General Mathematics Vol. 31, No. 2 (2023), 1–16

DOI: 10.2478/gm-2023-0008

sciendo

Unique common fixed point results for two maps in incomplete spaces [1]

Hakima Bouhadjera, Aya Dari, Ryme Zouaimia

Abstract

In this paper, in the first part, we will improve and extend the main results 1 and 2 of Mishra et al. [6] in a b-metric space, and in the second and last part, we will improve and extend Theorem 5 and Theorem 6 of Shah [7] in a b-metric-like space, in both sections, we will use our new notion of occasionally weakly biased maps of type $(\mathcal{A})$. Some illustrative examples are given in order to clarify and support our findings.

2010 Mathematics Subject Classification: 47H10, 54H25.
Key words and phrases: Occasionally weakly biased maps of type (A), b-metric spaces, b-metric-like spaces.

1 Unique common fixed points of occasionally weakly biased maps in b-metric spaces

In this first section, we will improve and extend the main results of Mishra et al. [6] using the concept of occasionally weakly biased maps of type $(\mathcal{A})$ in a b-metric space.

[1] *Received 16 July, 2023*
Accepted for publication (in revised form) 12 August, 2023

1.1 Introduction

Banach's theorem is considered as the guarantee of the existence and uniqueness of fixed points. Several mathematicians made some changes to this distinguished result, in order to improve, extend and generalize this important theorem. On the one hand, they increased the number of maps to investigate common fixed points. On the other hand, they concentrated on the complete metric space. For this end, many authors generalized the metric space to other spaces such as, symmetric, fuzzy, generalized, partial, probabilistic, dislocated, Menger, ultra metric spaces, and so on. Particularly, in 1985, in his dissertation [5], Matthews suggested the class of metric domains. According to him, metric domain has been introduced so as to boost the concept of completeness in domain theory and, he pointed out that there exists a bijection between the category of metric domains and the category of metric spaces. Now, the b-metric's definition was established by Bakhtin [2] and Czerwik [4] as a generalization of the metric. In addition, fixed points can exist under several different conditions. For this end, in a few months we introduced a new type of compatibility called occasionally weakly biased maps of type (A) [3] for generalizing some compatibility's notions. In this section, we will especially improve and expand the principal results of [6] and some similar results.

1.2 Preliminary notes

Definition 1 *([5]) A* ***metric domain*** *is a pair $< \mathcal{D}, \mathfrak{D} >$ where $\mathcal{D}$ is a non-empty set, and $\mathfrak{D}$ is a function from $\mathcal{D} \times \mathcal{D}$ to $\mathbb{R}_+$ such that*

1. *for all x, y in $\mathcal{D}$, $\mathfrak{D}(x, y) = 0 \Rightarrow x = y$,*
2. *for all x, y in $\mathcal{D}$, $\mathfrak{D}(x, y) = \mathfrak{D}(y, x)$,*
3. *for all x, y, z in $\mathcal{D}$, $\mathfrak{D}(x, y) \leq \mathfrak{D}(x, z) + \mathfrak{D}(z, y)$.*

Definition 2 *A b-metric on a nonempty set $\mathcal{X}$ is a function $d' : \mathcal{X} \times \mathcal{X} \to [0, +\infty)$ such that for all f, g, $h \in \mathcal{X}$ and a constant $\xi \geq 1$ the following three conditions hold true:*

1. *$d'(f, g) = 0 \Leftrightarrow f = g$;*
2. *$d'(f, g) = d'(g, f)$;*
3. *$d'(f, g) \leq \xi[d'(f, h) + d'(h, g)]$.*

The pair $(\mathcal{X}, d')$ is called a ***b-metric space****.*

Definition 3 *([3]) Let $\mathcal{S}$ and $\mathcal{T}$ be maps from a b-metric space into itself. $\mathcal{S}$ and $\mathcal{T}$ are* ***occasionally weakly $\mathcal{S}$-biased of type*** $(\mathcal{A})$ *and* ***occasionally weakly $\mathcal{T}$-biased of type*** $(\mathcal{A})$*, respectively, iff, there is an element $\iota \in \mathcal{X}$ such that $\mathcal{S}\iota = \mathcal{T}\iota$ implies*

$$d(\mathcal{S}\mathcal{S}\iota, \mathcal{T}\iota) \leq d(\mathcal{T}\mathcal{S}\iota, \mathcal{S}\iota),$$
$$d(\mathcal{T}\mathcal{T}\iota, \mathcal{S}\iota) \leq d(\mathcal{S}\mathcal{T}\iota, \mathcal{T}\iota),$$

respectively.

In their paper [6], Mishra et al. proved the next theorems:

Theorem 1 *Let $\mathcal{M}$ be a complete b-metric space with metric ρ and let $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be a function with the following*

$$\rho(\mathcal{T}x, \mathcal{T}y) \leq a\rho(x, \mathcal{T}x) + b\rho(y, \mathcal{T}y) + c\rho(x, y)$$

$\forall$ x, $y \in \mathcal{M}$, where a, b, c are non-negative real numbers and satisfy $a + s(b + c) < 1$ for $s \geq 1$ then $\mathcal{T}$ has a unique fixed point.

Theorem 2 *Let $(\mathcal{M}, \rho)$ be a complete b-metric space and a map $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ satisfying the following condition for all x, $y \in \mathcal{M}$*

$$\rho(\mathcal{T}x, \mathcal{T}y) \leq a\rho(x, \mathcal{T}x) + b\rho(y, \mathcal{T}y) + c\rho(x, \mathcal{T}y) + d\rho(y, \mathcal{T}x) + e\rho(x, y)$$

where a, b, c, d, e are non-negative and we set $\alpha = a + b + c + d + e$, such that $\alpha \in \left(0, \frac{1}{2s}\right)$ then $\mathcal{T}$ has a unique fixed point.

We will improve and extend the above theorems.

1.3 Unique common fixed points for two maps

Theorem 3 *Let $\mathcal{M}$ be a b-metric space with metric d and let $\mathcal{C}$, $\mathcal{D} : \mathcal{M} \to \mathcal{M}$ be occasionally weakly $\mathcal{D}$-biased maps of type $(\mathcal{A})$ satisfying the following condition*

$$d(\mathcal{C}x, \mathcal{C}y) \leq \alpha d(\mathcal{D}x, \mathcal{C}x) + \beta d(\mathcal{D}y, \mathcal{C}y) + \gamma d(\mathcal{D}x, \mathcal{D}y) \tag{1}$$

$\forall$ x, $y \in \mathcal{M}$, where α, β, γ are non-negative real numbers and satisfy $2s\alpha + \gamma < 1$ or $2s\beta + \gamma < 1$ for $s \geq 1$, then, $\mathcal{C}$ and $\mathcal{D}$ have a unique common fixed point.

