \tag{67}$$

We use (36), (28) and (29) to deduce that

$$j_2(\boldsymbol{v}_1,\boldsymbol{v}_1-\boldsymbol{v}_2)-j_2(\boldsymbol{v}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)\ge 0. \tag{68}$$

It follows from (67) and (68) that

$$(\dot{\boldsymbol{v}}_1-\dot{\boldsymbol{v}}_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_1^\alpha)-\mathcal{A}^\alpha\varepsilon(\boldsymbol{v}_2^\alpha),\varepsilon(\boldsymbol{v}_1^\alpha-\boldsymbol{v}_2^\alpha))_{Q^\alpha} \le -(\eta_1-\eta_2,\boldsymbol{v}_1-\boldsymbol{v}_2)_{\boldsymbol{V}'\times\boldsymbol{V}}. \tag{69}$$

We integrate this equality with respect to time, use the initial conditions $\boldsymbol{v}_1(0)=\boldsymbol{v}_2(0)=\boldsymbol{v}_0$ and condition (23) to find

$$m\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds \le -\int_0^t (\eta_1(s)-\eta_2(s),\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))_{\boldsymbol{V}'\times\boldsymbol{V}}\,ds,$$

for all $t\in[0,T]$. Then, using the inequality $2ab\le\frac{a^2}{m}+mb^2$ we obtain

$$\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds \le C\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_{\boldsymbol{V}'}\,ds \quad \forall t\in[0,T]. \tag{70}$$

We use now (58), (20), (26) and (27) to find

$$\|\varphi_1(t)-\varphi_2(t)\|^2_W \le C\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_V. \tag{71}$$

We substitute (71) in (66) to obtain

$$\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|^2_{\boldsymbol{V}'} \le C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|^2_{\boldsymbol{V}}+\int_0^t \|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s))\|^2_{\boldsymbol{V}}\,ds\right)$$
$$\le C\int_0^t \|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\|^2_{\boldsymbol{V}}\,ds.$$

It follows now from the previous inequality and the estimate (70) that

$$\|\Lambda(\eta_1)(t)-\Lambda(\eta_2)(t)\|^2_{\boldsymbol{V}'} \le C\int_0^t \|\eta_1(s)-\eta_2(s))\|^2_{\boldsymbol{V}'}\,ds.$$

Reiterating this inequality n times we obtain

$$\|\Lambda^n(\eta_1)-\Lambda^n(\eta_2)\|^2_{L^2(0,T;\boldsymbol{V}')} \le \frac{C^nT^n}{n!}\|\eta_1-\eta_2\|^2_{L^2(0,T;\boldsymbol{V}')}. \tag{72}$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $L^2(0,T;\boldsymbol{V}')$, and so Λ has a unique fixed point.

Now, we have all the ingredients to prove Theorem 1.

Proof. Existence. Let $\eta^* \in L^2(0,T;\boldsymbol{V}')$ be the fixed point of Λ and denote

$$\boldsymbol{u}_* = \boldsymbol{u}_{\eta^*}, \quad \varphi_* = \varphi_{\eta^*}, \tag{73}$$

$$\boldsymbol{\sigma}^\alpha_* = \mathcal{A}^\alpha\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^\alpha_*) + (\mathcal{E}^\alpha)^*\nabla\varphi^\alpha_* + \boldsymbol{\sigma}^\alpha_{\eta^*} \quad \forall t\in[0,T], \tag{74}$$

$$\boldsymbol{D}^\alpha_* = \mathcal{E}^\alpha\varepsilon(\boldsymbol{u}^\alpha_*) - \mathcal{B}^\alpha\nabla\varphi^\alpha_*, \quad \forall t\in[0,T]. \tag{75}$$

We prove that $(\boldsymbol{u}_*,\boldsymbol{\sigma}_*,\varphi_*,\boldsymbol{D}_*)$ satisfies (37)–(41) and regularity (42). Indeed, we write (46) for $\eta=\eta^*$ and use (73) to find

$$(\ddot{u}_*(t),w)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\alpha=1}^2(\mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}^\alpha_*(t)),\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + j(\dot{\boldsymbol{u}}_*(t),\boldsymbol{w}) \tag{76}$$
$$+(\eta^*(t),w)_{\boldsymbol{V}'\times\boldsymbol{V}} = (\mathbf{f}(t),\boldsymbol{w})_{\boldsymbol{V}'\times\boldsymbol{V}} \quad \forall\boldsymbol{w}\in\boldsymbol{V}, a.e.\ t\in[0,T].$$

Using equality $\Lambda\eta^*=\eta^*$ it follows that

$$(\eta^*(t),w)_{\boldsymbol{V}'\times\boldsymbol{V}} = \sum_{\alpha=1}^2\left(\mathcal{G}^\alpha\varepsilon(\boldsymbol{u}^\alpha_*(t)),\ \varepsilon(\boldsymbol{w}^\alpha)\right)_{Q^\alpha} + \sum_{\alpha=1}^2\left((\mathcal{E}^\alpha)^*\nabla\varphi^\alpha_*,\ \varepsilon(\boldsymbol{w}^\alpha)\right)_{Q^\alpha} \tag{77}$$
$$+\sum_{\alpha=1}^2\Big(\int_0^t \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha_*(s)-\mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}^\alpha_*(s))-(\mathcal{E}^\alpha)^*\nabla\varphi^\alpha,\varepsilon(\boldsymbol{u}^\alpha_*(s))\Big)\,ds,\ \varepsilon(\boldsymbol{w}^\alpha)\Big)_{Q^\alpha}, \quad \forall\boldsymbol{w}\in\boldsymbol{V}.$$