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are occasionally weakly $\mathcal{D}$-biased of type $(\mathcal{A})$, then, there is an element $p \in \mathcal{M}$ such that $\mathcal{C}p = \mathcal{D}p$ implies that $d(\mathcal{D}\mathcal{D}p, \mathcal{C}p) \leq d(\mathcal{C}\mathcal{D}p, \mathcal{D}p)$.

First version: We claim that $\mathcal{C}\mathcal{C}p = \mathcal{C}p$. If not, the use of inequality (1) gives

$$\begin{aligned}
d(\mathcal{C}\mathcal{C}p, \mathcal{C}p) &\leq \alpha d(\mathcal{D}\mathcal{C}p, \mathcal{C}\mathcal{C}p) + \beta d(\mathcal{D}p, \mathcal{C}p) + \gamma d(\mathcal{D}\mathcal{C}p, \mathcal{D}p) \\
&= \alpha d(\mathcal{D}\mathcal{C}p, \mathcal{C}\mathcal{C}p) + \gamma d(\mathcal{D}\mathcal{C}p, \mathcal{D}p) \\
&= \alpha d(\mathcal{D}\mathcal{D}p, \mathcal{C}\mathcal{C}p) + \gamma d(\mathcal{D}\mathcal{D}p, \mathcal{C}p) \\
&\leq \alpha s[d(\mathcal{D}\mathcal{D}p, \mathcal{C}p) + d(\mathcal{C}p, \mathcal{C}\mathcal{C}p)] + \gamma d(\mathcal{C}\mathcal{D}p, \mathcal{D}p) \\
&\leq \alpha s[d(\mathcal{C}\mathcal{D}p, \mathcal{D}p) + d(\mathcal{C}p, \mathcal{C}\mathcal{C}p)] + \gamma d(\mathcal{C}\mathcal{D}p, \mathcal{D}p) \\
&= [2s\alpha + \gamma] d(\mathcal{C}\mathcal{C}p, \mathcal{C}p) \\
&< d(\mathcal{C}\mathcal{C}p, \mathcal{C}p)
\end{aligned}$$

a contradiction, hence $\mathcal{C}\mathcal{C}p = \mathcal{C}p$, consequently $\mathcal{D}\mathcal{C}p = \mathcal{C}p$. Setting $\mathcal{C}p = \mathcal{D}p = w$, therefore, w is a common fixed point of $\mathcal{C}$ and $\mathcal{D}$.

Now, suppose there exists another common fixed point say w', using condition (1), we get

$$\begin{aligned} d(w,w') = d(\mathcal{C}w,\mathcal{C}w') &\leq \alpha d(\mathcal{D}w,\mathcal{C}w) + \beta d(\mathcal{D}w',\mathcal{C}w') + \gamma d(\mathcal{D}w,\mathcal{D}w') \\ &= \gamma d(w,w') \\ &< d(w,w') \end{aligned}$$

a contradiction, thus, $w' = w$.

Second version: Assume that $d(\mathcal{C}\mathcal{C}p,\mathcal{C}p) > 0$, then, by inequality (1), we find

$$\begin{aligned} d(\mathcal{C}p,\mathcal{C}\mathcal{C}p) &\leq \alpha d(\mathcal{D}p,\mathcal{C}p) + \beta d(\mathcal{D}\mathcal{C}p,\mathcal{C}\mathcal{C}p) + \gamma d(\mathcal{D}p,\mathcal{D}\mathcal{C}p) \\ &= \beta d(\mathcal{D}\mathcal{C}p,\mathcal{C}\mathcal{C}p) + \gamma d(\mathcal{D}p,\mathcal{D}\mathcal{C}p) \\ &= \beta d(\mathcal{D}\mathcal{D}p,\mathcal{C}\mathcal{C}p) + \gamma d(\mathcal{C}p,\mathcal{D}\mathcal{D}p) \\ &\leq \beta s[d(\mathcal{D}\mathcal{D}p,\mathcal{C}p) + d(\mathcal{C}p,\mathcal{C}\mathcal{C}p)] + \gamma d(\mathcal{C}\mathcal{D}p,\mathcal{D}p) \\ &\leq \beta s[d(\mathcal{C}\mathcal{D}p,\mathcal{D}p) + d(\mathcal{C}p,\mathcal{C}\mathcal{C}p)] + \gamma d(\mathcal{C}\mathcal{D}p,\mathcal{D}p) \\ &= [2s\beta + \gamma] d(\mathcal{C}\mathcal{C}p,\mathcal{C}p) \\ &< d(\mathcal{C}\mathcal{C}p,\mathcal{C}p) \end{aligned}$$

which is a contradiction, hence $d(\mathcal{C}\mathcal{C}p,\mathcal{C}p) = 0$; i.e., $\mathcal{C}\mathcal{C}p = \mathcal{C}p$, consequently $\mathcal{D}\mathcal{C}p = \mathcal{C}p$. Putting $\mathcal{C}p = \mathcal{D}p = w$, therefore, w is a common fixed point of $\mathcal{C}$ and $\mathcal{D}$.

Now, suppose there is another common fixed point w', from condition (1), we have

$$\begin{aligned} d(w,w') = d(\mathcal{C}w,\mathcal{C}w') &\leq \alpha d(\mathcal{D}w,\mathcal{C}w) + \beta d(\mathcal{D}w',\mathcal{C}w') + \gamma d(\mathcal{D}w,\mathcal{D}w') \\ &= \gamma d(w,w') \\ &< d(w,w') \end{aligned}$$

this contradiction implies that $w' = w$.

Now, we are ready to state and prove our second theorem.

Theorem 4 *Let $(\mathcal{M},d)$ be a b-metric space and let $\mathcal{S}$, $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ be occasionally weakly $\mathcal{T}$-biased maps of type $(\mathcal{A})$, satisfying the following condition*

$$\begin{aligned} (2) \qquad d(\mathcal{S}x,\mathcal{S}y) &\leq \alpha d(\mathcal{T}x,\mathcal{S}x) + \beta d(\mathcal{T}y,\mathcal{S}y) + \gamma d(\mathcal{T}x,\mathcal{S}y) + \delta d(\mathcal{T}y,\mathcal{S}x) \\ &\quad + \lambda d(\mathcal{T}x,\mathcal{T}y) \end{aligned}$$

for all x, $y \in \mathcal{M}$, where α, β, γ, δ, λ are non-negative real numbers such that $2s\alpha + \gamma + \delta + \lambda < 1$ or $2s\beta + \gamma + \delta + \lambda < 1$, then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. As in the proof of the first theorem, since $\mathcal{S}$ and $\mathcal{T}$ are occasionally weakly $\mathcal{T}$-biased of type $(\mathcal{A})$, then, there exists a point $t \in \mathcal{M}$ such that $\mathcal{S}t = \mathcal{T}t$ implies that $d(\mathcal{T}\mathcal{T}t,\mathcal{S}t) \leq d(\mathcal{S}\mathcal{T}t,\mathcal{T}t)$.

First version: We claim that $\mathcal{S}\mathcal{S}t = \mathcal{S}t$. If not, the use of inequality (2) gives

$$\begin{aligned} d(\mathcal{S}\mathcal{S}t, \mathcal{S}t) &\leq \alpha d(\mathcal{T}\mathcal{S}t, \mathcal{S}\mathcal{S}t) + \beta d(\mathcal{T}t, \mathcal{S}t) + \gamma d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) \\ &\quad + \delta d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) + \lambda d(\mathcal{T}\mathcal{S}t, \mathcal{T}t) \\ &= \alpha d(\mathcal{T}\mathcal{S}t, \mathcal{S}\mathcal{S}t) + \gamma d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) + \delta d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) \\ &\quad + \lambda d(\mathcal{T}\mathcal{S}t, \mathcal{T}t) \\ &\leq s\alpha[d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) \\ &\quad + \delta d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) + \lambda d(\mathcal{T}\mathcal{S}t, \mathcal{T}t) \\ &= s\alpha[d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) \\ &\quad + \delta d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) + \lambda d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) \\ &\leq s\alpha[d(\mathcal{S}\mathcal{T}t, \mathcal{T}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{S}\mathcal{T}t, \mathcal{T}t) \\ &\quad + \delta d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) + \lambda d(\mathcal{S}\mathcal{T}t, \mathcal{T}t) \\ &= [2s\alpha + \gamma + \delta + \lambda] d(\mathcal{S}\mathcal{S}t, \mathcal{S}t) \\ &< d(\mathcal{S}\mathcal{S}t, \mathcal{S}t) \end{aligned}$$

a contradiction, hence $\mathcal{S}\mathcal{S}t = \mathcal{S}t$, consequently $\mathcal{T}\mathcal{S}t = \mathcal{S}t$. Setting $\mathcal{S}t = \mathcal{T}t = w$, therefore, w is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Now, suppose there exists another common fixed point say w', using condition (2), we get

$$\begin{aligned} d(w, w') = d(\mathcal{S}w, \mathcal{S}w') &\leq \alpha d(\mathcal{T}w, \mathcal{S}w) + \beta d(\mathcal{T}w', \mathcal{S}w') + \gamma d(\mathcal{T}w, \mathcal{S}w') \\ &\quad + \delta d(\mathcal{T}w', \mathcal{S}w) + \lambda d(\mathcal{T}w, \mathcal{T}w') \\ &= [\gamma + \delta + \lambda] d(w, w') \\ &< d(w, w') \end{aligned}$$

a contradiction, thus, $w' = w$.

Second version: Suppose that $d(\mathcal{S}\mathcal{S}t, \mathcal{S}t)$ is positive, then, by inequality (2), we obtain

$$\begin{aligned} d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) &\leq \alpha d(\mathcal{T}t, \mathcal{S}t) + \beta d(\mathcal{T}\mathcal{S}t, \mathcal{S}\mathcal{S}t) + \gamma d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) \\ &\quad + \delta d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) + \lambda d(\mathcal{T}t, \mathcal{T}\mathcal{S}t) \\ &= \beta d(\mathcal{T}\mathcal{T}t, \mathcal{S}\mathcal{S}t) + \gamma d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) + \delta d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) \\ &\quad + \lambda d(\mathcal{T}t, \mathcal{T}\mathcal{S}t) \\ &\leq s\beta[d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{T}t, \mathcal{S}\mathcal{S}t) \\ &\quad + \delta d(\mathcal{T}\mathcal{S}t, \mathcal{S}t) + \lambda d(\mathcal{T}t, \mathcal{T}\mathcal{S}t) \\ &= s\beta[d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) \\ &\quad + \delta d(\mathcal{T}\mathcal{T}t, \mathcal{S}t) + \lambda d(\mathcal{S}t, \mathcal{T}\mathcal{T}t) \\ &\leq s\beta[d(\mathcal{S}\mathcal{T}t, \mathcal{T}t) + d(\mathcal{S}t, \mathcal{S}\mathcal{S}t)] + \gamma d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) \\ &\quad + \delta d(\mathcal{S}\mathcal{T}t, \mathcal{T}t) + \lambda d(\mathcal{T}t, \mathcal{S}\mathcal{T}t) \\ &= [2s\beta + \gamma + \delta + \lambda] d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) \\ &< d(\mathcal{S}t, \mathcal{S}\mathcal{S}t) \end{aligned}$$

a contradiction, hence $\mathcal{S}\mathcal{S}t = \mathcal{S}t$, consequently $\mathcal{T}\mathcal{S}t = \mathcal{S}t$. Taking $\mathcal{S}t = \mathcal{T}t = w$, therefore, w is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Now, suppose the existence of another common fixed point w', using condition (2), we get

$$\begin{aligned} d(w,w') = d(\mathcal{S}w,\mathcal{S}w') &\leq \alpha d(\mathcal{T}w,\mathcal{S}w) + \beta d(\mathcal{T}w',\mathcal{S}w') + \gamma d(\mathcal{T}w,\mathcal{S}w') \\ &\quad + \delta d(\mathcal{T}w',\mathcal{S}w) + \lambda d(\mathcal{T}w,\mathcal{T}w') \\ &= [\gamma+\delta+\lambda]d(w,w') \\ &< d(w,w') \end{aligned}$$

which is a contradiction, hence $w' = w$.