We now substitute (77) in (76) to obtain

$$(\ddot{\boldsymbol{u}}_*(t), w)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{\alpha=1}^{2}(\mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}_*^\alpha(t)),\ \varepsilon(\boldsymbol{w}^\alpha))_{Q^\alpha} + \sum_{\alpha=1}^{2}\left(\mathcal{G}^\alpha\varepsilon(\boldsymbol{u}_*^\alpha(t)),\ \varepsilon(\boldsymbol{w}^\alpha)\right)_{Q^\alpha}$$
$$+\sum_{\alpha=1}^{2}\Big(\int_0^t \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_*^\alpha(s) - \mathcal{A}^\alpha\varepsilon(\dot{\boldsymbol{u}}_*^\alpha(s)) - (\mathcal{E}^\alpha)^*\nabla\varphi^\alpha, \varepsilon(\boldsymbol{u}_*^\alpha(s))\Big)\,ds,\ \varepsilon(\boldsymbol{w}^\alpha)\Big)_{Q^\alpha}$$
$$+\sum_{\alpha=1}^{2}\left((\mathcal{E}^\alpha)^*\nabla\varphi_*^\alpha,\ \varepsilon(\boldsymbol{w}^\alpha)\right)_{Q^\alpha} + j(\boldsymbol{u}_*(t), \boldsymbol{w}),\quad \forall \boldsymbol{w}\in\boldsymbol{V}. \tag{78}$$

Using $\boldsymbol{u}_{\eta^*}$ in (58), by (73) we have:

$$\sum_{\alpha=1}^{2}(\mathcal{B}^\alpha\nabla\varphi_*^\alpha(t), \nabla\phi^\alpha)_{H^\alpha} - \sum_{\alpha=1}^{2}(\mathcal{E}^\alpha\varepsilon(\boldsymbol{u}_*^\alpha(t)), \nabla\phi^\alpha)_{H^\alpha} = (q(t), \phi)_W \tag{79}$$
$$\forall\phi\in W,\ t\in[0,T].$$

The relations (73), (74), (78) and (79) allow us to conclude now that $(\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \boldsymbol{D}_*)$ satisfy (37). Next, (41) and the regularity (42) follow from Lemmas 1, 3 and 2 . Since $\boldsymbol{u}_*$ and φ_* satisfy (42), it follows from Lemma 3 and (74) that

$$\sigma_* \in L^2(0,T;Q). \tag{80}$$

We choose $w = (w^1, w^2)$ with $w^\alpha = \omega^\alpha \in D(\Omega^\alpha)^d$ and $w^{3-\alpha} = 0$ in (78) and by (73) and (34):

$$\rho^\alpha\ddot{\boldsymbol{u}}_*^\alpha = \mathrm{Div}\,\boldsymbol{\sigma}_*^\alpha + \boldsymbol{f}_0^\alpha,\ \text{a.e. } t\in[0,T],\quad \alpha = 1,2.$$

Also, by (30), (31), and (80) we have:

$$(\mathrm{Div}\,\boldsymbol{\sigma}_*^1, \mathrm{Div}\,\boldsymbol{\sigma}_*^2) \in L^2(0,T;\boldsymbol{V}')$$

Let $t_1, t_2\in[0,T]$, by (26), (27), (20) and (75), we deduce that

$$\|\boldsymbol{D}_*(t_1) - \boldsymbol{D}_*(t_2)\|_H \le C\left(\|\varphi_*(t_1) - \varphi_*(t_2)\|_W + \|\boldsymbol{u}_*(t_1) - \boldsymbol{u}_*(t_2)\|_{\boldsymbol{V}}\right).$$

The regularity of $\boldsymbol{u}_*$ and φ_* given by (42) and (43) implies

$$\boldsymbol{D}_* \in C(0,T;Q). \tag{81}$$

We choose $\phi = (\phi^1, \phi^2)$ with $\phi^\alpha \in D(\Omega^\alpha)^d$ and $\phi^{3-\alpha} = 0$ in (79) and using (35) we find

$$div\boldsymbol{D}_*^\alpha(t) = q_0^\alpha(t)\quad \forall t\in[0,T],\quad \alpha = 1,2.$$

By (31) and (81) we obtain

$$\boldsymbol{D}_* \in C(0,T;\mathcal{W}).$$

Finally we conclude that the weak solution $(\boldsymbol{u}_*, \boldsymbol{\sigma}_*, \varphi_*, \boldsymbol{D}_*)$ of the piezoelectric contact problem **P** has the regularity (42)–(45), which concludes the existence part of Theorem 1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (64).

5 Numerical analysis (Fully discrete approximations: error estimates)

In this section, we present a finite element algorithm to approximate solutions of Problem **PV** and provide an error estimate for these approximations. The discretization of equations (39) to (57) is carried out as follows. Initially, we consider two finite dimensional spaces $\boldsymbol{V}^h \subset \boldsymbol{V}$ and $W^h \subset W$ that approximate the spaces $\boldsymbol{V}$ and W, respectively. The parameter $h > 0$ represents the spatial discretization.