1.4 Illustrative examples

Example 1 *Equip $\mathcal{X} = (0,+\infty)$ with the b-metric $d(x,y) = (x-y)^2$, where $s = 2$. Taking $\alpha = \beta = \frac{1}{17}$, $\gamma = \frac{3}{4}$ and define*

$$\mathcal{C}x = \begin{cases} 1 \text{ if } x \in (0,1] \\ \frac{1}{4x} \text{ if } x \in (1,+\infty), \end{cases} \quad \mathcal{D}x = \begin{cases} \frac{1}{x} \text{ if } x \in (0,1] \\ 40 \text{ if } x \in (1,+\infty). \end{cases}$$

Before starting, note that $\mathcal{C}$ and $\mathcal{D}$ are occasionally weakly $\mathcal{D}$-biased of type $(\mathcal{A})$. We have

1. *for x, $y \in (0,1]$, $\mathcal{C}x = \mathcal{C}y = 1$, $\mathcal{D}x = \frac{1}{x}$, $\mathcal{D}y = \frac{1}{y}$ and*

$$\begin{aligned} d(\mathcal{C}x,\mathcal{C}y) &= 0 \\ &\leq \frac{1}{17}\left(1-\frac{1}{x}\right)^2 + \frac{1}{17}\left(1-\frac{1}{y}\right)^2 + \frac{3}{4}\left(\frac{1}{x}-\frac{1}{y}\right)^2 \\ &= \alpha d(\mathcal{D}x,\mathcal{C}x) + \beta d(\mathcal{D}y,\mathcal{C}y) + \gamma d(\mathcal{D}x,\mathcal{D}y), \end{aligned}$$

2. *for x, $y \in (1,+\infty)$, $\mathcal{C}x = \frac{1}{4x}$, $\mathcal{C}y = \frac{1}{4y}$, $\mathcal{D}x = \mathcal{D}y = 40$ and*

$$\begin{aligned} d(\mathcal{C}x,\mathcal{C}y) &= \left(\frac{1}{4x}-\frac{1}{4y}\right)^2 \\ &\leq \frac{1}{17}\left(\frac{1}{4x}-40\right)^2 + \frac{1}{17}\left(\frac{1}{4y}-40\right)^2 \\ &= \alpha d(\mathcal{D}x,\mathcal{C}x) + \beta d(\mathcal{D}y,\mathcal{C}y) + \gamma d(\mathcal{D}x,\mathcal{D}y), \end{aligned}$$

3. *for $x \in (0,1]$ and $y \in (1,+\infty)$, $\mathcal{C}x = 1$, $\mathcal{C}y = \frac{1}{4y}$, $\mathcal{D}x = \frac{1}{x}$, $\mathcal{D}y = 40$ and*

$$\begin{aligned} d(\mathcal{C}x,\mathcal{C}y) &= \left(1-\frac{1}{4y}\right)^2 \\ &\leq \frac{1}{17}\left(1-\frac{1}{x}\right)^2 + \frac{1}{17}\left(40-\frac{1}{4y}\right)^2 + \frac{3}{4}\left(40-\frac{1}{x}\right)^2 \\ &= \alpha d(\mathcal{D}x,\mathcal{C}x) + \beta d(\mathcal{D}y,\mathcal{C}y) + \gamma d(\mathcal{D}x,\mathcal{D}y), \end{aligned}$$

4. for $x \in (1, +\infty)$ *and* $y \in (0, 1]$, $\mathcal{C}x = \frac{1}{4x}$, $\mathcal{C}y = 1$, $\mathcal{D}x = 40$, $\mathcal{D}y = \frac{1}{y}$ *and*

$$\begin{aligned} d(\mathcal{C}x, \mathcal{C}y) &= \left(1 - \frac{1}{4x}\right)^2 \\ &\leq \frac{1}{17}\left(40 - \frac{1}{4x}\right)^2 + \frac{1}{17}\left(1 - \frac{1}{y}\right)^2 + \frac{3}{4}\left(40 - \frac{1}{y}\right)^2 \\ &= \alpha d(\mathcal{D}x, \mathcal{C}x) + \beta d(\mathcal{D}y, \mathcal{C}y) + \gamma d(\mathcal{D}x, \mathcal{D}y), \end{aligned}$$

thereby, all hypotheses of the first theorem are satisfied; the two maps have 1 *as the only common fixed point.*

Example 2 *Equip* $\mathcal{X} = (0, +\infty)$ *with the b-metric* $d(x, y) = (x - y)^2$, *where* $s = 2$. *Taking* $\alpha = \beta = \gamma = \delta = \frac{1}{33}$, $\lambda = \frac{3}{4}$ *and define*

$$\mathcal{S}x = \begin{cases} 1 \text{ if } x \in (0, 1] \\ \frac{1}{9x} \text{ if } x \in (1, +\infty), \end{cases} \quad \mathcal{T}x = \begin{cases} \frac{1}{x} \text{ if } x \in (0, 1] \\ 90 \text{ if } x \in (1, +\infty). \end{cases}$$

Before starting, note that $\mathcal{S}$ *and* $\mathcal{T}$ *are occasionally weakly* $\mathcal{T}$*-biased of type* $(\mathcal{A})$. *We have*

1. for $x, y \in (0, 1]$, $\mathcal{S}x = \mathcal{S}y = 1$, $\mathcal{T}x = \frac{1}{x}$, $\mathcal{T}y = \frac{1}{y}$ *and*

$$\begin{aligned} d(\mathcal{S}x, \mathcal{S}y) &= 0 \\ &\leq \frac{2}{33}\left[\left(\frac{1}{x} - 1\right)^2 + \left(\frac{1}{y} - 1\right)^2\right] + \frac{3}{4}\left(\frac{1}{x} - \frac{1}{y}\right)^2 \\ &= \alpha d(\mathcal{T}x, \mathcal{S}x) + \beta d(\mathcal{T}y, \mathcal{S}y) + \gamma d(\mathcal{T}x, \mathcal{S}y) + \delta d(\mathcal{T}y, \mathcal{S}x) \\ &\quad + \lambda d(\mathcal{T}x, \mathcal{T}y), \end{aligned}$$

2. for $x, y \in (1, +\infty)$, $\mathcal{S}x = \frac{1}{9x}$, $\mathcal{S}y = \frac{1}{9y}$, $\mathcal{T}x = \mathcal{T}y = 90$ *and*

$$\begin{aligned} d(\mathcal{S}x, \mathcal{S}y) &= \left(\frac{1}{9x} - \frac{1}{9y}\right)^2 \\ &\leq \frac{2}{33}\left[\left(90 - \frac{1}{9x}\right)^2 + \left(90 - \frac{1}{9y}\right)^2\right] \\ &= \alpha d(\mathcal{T}x, \mathcal{S}x) + \beta d(\mathcal{T}y, \mathcal{S}y) + \gamma d(\mathcal{T}x, \mathcal{S}y) + \delta d(\mathcal{T}y, \mathcal{S}x) \\ &\quad + \lambda d(\mathcal{T}x, \mathcal{T}y), \end{aligned}$$

3. for $x \in (0,1]$ *and* $y \in (1,+\infty)$, $\mathcal{S}x = 1$, $\mathcal{S}y = \frac{1}{9y}$, $\mathcal{T}x = \frac{1}{x}$, $\mathcal{T}y = 90$ *and*

$$\begin{aligned} d(\mathcal{S}x,\mathcal{S}y) &= \left(1-\frac{1}{9y}\right)^2 \\ &\leq \frac{1}{33}\left(1-\frac{1}{x}\right)^2+\frac{1}{33}\left(90-\frac{1}{9y}\right)^2+\frac{1}{33}\left(\frac{1}{x}-\frac{1}{9y}\right)^2 \\ &\quad +\frac{1}{33}(89)^2+\frac{3}{4}\left(90-\frac{1}{x}\right)^2 \\ &= \alpha d(\mathcal{T}x,\mathcal{S}x)+\beta d(\mathcal{T}y,\mathcal{S}y)+\gamma d(\mathcal{T}x,\mathcal{S}y)+\delta d(\mathcal{T}y,\mathcal{S}x) \\ &\quad +\lambda d(\mathcal{T}x,\mathcal{T}y), \end{aligned}$$

4. for $x \in (1,+\infty)$ *and* $y \in (0,1]$, $\mathcal{S}x = \frac{1}{9x}$, $\mathcal{S}y = 1$, $\mathcal{T}x = 90$, $\mathcal{T}y = \frac{1}{y}$ *and*

$$\begin{aligned} d(\mathcal{S}x,\mathcal{S}y) &= \left(1-\frac{1}{9x}\right)^2 \\ &\leq \frac{1}{33}\left(90-\frac{1}{9x}\right)^2+\frac{1}{33}\left(1-\frac{1}{y}\right)^2+\frac{1}{33}(89)^2 \\ &\quad +\frac{1}{33}\left(\frac{1}{y}-\frac{1}{9x}\right)^2+\frac{3}{4}\left(90-\frac{1}{y}\right)^2 \\ &= \alpha d(\mathcal{T}x,\mathcal{S}x)+\beta d(\mathcal{T}y,\mathcal{S}y)+\gamma d(\mathcal{T}x,\mathcal{S}y)+\delta d(\mathcal{T}y,\mathcal{S}x) \\ &\quad +\lambda d(\mathcal{T}x,\mathcal{T}y), \end{aligned}$$

thereby, all hypotheses of the second theorem are fulfilled; the two maps possess 1 *as the only common fixed point.*

2 Existence and uniqueness of common fixed points in b-metric-like spaces

In this second section, we will improve and extend Theorem 5 and Theorem 6 of Shah [7] using our new notion of occasionally weakly biased maps of type $(\mathcal{A})$ in a b-metric-like space.

2.1 Introduction

In 2013, Alghamdi et al. [1] suggested a recent generalization of metric domains and b-metric spaces named a b-metric-like space. Thereafter, they provided several results in these current spaces. According to them, their theorems generalize and ameliorate many prominent results in fixed point domain.

2.2 Preliminaries

Definition 4 *([1]) A b-metric-like on a nonempty set* $\mathbf{X}$ *is a function* $\mathcal{D} : \mathbf{X} \times \mathbf{X} \to [0, +\infty)$ *such that for all* r, s, $t \in \mathbf{X}$ *and a constant* $\mathcal{K} \geq 1$ *the following three conditions hold true:*

1. $(\mathcal{D}1)$ *if* $\mathcal{D}(r, s) = 0 \Rightarrow r = s$;
2. $(\mathcal{D}2)$ $\mathcal{D}(r, s) = \mathcal{D}(s, r)$;
3. $(\mathcal{D}3)$ $\mathcal{D}(r, s) \leq \mathcal{K}[\mathcal{D}(r, t) + \mathcal{D}(t, s)]$.

The pair $(\mathbf{X}, \mathcal{D})$ *is called a* ***b-metric-like space****.*

Definition 5 *([3]) Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be maps from a b-metric-like space into itself.* $\mathcal{S}$ *and* $\mathcal{T}$ *are* ***occasionally weakly*** $\mathcal{S}$***-biased of type*** $(\mathcal{A})$ *and* ***occasionally weakly*** $\mathcal{T}$***-biased of type*** $(\mathcal{A})$*, respectively, iff, there is an element* $\iota \in \mathcal{X}$ *such that* $\mathcal{S}\iota = \mathcal{T}\iota$ *implies*

$$d(\mathcal{S}\mathcal{S}\iota, \mathcal{T}\iota) \leq d(\mathcal{T}\mathcal{S}\iota, \mathcal{S}\iota),$$
$$d(\mathcal{T}\mathcal{T}\iota, \mathcal{S}\iota) \leq d(\mathcal{S}\mathcal{T}\iota, \mathcal{T}\iota),$$

respectively.

In his paper [7], Shah proved the below results.

Theorem 5 *Let* $(\mathcal{X}, d)$ *be a complete b-metric-like space with* $b \geq 1$ *be a constant. Let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a surjection such that,*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \quad \geq \quad & a_1 d(\xi, \eta) + a_2 d(\xi, \mathcal{T}\xi) + a_3 d(\eta, \mathcal{T}\eta) + a_4 d(\xi, \mathcal{T}\eta) + a_5 d(\eta, \mathcal{T}\xi) \\ & + a_6 d(\xi, \xi) + a_7 d(\eta, \eta), \end{aligned}$$

for all ξ, $\eta \in \mathcal{X}$, a_1, a_2, a_3, a_4, a_5, a_6, $a_7 \geq 0$, *satisfy* $b(a_1 + a_2 + a_5 + 2a_6 + a_7) + a_4 + b^2(a_3 - a_4 + a_5 + a_7) > b^2$ *and* $1 - a_3 + a_4 - a_5 - a_7 > 0$, *then* $\mathcal{T}$ *has a fixed point.*