Next, we discretize the time derivatives by using a uniform partition of $[0, T]$, represented by $0 = t_0 < t_1 < ... < t_N = T$. Let k be the time step size, where $k = T/N$, and for a continuous function $f(t)$, we denote $f_n = f(t_n)$ as f evaluated at tn. Finally, for a sequence $\{w_n\}_{n=0}^N$, we define $\delta w_n = (w_n - w_{n-1})/k$ as the divided differences.

It is important to note that in this section, no summation is assumed over a repeated index. The constant c denotes a positive value that depends on the problem's data, but it is independent of the discretization parameters h and k. Thus, by utilizing the backward Euler scheme, the fully discrete approximation of Problem **PV** is given by the following description.

Problem PV^{hk}. Find a discrete velocity field $\boldsymbol{v}^{hk} = \{(\boldsymbol{v}_n^{1,hk}, \boldsymbol{v}_n^{2,hk})\}_{n=0}^N \subset \boldsymbol{V}^h$, a discrete stress field $\boldsymbol{\sigma}^{hk} = \{(\boldsymbol{\sigma}_n^{1,hk}, \boldsymbol{\sigma}_n^{2,hk})\}_{n=1}^N \subset Q^h$ and a discrete electric potential field
$\varphi^{hk} = \{(\varphi_n^{1,hk}, \varphi_n^{2,hk})\}_{n=1}^N \subset W^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$ and for all $n = 1, ..., N$

$$\begin{aligned} \boldsymbol{\sigma}^{\alpha,hk} &= \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^{\alpha,hk}) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^{\alpha,hk}) + (\mathcal{E}^\alpha)^* \nabla \varphi^{\alpha,hk} + \\ &\sum_{j=0}^{n-1} k\, \mathcal{F}^\alpha \Big(\boldsymbol{\sigma}_j^{\alpha,hk} - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk}) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^{\alpha,hk},\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk}) \Big), \end{aligned} \tag{82}$$

$$\begin{aligned} (\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_{\boldsymbol{V}' \times \boldsymbol{V}} + \sum_{\alpha=1}^{2} (\boldsymbol{\sigma}_n^{\alpha,hk}, \boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}))_{Q^\alpha} + j(\boldsymbol{v}_n^{hk}, \boldsymbol{w}^h) = (\mathbf{f}_n, \boldsymbol{w}^h)_{\boldsymbol{V}' \times \boldsymbol{V}} \\ \forall \boldsymbol{w}^h \in \boldsymbol{V}^h, \end{aligned} \tag{83}$$

$$\sum_{\alpha=1}^{2}(\mathcal{B}^{\alpha}\nabla\varphi_n^{\alpha,hk},\nabla\phi^{\alpha,h})_{H^{\alpha}}-\sum_{\alpha=1}^{2}(\mathcal{E}^{\alpha}\varepsilon(\boldsymbol{u}_n^{\alpha,hk}),\nabla\phi^{\alpha,h})_{H^{\alpha}}=(q_n,\phi^h)_W \quad \forall\phi^h\in W^h, \tag{84}$$

where the discrete displacement field $\boldsymbol{u}^{hk}=\{(\boldsymbol{u}_n^{1,hk},\boldsymbol{u}_n^{2,hk})\}_{n=1}^{N}\subset\boldsymbol{V}^h$ is given by

$$\boldsymbol{u}_n^{hk}=\boldsymbol{u}_0^h+\sum_{j=0}^{n}k\boldsymbol{v}_j^{hk}, \tag{85}$$

and $\boldsymbol{u}_0^h$ and $\boldsymbol{v}_0^h$ are appropriate approximations of the initial conditions $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$, respectively.

We notice that the fully discrete problem $\mathbf{PV}^{hk}$ can be seen as a coupled system of variational equations. Using classical results of nonlinear variational equations (see [8]), we obtain that Problem $\mathbf{PV}^{hk}$ admits a unique solution $\boldsymbol{v}^{hk}\subset\boldsymbol{V}^h$, $\boldsymbol{\sigma}^{hk}\subset Q^h$ and $\varphi^{hk}\subset W^h$.

Our interest in this section lies in estimating the numerical errors $\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H$, $\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q$ and $\|\varphi_n-\varphi_n^{hk}\|_W$. We have the following main error estimates result.

Theorem 3 *Let the assumptions of Theorem* (1) *hold. Let* $(\boldsymbol{v},\boldsymbol{\sigma},\varphi)$ *and* $(\boldsymbol{v}^{hk},\boldsymbol{\sigma}^{hk},\varphi^{hk})$ *denote the solutions to problems* $\boldsymbol{PV}$ *and* $\boldsymbol{PV}^{hk}$*, respectively. Then, the following error estimates hold for all* $\boldsymbol{w}^h=\{\boldsymbol{w}_n^h\}_{j-1}^{N}\subset\boldsymbol{V}^h$ *and* $\phi^h=\{\phi_n^h\}_{j-1}^{N}\subset W^h$*,*

$$\begin{aligned}&\max_{1\le n\le N}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2+\|\varphi_n-\varphi_n^{hk}\|_W^2\right\}+k\sum_{n-1}^{N}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2\\ &\le c\left(\begin{array}{l}\max\limits_{1\le n\ \le N}\|\varphi_n-\phi_n^h\|_W^2+k\sum\limits_{j=1}^{N}\left[\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_{\boldsymbol{V}}^2+\right]+\max\limits_{1\leqslant n\leqslant N}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2\\ +k^2+\dfrac{1}{k}\sum\limits_{j=1}^{N-1}\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_{\boldsymbol{V}}^2\end{array}\right).\end{aligned} \tag{86}$$