Theorem 6 *Let* $(\mathcal{X}, d)$ *be a complete b-metric-like space with* $b \geq 1$ *be a constant. Let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a surjection such that,*

$$\begin{aligned} d(\mathcal{T}\xi, \mathcal{T}\eta) \quad \geq \quad & a_1 d(\xi, \eta) + a_2[d(\xi, \mathcal{T}\xi) + d(\eta, \mathcal{T}\eta)] + a_3[d(\xi, \mathcal{T}\eta) + d(\eta, \mathcal{T}\xi)] \\ & + a_4[d(\xi, \mathcal{T}\xi) + d(\xi, \eta)] + a_5[d(\xi, \mathcal{T}\eta) + d(\xi, \eta)], \end{aligned}$$

for all ξ, $\eta \in \mathcal{X}$, a_1, a_2, a_3, a_4, a_5, a_6, $a_7 \geq 0$, *satisfy* $b(a_1 + a_2 + a_3 + 2a_4 + a_5) + a_3 + b^2(a_2 + a_5) > b^2$ *and* $1 - (a_2 + a_5) > 0$, *then* $\mathcal{T}$ *has a fixed point.*

We will improve the above theorems by removing the completeness and the surjection, and we will extend them by increasing the number of maps, of course, using our new concept of occasionally weakly biased maps of type $(\mathcal{A})$.

2.3 Unique common fixed points for two maps

Theorem 7 *Let $(\mathcal{X}, d)$ be a b-metric-like space with $s \geq 1$ be a constant. Let $\mathcal{S}$, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be occasionally weakly $\mathcal{S}$-biased maps of type $(\mathcal{A})$, satisfying the following inequality*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) \geq\ & a_1 d(\mathcal{S}x, \mathcal{S}y) + a_2 d(\mathcal{S}x, \mathcal{T}x) + a_3 d(\mathcal{S}y, \mathcal{T}y) \\ & + a_4 d(\mathcal{S}x, \mathcal{T}y) + a_5 d(\mathcal{S}y, \mathcal{T}x) + a_6 d(\mathcal{S}x, \mathcal{S}x) \\ & + a_7 d(\mathcal{S}y, \mathcal{S}y), \end{aligned} \tag{3}$$

for all x, $y \in \mathcal{X}$, a_1, a_2, a_3, a_4, a_5, a_6, $a_7 \geq 0$, satisfy $[1 - (a_1 + a_4 + a_5) - 2s(a_2 + a_3 + a_6 + a_7)] < 0$, then, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. By hypotheses, there exists an element e in $\mathcal{X}$ such that $\mathcal{S}e = \mathcal{T}e$ implies $d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) \leq d(\mathcal{T}\mathcal{S}e, \mathcal{S}e)$.

Suppose that $\mathcal{T}\mathcal{T}e \neq \mathcal{T}e$, then, inequality (3) gives

$$\begin{aligned} d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) \geq\ & a_1 d(\mathcal{S}\mathcal{T}e, \mathcal{S}e) + a_2 d(\mathcal{S}\mathcal{T}e, \mathcal{T}\mathcal{T}e) + a_3 d(\mathcal{S}e, \mathcal{T}e) + a_4 d(\mathcal{S}\mathcal{T}e, \mathcal{T}e) \\ & + a_5 d(\mathcal{S}e, \mathcal{T}\mathcal{T}e) + a_6 d(\mathcal{S}\mathcal{T}e, \mathcal{S}\mathcal{T}e) + a_7 d(\mathcal{S}e, \mathcal{S}e) \\ =\ & a_1 d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + a_2 d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) + a_3 d(\mathcal{T}e, \mathcal{T}e) + a_4 d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) \\ & + a_5 d(\mathcal{T}e, \mathcal{T}\mathcal{T}e) + a_6 d(\mathcal{S}\mathcal{S}e, \mathcal{S}\mathcal{S}e) + a_7 d(\mathcal{T}e, \mathcal{T}e) \\ =\ & [a_1 + a_4] d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + a_2 d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) \\ & + [a_3 + a_7] d(\mathcal{T}e, \mathcal{T}e) + a_5 d(\mathcal{T}e, \mathcal{T}\mathcal{T}e) + a_6 d(\mathcal{S}\mathcal{S}e, \mathcal{S}\mathcal{S}e). \end{aligned}$$

As we have

$$\begin{aligned} [a_1 + a_4] d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) &\geq [a_1 + a_4] d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) \\ 2s a_2 d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) &\geq a_2 d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) \\ 2s[a_3 + a_7] d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) &\geq [a_3 + a_7] d(\mathcal{T}e, \mathcal{T}e) \\ 2s a_6 d(\mathcal{T}e, \mathcal{T}\mathcal{T}e) &\geq a_6 d(\mathcal{S}\mathcal{S}e, \mathcal{S}\mathcal{S}e) \end{aligned}$$

then

$$[1 - (a_1 + a_4 + a_5) - 2s(a_2 + a_3 + a_6 + a_7)] d(\mathcal{T}e, \mathcal{T}\mathcal{T}e) \geq 0$$

hence, $d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) = 0 \Rightarrow \mathcal{T}\mathcal{T}e = \mathcal{T}e$, consequently, $d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) = 0 \Rightarrow \mathcal{S}\mathcal{T}e = \mathcal{T}e$, therefore, $\mathcal{T}e = \mathcal{S}e = w$ is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Now, suppose there is another common fixed point (say w'), then, using condition (3), we get

$$\begin{aligned} d(w, w') = d(\mathcal{T}w, \mathcal{T}w') \geq\ & a_1 d(\mathcal{S}w, \mathcal{S}w') + a_2 d(\mathcal{S}w, \mathcal{T}w) + a_3 d(\mathcal{S}w', \mathcal{T}w') \\ & + a_4 d(\mathcal{S}w, \mathcal{T}w') + a_5 d(\mathcal{S}w', \mathcal{T}w) + a_6 d(\mathcal{S}w, \mathcal{S}w) \\ & + a_7 d(\mathcal{S}w', \mathcal{S}w') \\ =\ & [a_1 + a_4 + a_5] d(w, w') + [a_2 + a_6] d(w, w) \\ & + [a_3 + a_7] d(w', w'). \end{aligned}$$

As we have

$$\begin{aligned} 2s[a_2+a_6]d(w,w') &\geq [a_2+a_6]d(w,w) \\ 2s[a_3+a_7]d(w,w') &\geq [a_3+a_7]d(w',w') \end{aligned}$$

then

$$[1-(a_1+a_4+a_5)-2s(a_2+a_3+a_6+a_7)]d(w,w') \geq 0$$

hence $w'=w$.

Example 3 *Equip $\mathcal{X}=(-1,10)$ with the b-metric-like $d(x,y)=(|x|+|y|)^2$, where $s=2$. Taking $a_1=\ldots=a_7=\frac{1}{18}$ and define*

$$\mathcal{S}x=\begin{cases} 0 \text{ if } x\in(-1,0] \\ \frac{x-10}{100} \text{ if } x\in(0,10), \end{cases} \quad \mathcal{T}x=\begin{cases} x \text{ if } x\in(-1,0] \\ 9 \text{ if } x\in(0,10). \end{cases}$$

Before starting, note that $\mathcal{S}$ and $\mathcal{T}$ are occasionally weakly $\mathcal{S}$-biased of type $(\mathcal{A})$. We have

1. *for $x,\ y\in(-1,0]$, $\mathcal{S}x=\mathcal{S}y=0$, $\mathcal{T}x=x$, $\mathcal{T}y=y$ and*

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &= (|x|+|y|)^2 \\ &\geq \frac{1}{9}\left[x^2+y^2\right] \\ &= a_1d(\mathcal{S}x,\mathcal{S}y)+a_2d(\mathcal{S}x,\mathcal{T}x)+a_3d(\mathcal{S}y,\mathcal{T}y)+a_4d(\mathcal{S}x,\mathcal{T}y) \\ &\quad +a_5d(\mathcal{S}y,\mathcal{T}x)+a_6d(\mathcal{S}x,\mathcal{S}x)+a_7d(\mathcal{S}y,\mathcal{S}y), \end{aligned}$$

2. *for $x,\ y\in(0,10)$, $\mathcal{S}x=\frac{x-10}{100}$, $\mathcal{S}y=\frac{y-10}{100}$, $\mathcal{T}x=\mathcal{T}y=9$ and*

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &= (18)^2 \\ &\geq \frac{1}{18}\left(\left|\frac{x-10}{100}\right|+\left|\frac{y-10}{100}\right|\right)^2 \\ &\quad +\frac{1}{9}\left[\left(\left|\frac{x-10}{100}\right|+9\right)^2+\left(\left|\frac{y-10}{100}\right|+9\right)^2\right] \\ &\quad +\frac{2}{9}\left(\left|\frac{x-10}{100}\right|^2+\left|\frac{y-10}{100}\right|^2\right) \\ &= a_1d(\mathcal{S}x,\mathcal{S}y)+a_2d(\mathcal{S}x,\mathcal{T}x)+a_3d(\mathcal{S}y,\mathcal{T}y)+a_4d(\mathcal{S}x,\mathcal{T}y) \\ &\quad +a_5d(\mathcal{S}y,\mathcal{T}x)+a_6d(\mathcal{S}x,\mathcal{S}x)+a_7d(\mathcal{S}y,\mathcal{S}y), \end{aligned}$$

3. for $x \in (-1,0]$ *and* $y \in (0,10)$, $\mathcal{S}x = 0$, $\mathcal{S}y = \frac{y-10}{100}$, $\mathcal{T}x = x$, $\mathcal{T}y = 9$ *and*

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &= (|x|+9)^2 \\ &\geq \frac{1}{18}\left(\left|\frac{y-10}{100}\right|\right)^2 + \frac{1}{18}(|x|)^2 + \frac{1}{18}\left(9+\left|\frac{y-10}{100}\right|\right)^2 \\ &\quad +\frac{1}{18}(81) + \frac{1}{18}\left(x+\left|\frac{y-10}{100}\right|\right)^2 + \frac{2}{9}\left(\left|\frac{y-10}{100}\right|\right)^2 \\ &= a_1 d(\mathcal{S}x,\mathcal{S}y) + a_2 d(\mathcal{S}x,\mathcal{T}x) + a_3 d(\mathcal{S}y,\mathcal{T}y) + a_4 d(\mathcal{S}x,\mathcal{T}y) \\ &\quad +a_5 d(\mathcal{S}y,\mathcal{T}x) + a_6 d(\mathcal{S}x,\mathcal{S}x) + a_7 d(\mathcal{S}y,\mathcal{S}y), \end{aligned}$$

4. for $x \in (0,10)$ *and* $y \in (-1,0]$, $\mathcal{S}x = \frac{x-10}{100}$, $\mathcal{S}y = 0$, $\mathcal{T}x = 9$, $\mathcal{T}y = y$ *and*

$$\begin{aligned} d(\mathcal{T}x,\mathcal{T}y) &= (9+|y|)^2 \\ &\geq \frac{1}{18}\left(\left|\frac{x-10}{100}\right|\right)^2 + \frac{1}{18}\left(9+\left|\frac{x-10}{100}\right|\right)^2 + \frac{1}{18}(|y|)^2 \\ &\quad +\frac{1}{18}\left(\left|\frac{x-10}{100}\right|+y\right)^2 + \frac{1}{18}(81) + \frac{2}{9}\left(\left|\frac{x-10}{100}\right|\right)^2 \\ &= a_1 d(\mathcal{S}x,\mathcal{S}y) + a_2 d(\mathcal{S}x,\mathcal{T}x) + a_3 d(\mathcal{S}y,\mathcal{T}y) + a_4 d(\mathcal{S}x,\mathcal{T}y) \\ &\quad +a_5 d(\mathcal{S}y,\mathcal{T}x) + a_6 d(\mathcal{S}x,\mathcal{S}x) + a_7 d(\mathcal{S}y,\mathcal{S}y), \end{aligned}$$

thereby, all hypotheses of the above theorem are satisfied; the two maps admit 0 *as the only common fixed point.*