Proof. First, let us obtain an error estimation on the electric potential. We proceed as in [3] where further details can be found. Taking (40) at time $t=t_n$ for $\phi=\phi^h\in W^h$ and substracting it to (84), it follows that

$$\sum_{\alpha=1}^{2}(\mathcal{B}^{\alpha}\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk}),\nabla\phi^{\alpha,h})_{H^{\alpha}}-\sum_{\alpha=1}^{2}(\mathcal{E}^{\alpha}\varepsilon(\boldsymbol{u}_n^{\alpha}-\boldsymbol{u}_n^{\alpha,hk}),\nabla\phi^{\alpha,h})_{H^{\alpha}}=0,\quad \phi^h\in W^h,$$

Thus, we have

$$\sum_{\alpha=1}^{2}(\mathcal{B}^{\alpha}\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk}),\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk}))_{H^{\alpha}}-\sum_{\alpha=1}^{2}(\mathcal{E}^{\alpha}\varepsilon(\boldsymbol{u}_n^{\alpha}-\boldsymbol{u}_n^{\alpha,hk}),\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk}))_{H^{\alpha}}=$$

$$\sum_{\alpha=1}^{2}(\mathcal{B}^{\alpha}\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk}),\nabla(\varphi_n^{\alpha}-\phi^{\alpha,h}))_{H^{\alpha}}-\sum_{\alpha=1}^{2}(\mathcal{E}^{\alpha}\varepsilon(\boldsymbol{u}_n^{\alpha}-\boldsymbol{u}_n^{\alpha,hk}),\nabla(\varphi_n^{\alpha}-\phi^{\alpha,h}))_{H^{\alpha}}\quad\forall\phi^h\in W^h,$$

using properties (26) and (27), after some algebra we find that

$$\|\varphi_n - \varphi_n^{hk}\|_W^2 \leq c\Big(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}\|\varphi_n - \varphi_n^{hk}\|_W + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}\|\varphi_n - \phi^h\|_W$$

$$+\|\varphi_n - \varphi_n^{hk}\|_W\|\varphi_n - \phi^h\|_W\Big) \quad \forall \phi^h \in W^h.$$

We apply the following elementary inequality:

$$a.b \leq \beta a^2 + \frac{a}{\beta^2}b, \quad a, b, \beta \in \mathbb{R}, \beta > 0. \tag{87}$$

So we find

$$\|\varphi_n - \varphi_n^{hk}\|_W^2 \leq c\Big(\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_n - \phi^h\|_W^2\Big) \quad \forall \phi^h \in W^h. \tag{88}$$

Secondly, we estimate the error for a stress field for $n = 1, 2, ..., N$. By induction, we obtain the following formula:

$$\begin{aligned}\boldsymbol{\sigma}^{\alpha,hk} &= \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^{\alpha,hk}) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}^{\alpha,hk}) + (\mathcal{E}^\alpha)^* \nabla \varphi^{\alpha,hk} + \\ &\sum_{j=0}^{n-1} k\ \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha,hk} - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk}) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^{\alpha,hk},\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk})\Big).\end{aligned} \tag{89}$$

Next, we integrate (5) from $t = 0$ to $t = t_n$ to obtain

$$\begin{aligned}\boldsymbol{\sigma}_n^\alpha &= \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_n^\alpha) + (\mathcal{E}^\alpha)^* \nabla \varphi_n^\alpha + \\ &\int_0^{t_n} \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \varphi^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s))\Big)\, ds.\end{aligned} \tag{90}$$

Combining (89) and (90), we find

$$\begin{aligned}&\boldsymbol{\sigma}^{\alpha,hk} - \boldsymbol{\sigma}_n^\alpha = \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}) + \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_n^\alpha) - \mathcal{G}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha,hk}) + \\ &(\mathcal{E}^\alpha)^* \nabla(\varphi_n^\alpha - \varphi_n^{\alpha,hk}) + \sum_{j=0}^{n-1} k\ \Big[\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^\alpha) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^\alpha)\Big) - \\ &\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha,hk} - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk}) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^{\alpha,hk},\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk})\Big)\Big] + \boldsymbol{M}_n^k(\mathcal{F}^\alpha),\end{aligned} \tag{91}$$

where, for $n = 1, ..., N$, the quadrature error is

$$\begin{aligned}\boldsymbol{M}_n^k(\mathcal{F}^\alpha) &= \int_0^{t_n} \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha(s) - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha(s)) - (\mathcal{E}^\alpha)^* \nabla \varphi^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(s))\Big)\, ds \\ &- \sum_{j=0}^{n-1} k\ \Big(\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^\alpha) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^\alpha)\Big).\end{aligned}$$

By the Lipschitz continuity of $\mathcal{F}$ we have

$$\Big\|\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^\alpha - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^\alpha) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^\alpha,\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^\alpha)\Big) - \mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha,hk} - \mathcal{A}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk}) - (\mathcal{E}^\alpha)^* \nabla \varphi_j^{\alpha,hk},\, \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk})\Big)\Big\|_Q$$

$$\leq c\Big(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}\Big).$$

Then (91) implies

$$\begin{aligned}\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q\leq c\Big(&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}+\|\varphi_n-\varphi_n^{hk}\|_W+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}+\\ &\|\boldsymbol{M}_n^k(\mathcal{F})\|_Q+\sum_{j=0}^{n-1}k\,\Big[\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\\ &\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}\Big]\Big).\end{aligned}\tag{92}$$

Again, by the Lipschitz continuity of $\mathcal{F}$ we find

$$\begin{aligned}\|\boldsymbol{M}_n^k(\mathcal{F})\|_Q&\leq\sum_{j=0}^{n-1}\int_{t_j}^{t_{j-1}}\Big\|\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}^\alpha(t)-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha(t))-(\mathcal{E}^\alpha)^*\nabla\varphi^\alpha,\,\boldsymbol{\varepsilon}(\boldsymbol{u}^\alpha(t))\Big)-\\ &\qquad\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^\alpha-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_j^\alpha)-(\mathcal{E}^\alpha)^*\nabla\varphi_j^\alpha,\,\boldsymbol{\varepsilon}(\boldsymbol{u}_j^\alpha)\Big)\Big\|_Q dt\\ &\leq c\sum_{j=0}^{n-1}\int_{t_j}^{t_{j-1}}\Big(\|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}_j\|_Q+\|\boldsymbol{v}(t)-\boldsymbol{v}_j\|_{\boldsymbol{V}}+\|\varphi(t)-\varphi_j\|_W+\|\boldsymbol{u}(t)-\boldsymbol{u}_j\|_{\boldsymbol{V}}\Big)dt.\end{aligned}$$

Using the assumptions on the regularity of the solution we deduce

$$\|\boldsymbol{M}_n^k(\mathcal{F})\|_Q\leq ck,\tag{93}$$

where the constant c depends on $\|\boldsymbol{\sigma}\|_{L^2(0,T;Q)}$, $\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{V})\cap C(0,T;H)}$ and $\|\boldsymbol{u}_0\|_{L^2(0,T;\boldsymbol{V})\cap C^1(0,T;H)}$. From (92) and (93) we also obtain the inequality

$$\begin{aligned}\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2\leq c\Big(&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2+\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2+\\ &k^2+\sum_{j=0}^{n-1}k\,\Big[\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2+\\ &\|\varphi_j-\varphi_j^{hk}\|_W^2+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2\Big]\Big).\end{aligned}\tag{94}$$

In view of the applications involving elasto-viscoplastic piezoelectricmaterials, it is convenient to consider the quantity

$$\boldsymbol{N}_n^{\alpha,hk}(\boldsymbol{v})=\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha)-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha-\boldsymbol{v}_n^{\alpha,hk})\Big)_{Q^\alpha}.$$

By the properties (23) and Korn's inequality , we have the following lower bound:

$$\boldsymbol{N}_n^{\alpha,hk}(\boldsymbol{v})\geq m_{\mathcal{A}}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Q^2\geq c\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2.\tag{95}$$

We turn to derive an upper bound for $\boldsymbol{N}_n^{\alpha,hk}(\boldsymbol{v})$. For any $\boldsymbol{w}^h \in \boldsymbol{V}^h$, we may write

$$\begin{aligned}\boldsymbol{N}_n^{\alpha,hk}(\boldsymbol{v}) = &\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha)-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha-\boldsymbol{w}^{\alpha,h})\Big)_{Q^\alpha}+\\ &\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^\alpha)-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Big)_{Q^\alpha}.\end{aligned}\tag{96}$$

By (89), we find

$$\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Big)_{Q^\alpha}=\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}-\sum_{\alpha=1}^{2}\Bigg((\mathcal{E}^\alpha)^*\nabla(\varphi_n^{\alpha,hk})+$$
$$\mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha,hk})+\sum_{j=0}^{n-1}k\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha,hk}-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk})-(\mathcal{E}^\alpha)^*\nabla(\varphi_j^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk})\Big)ds,\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Bigg)_{Q^\alpha}.$$

It follows from (90) that

$$\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha}),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Big)_{Q^\alpha}=\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}-\sum_{\alpha=1}^{2}\Bigg((\mathcal{E}^\alpha)^*\nabla(\varphi_n^{\alpha})+$$
$$\mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha})+\int_0^{t_n}\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_n^{\alpha}-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha})-(\mathcal{E}^\alpha)^*\nabla(\varphi_n^{\alpha}),\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha})\Big)ds,\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Bigg)_{Q^\alpha}.$$

Subtracting these two equalities, we obtain

$$\sum_{\alpha=1}^{2}\Big(\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha})-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Big)_{Q^\alpha}=\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha}-\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}$$
$$-\sum_{\alpha=1}^{2}\Bigg((\mathcal{E}^\alpha)^*\nabla(\varphi_n^{\alpha}-\varphi_n^{\alpha,hk})+\mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha})-\mathcal{G}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{\alpha,hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk})\Bigg)_{Q^\alpha}-\boldsymbol{P}_n(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}).$$

where

$$\boldsymbol{P}_n(\boldsymbol{w}^{\alpha,h})=\sum_{\alpha=1}^{2}\Big(\boldsymbol{M}_n^k(\mathcal{F}^\alpha),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h})\Big)_{Q^\alpha}+\sum_{\alpha=1}^{2}\sum_{j=0}^{n-1}k\Bigg(\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha}-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha})-(\mathcal{E}^\alpha)^*\nabla\varphi_j^{\alpha},\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha})\Big)$$
$$-\mathcal{F}^\alpha\Big(\boldsymbol{\sigma}_j^{\alpha,hk}-\mathcal{A}^\alpha\boldsymbol{\varepsilon}(\boldsymbol{v}_j^{\alpha,hk})-(\mathcal{E}^\alpha)^*\nabla\varphi_j^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_j^{\alpha,hk})\Big),\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h})\Bigg)_{Q^\alpha}.$$

We bound this term as follows:

$$|\boldsymbol{P}_n(\boldsymbol{w}^{\alpha,h})|\le c\Big(\sum_{j=0}^{n-1}k(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}})$$

$$+\|\boldsymbol{M}_n^k(\mathcal{F})\|_Q\Big)\|\boldsymbol{w}^h\|_{\boldsymbol{V}}$$

$$\leq c\Big(\sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}})+k\Big)\|\boldsymbol{w}^h\|_{\boldsymbol{V}}.$$

Then from the expression (96), we get the upper bound,

$$\begin{aligned}\boldsymbol{N}_n^{\alpha,hk}(\boldsymbol{v}) \leq c\Big(&\|\varphi_n-\varphi_n^{hk}\|_W+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}+\sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\\ &\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}})+k\Big)\|\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}+\\ &c\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_{\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha}-\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}.\end{aligned} \tag{97}$$

Combining the lower bound (95) and the upper bound (97), we obtain

$$\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \leq c\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_{\boldsymbol{V}}+\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha}-\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}+c\Big(\|\varphi_n-\varphi_n^{hk}\|_W$$

$$+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}+\sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}+\|\varphi_j-\varphi_j^{hk}\|_W+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}})+k\Big)\|\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}.$$

Next we bound $\|\boldsymbol{w}^h-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}$ with $\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_{\boldsymbol{V}}+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}$, Hence, finally, we have the estimate

$$\begin{aligned}\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \leq\ & c\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_{\boldsymbol{V}}^2+c\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha}-\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}-\boldsymbol{v}_n^{\alpha,hk}))_{Q^\alpha}+\\ & c\Big(\|\varphi_n-\varphi_n^{hk}\|_W^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2+\sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j-\boldsymbol{\sigma}_j^{hk}\|_Q^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2+\\ &\|\varphi_j-\varphi_j^{hk}\|_W^2+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2)+k^2\Big).\end{aligned} \tag{98}$$

These inequalities will be used in the numerical analysis of contact problems for elasto-viscoplastic piezoelectric materials.

Third, we proceed now to estimate the numerical errors on the velocity field. We rewrite variational Eq. (39) at time $t=t_n$ for $\boldsymbol{w}=\boldsymbol{w}^h\in\boldsymbol{V}^h$ and we substract it to (83) to obtain

$$\sum_{\alpha=1}^{2}(\boldsymbol{\sigma}_n^{\alpha}-\boldsymbol{\sigma}_n^{\alpha,hk},\boldsymbol{\varepsilon}(\boldsymbol{w}^{\alpha,h}))_{Q^\alpha}=-(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}}-j(\boldsymbol{v}_n,\boldsymbol{w}^h)+j(\boldsymbol{v}_n^{hk},\boldsymbol{w}^h).$$

Then, (98) implies

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \le c\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{\boldsymbol{V}}^2 + c\Big((\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}} + j(\boldsymbol{v}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)\Big) + c\Big(\|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + \sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_j - \varphi_j^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2) + k^2\Big). \tag{99}$$

Next, we write

$$(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}} = -(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_{\boldsymbol{V}'\times\boldsymbol{V}} + (\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}} + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}}.$$

Then, it follows from (99) that

$$(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_{\boldsymbol{V}'\times\boldsymbol{V}} + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \le c\Big(\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{\boldsymbol{V}}^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}\|_H^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + |j(\boldsymbol{v}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h)| + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}} + \sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_j - \varphi_j^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2) + k^2\Big). \tag{100}$$

From properties (28)–(29), we have

$$j(\boldsymbol{v}_n, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) - j(\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk} - \boldsymbol{w}^h) \le \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}\|\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h\|_{\boldsymbol{V}} \qquad \forall \boldsymbol{w}^h \in \boldsymbol{V}^h,$$

where the term $\|\boldsymbol{v}_n^{hk} - \boldsymbol{w}^h\|_{\boldsymbol{V}}$ is further bounded by

$$\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{\boldsymbol{V}} + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}.$$

Using repeatedly inequality (87) and then (100) implies that

$$(\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_{\boldsymbol{V}'\times\boldsymbol{V}} + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \le c\Big(\|\boldsymbol{v}_n - \boldsymbol{w}^h\|_{\boldsymbol{V}}^2 + \|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}\|_H^2 + \|\varphi_n - \varphi_n^{hk}\|_W^2 + \sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_j - \varphi_j^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2) + k^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + (\delta\boldsymbol{v}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_{\boldsymbol{V}'\times\boldsymbol{V}}\Big), \tag{101}$$

we denote $\delta\boldsymbol{v}_n = (\boldsymbol{v}_n - \boldsymbol{v}_{n-1})/k$.

Next, a lower bound for $(\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_{\boldsymbol{V}' \times \boldsymbol{V}}$ is

$$(\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_{\boldsymbol{V}' \times \boldsymbol{V}} \geq \frac{1}{2k}\Big(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2 \Big).$$

Therefore, for $\boldsymbol{w}_n^h \in \boldsymbol{V}^h$, we obtain from (101) that

$$\frac{1}{2k}\Big(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2 \Big) + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \leq c\Big(\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_{\boldsymbol{V}}^2 + \|\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}\|_H^2$$

$$+ \|\varphi_n - \varphi_n^{hk}\|_W^2 + \sum_{j=0}^{n-1} k(\|\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j^{hk}\|_Q^2 + \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_j - \varphi_j^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2) + k^2$$

$$+ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_{\boldsymbol{V}}^2 + (\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_{\boldsymbol{V}' \times \boldsymbol{V}}\Big). \tag{102}$$

Multiplying this inequality by $2k$, changing n to j, replacing $\boldsymbol{w}^h$ by $\boldsymbol{w}_j^h$, and performing the summation over $j = 1$ to n, yields

$$\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_0 - \boldsymbol{v}_0^{hk}\|_H^2 + 2\sum_{j=1}^{n} k\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_{\boldsymbol{V}}^2 \leq c\sum_{j=1}^{n} k\Big(\|\dot{\boldsymbol{v}}_j - \delta \boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_{\boldsymbol{V}}^2$$

$$+ \sum_{i=0}^{n-1} k(\|\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i^{hk}\|_Q^2 + \|\boldsymbol{v}_i - \boldsymbol{v}_i^{hk}\|_{\boldsymbol{V}}^2 + \|\varphi_i - \varphi_i^{hk}\|_W^2 + \|\boldsymbol{u}_i - \boldsymbol{u}_i^{hk}\|_{\boldsymbol{V}}^2) + k^2$$

$$+ \|\varphi_j - \varphi_j^{hk}\|_W^2 + \|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}}^2 + (\delta \boldsymbol{v}_j - \delta \boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_{\boldsymbol{V}' \times \boldsymbol{V}}\Big) \tag{103}$$

for all $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^n \subset \boldsymbol{V}^h$.

We estimate the second term on the right-hand side as follows:

$$\sum_{j=1}^{n} k(\delta \boldsymbol{v}_j - \delta \boldsymbol{v}_j^{hk}, \boldsymbol{v}_j - \boldsymbol{w}_j^h)_{\boldsymbol{V}' \times \boldsymbol{V}} = \sum_{j=1}^{n} (\boldsymbol{v}_j - \boldsymbol{v}_j^{hk} - (\boldsymbol{v}_{j-1} - \boldsymbol{v}_{j-1}^{hk}), \boldsymbol{v}_j - \boldsymbol{w}_j^h)_{\boldsymbol{V}' \times \boldsymbol{V}}$$

$$\leq \epsilon \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + c\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + c\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + c\|\boldsymbol{v}_1 - \boldsymbol{w}_1^h\|_H^2$$

$$+ c\sum_{j=1}^{n-1} \|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\|_H \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H.$$

Taking Eq. (57) at time $t = t_j$ and subtracting it to (85) we find that

$$\|\boldsymbol{u}_j - \boldsymbol{u}_j^{hk}\|_{\boldsymbol{V}} \leq c\Big(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_{\boldsymbol{V}} + \sum_{l=1}^{j} k\|\boldsymbol{v}_l - \boldsymbol{v}_l^{hk}\|_{\boldsymbol{V}} + I_j \Big), \tag{104}$$

where I_j is the integration error given by

$$I_j = \Big\| \int_0^{t_j} \boldsymbol{v}(s)ds - \sum_{l=1}^{j} k\boldsymbol{v}_l \Big\|_{\boldsymbol{V}}.$$

From Theorem (1) we have $\boldsymbol{u} \in H^2(0,T;\boldsymbol{V})$ and therefore, it is easy to check that

$$I_j \leqslant \|\boldsymbol{u}\|_{H^2(0,T;\boldsymbol{V})}.$$

Finally, combining (88), (94) and (103), taking into account estimation (104) and applying a discrete version of Gronwall's inequality (see [11] for details), we then obtain (86). We notice that the above error estimates are the basis for the analysis of the convergence rate of the algorithm. Thus, let Ω^α be a polyhedral domain and denote by $\mathfrak{F}^{\alpha,h}$ a triangulation of Ω^α compatible with the partition of the boundary $\Gamma^\alpha = \partial\Omega^\alpha$ into Γ_1^α, Γ_2^α and Γ_3 on one hand, and into Γ_a^α and Γ_b^α, on the other hand. Let $\boldsymbol{V}^h$, Q^h and W^h consist of continuous and piecewise affine functions; that is,

$$\boldsymbol{V}^h(\Omega^\alpha) = \{\boldsymbol{w}^{\alpha,h} \in [C(\bar{\Omega^\alpha})]^d; \boldsymbol{w}^{\alpha,h}_{|_{Tr}} \in [P_1(Tr)]^d, \forall Tr \in \mathfrak{F}^{\alpha,h}, \boldsymbol{w}^{\alpha,h}|_{\Gamma_1^\alpha} \equiv 0\},$$

$$Q^h(\Omega^\alpha) = \{\boldsymbol{\tau}^\alpha \in Q^\alpha; \boldsymbol{\tau}^{\alpha,h}_{|_{Tr}} \in [P_1(Tr)]^{d\times d}, \forall Tr \in \mathfrak{F}^{\alpha,h}\},$$

$$W^h(\Omega^\alpha) = \{\phi^{\alpha,h} \in [C(\bar{\Omega^\alpha})]^d; \phi^{\alpha,h}_{|_{Tr}} \in [P_1(Tr)]^d, \forall Tr \in \mathfrak{F}^{\alpha,h}, \phi^{\alpha,h}|_{\Gamma_a^\alpha} \equiv 0\},$$

where $P_1(Tr)$ represents the space of polynomials of global degree less or equal to one in Tr. Assume that the discrete initial conditions $\boldsymbol{u}_0^h$ and $\boldsymbol{v}_0^h$ are obtained by

$$\boldsymbol{u}_0^h = \Pi^h \boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \Pi^h \boldsymbol{v}_0, \tag{105}$$

Where $\Pi^h = (\pi^h)_{i=1}^d : [C(\bar{\Omega^\alpha})]^d \longrightarrow \boldsymbol{V}^h$, and π^h is the standard finite element interpolation operator (see, e.g. [6]).

Then, we have the following corollary which states the linear convergence of the algorithm under suitable regularity conditions.

Corollary 1 *Let the assumptions of Theorem* (1) *hold. Let* $(\boldsymbol{v},\boldsymbol{\sigma},\varphi)$ *and* $(\boldsymbol{v}^{hk},\boldsymbol{\sigma}^{hk},\varphi^{hk})$ *denote the solutions to problems* $\boldsymbol{PV}$ *and* $\boldsymbol{PV}^{hk}$*, respectively, and let the discrete initial conditions be given by* (105)*. Under the following regularity conditions:*

$$\boldsymbol{u} \in C^1(0,T;[H^2(\Omega)]^d) \cap H^3(0,T;\boldsymbol{V}), \;\; \boldsymbol{\sigma} \in C(0,T;H^1(\Omega)^{d\times d}), \;\; \varphi \in C(0,T;H^2(\Omega)),$$

the linear convergence of the algorithm is achieved; that is, there exists a positive constant $c > 0$*, independent of the discretization parameters* h *and* k*, such that*

$$\max_{1\leqslant n\leqslant N} \Big\{ \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|^2_{\boldsymbol{V}} + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|^2_Q + \|\varphi_n - \varphi_n^{hk}\|^2_W \Big\} \leqslant c(h+k). \tag{106}$$

Proof. We have the following approximation properties of the finite element spaces $\boldsymbol{V}^h$ and W^h (see [6]),

$$\max_{1\leqslant n\leqslant N}\inf_{\phi_n^h\in W^h}\|\varphi_n-\phi_n^h\|_W\leqslant ch\|\varphi\|_{C(0,T;H^2(\Omega))},$$

$$\max_{1\leqslant n\leqslant N}\inf_{\boldsymbol{v}_n^h\in \boldsymbol{V}^h}\|\boldsymbol{u}_n-\boldsymbol{w}_n^h\|_{\boldsymbol{V}}\leqslant ch\|\boldsymbol{u}\|_{C^1(0,T;[H^2(\Omega)]^d)},$$

$$\max_{1\leqslant n\leqslant N}\inf_{\boldsymbol{v}_n^h\in \boldsymbol{V}^h}\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_{H}\leqslant ch\|\boldsymbol{u}\|_{C^1(0,T;[H^2(\Omega)]^d)}.$$

Moreover, from the definition of the finite element interpolation operator Π^h it follows that

$$\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_{\boldsymbol{V}}\leqslant ch\|\boldsymbol{u}\|_{C^1(0,T;[H^2(\Omega)]^d)},\qquad \|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_{H}\leqslant ch^2\|\boldsymbol{u}\|_{C^1(0,T;[H^2(\Omega)]^d)},$$

and it is easy to check that

$$\sum_{j=1}^{N}k\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2\leqslant ch^2\|\boldsymbol{u}\|_{H^3(0,T;H)}^2.$$

We find that (see [10] for details)

$$\frac{1}{k}\sum_{j=1}^{N-1}\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2\leqslant ch^2\|\boldsymbol{u}\|_{H^2(0,T;\boldsymbol{V})}^2,$$

and combining the previous estimates, (86) and (104) it leads to (106).

Conclusion

We presented a model for the a friction problem between two piezoelectric bodies that are elastic- viscoplastic. The existence of the unique weak solution for the problem was established by using arguments from the parabolic inequalities, differential equations and fixed-point arguments. Additionally, a fully discrete approximation is investigated using the Euler scheme and finite element method for the spatial variable and time derivatives. And we calculated error estimates, leading to convergence results under certain regularity conditions.

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