Theorem 8 *Let* $(\mathcal{X},d)$ *be a b-metric-like space with* $s \geq 1$ *be a constant. Let* $\mathcal{T}$, $\mathcal{S}:\mathcal{X}\to\mathcal{X}$ *be occasionally weakly* $\mathcal{S}$*-biased maps of type* $(\mathcal{A})$*, such that,*

$$\begin{aligned} (4)\quad d(\mathcal{T}x,\mathcal{T}y) &\geq a_1 d(\mathcal{S}x,\mathcal{S}y) + a_2[d(\mathcal{S}x,\mathcal{T}x) + d(\mathcal{S}y,\mathcal{T}y)] \\ &\quad +a_3[d(\mathcal{S}x,\mathcal{T}y) + d(\mathcal{S}y,\mathcal{T}x)] + a_4[d(\mathcal{S}x,\mathcal{T}x) + d(\mathcal{S}x,\mathcal{S}y)] \\ &\quad +a_5[d(\mathcal{S}x,\mathcal{T}y) + d(\mathcal{S}x,\mathcal{S}y)], \end{aligned}$$

for all x, $y \in \mathcal{X}$, a_1, a_2, a_3, a_4, $a_5 \geq 0$, *satisfy* $1 < a_1+a_4+2(a_3+a_5)+2s(2a_2+a_4)$, *then,* $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ are occasionally weakly $\mathcal{S}$-biased of type $(\mathcal{A})$, then, there exists an element e in $\mathcal{X}$ such that $\mathcal{S}e = \mathcal{T}e$ implies $d(\mathcal{S}\mathcal{S}e,\mathcal{T}e) \leq d(\mathcal{T}\mathcal{S}e,\mathcal{S}e)$.

Assume that $d(\mathcal{T}\mathcal{T}e, \mathcal{T}e)$ is positive, then, inequality (4) gives

$$\begin{aligned} d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) &\geq a_1 d(\mathcal{S}\mathcal{T}e, \mathcal{S}e) + a_2[d(\mathcal{S}\mathcal{T}e, \mathcal{T}\mathcal{T}e) + d(\mathcal{S}e, \mathcal{T}e)] \\ &\quad + a_3[d(\mathcal{S}\mathcal{T}e, \mathcal{T}e) + d(\mathcal{S}e, \mathcal{T}\mathcal{T}e)] \\ &\quad + a_4[d(\mathcal{S}\mathcal{T}e, \mathcal{T}\mathcal{T}e) + d(\mathcal{S}\mathcal{T}e, \mathcal{S}e)] \\ &\quad + a_5[d(\mathcal{S}\mathcal{T}e, \mathcal{T}e) + d(\mathcal{S}\mathcal{T}e, \mathcal{S}e)] \\ &= a_1 d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + a_2[d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) + d(\mathcal{T}e, \mathcal{T}e)] \\ &\quad + a_3[d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + d(\mathcal{T}e, \mathcal{T}\mathcal{T}e)] \\ &\quad + a_4[d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) + d(\mathcal{S}\mathcal{S}e, \mathcal{T}e)] \\ &\quad + a_5[d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + d(\mathcal{S}\mathcal{S}e, \mathcal{T}e)] \\ &= [a_1 + a_3 + a_4 + 2a_5]d(\mathcal{S}\mathcal{S}e, \mathcal{T}e) + [a_2 + a_4]d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e) \\ &\quad + a_2 d(\mathcal{T}e, \mathcal{T}e) + a_3 d(\mathcal{T}e, \mathcal{T}\mathcal{T}e). \end{aligned}$$

As we have

$$[a_1 + a_3 + a_4 + 2a_5]d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) \geq [a_1 + a_3 + a_4 + 2a_5]d(\mathcal{S}\mathcal{S}e, \mathcal{T}e)$$

$$2s[a_2 + a_4]d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) \geq [a_2 + a_4]d(\mathcal{S}\mathcal{S}e, \mathcal{T}\mathcal{T}e)$$

$$2sa_2 d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) \geq a_2 d(\mathcal{T}e, \mathcal{T}e)$$

by substraction, we obtain

$$[1 - a_1 - a_4 - 2(a_3 + a_5) - 2s(2a_2 + a_4)]d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) \geq 0$$

a contradiction, hence, $d(\mathcal{T}\mathcal{T}e, \mathcal{T}e) = 0$ which implies that $\mathcal{T}\mathcal{T}e = \mathcal{T}e$, consequently, $\mathcal{S}\mathcal{T}e = \mathcal{T}e$.

Now, suppose there exists another common fixed point (say e'). By condition (4) we get

$$\begin{aligned} d(e, e') = d(\mathcal{T}e, \mathcal{T}e') &\geq a_1 d(\mathcal{S}e, \mathcal{S}e') + a_2[d(\mathcal{S}e, \mathcal{T}e) + d(\mathcal{S}e', \mathcal{T}e')] \\ &\quad + a_3[d(\mathcal{S}e, \mathcal{T}e') + d(\mathcal{S}e', \mathcal{T}e)] \\ &\quad + a_4[d(\mathcal{S}e, \mathcal{T}e) + d(\mathcal{S}e, \mathcal{S}e')] \\ &\quad + a_5[d(\mathcal{S}e, \mathcal{T}e') + d(\mathcal{S}e, \mathcal{S}e')] \\ &= a_1 d(e, e') + a_2[d(e, e) + d(e', e')] \\ &\quad + a_3[d(e, e') + d(e', e)] \\ &\quad + a_4[d(e, e) + d(e, e')] + a_5[d(e, e') + d(e, e')] \\ &= [a_1 + 2a_3 + a_4 + 2a_5]d(e, e') + [a_2 + a_4]d(e, e) \\ &\quad + a_2 d(e', e'). \end{aligned}$$

As we have

$$\begin{aligned} 2s[a_2 + a_4]d(e, e') &\geq [a_2 + a_4]d(e, e) \\ 2sa_2 d(e, e') &\geq a_2 d(e', e') \end{aligned}$$

by substraction, we get

$$[1 - a_1 - 2a_3 - a_4 - 2a_5 - 2s(2a_2 + a_4)]d(e, e') \geq 0$$

a contradiction, hence, $e' = e$.

Example 4 *Equip $\mathcal{X} = (-1, 10)$ with the b-metric-like $d(x, y) = (|x| + |y|)^2$, where $s = 2$. Taking $a_1 = \ldots = a_5 = \frac{1}{17}$ and define*

$$\mathcal{S}x = \begin{cases} 0 \text{ if } x \in (-1, 0] \\ \frac{x-10}{200} \text{ if } x \in (0, 10), \end{cases} \quad \mathcal{T}x = \begin{cases} -2x \text{ if } x \in (-1, 0] \\ 9 \text{ if } x \in (0, 10). \end{cases}$$

Before starting, note that $\mathcal{S}$ and $\mathcal{T}$ are occasionally weakly $\mathcal{S}$-biased of type $(\mathcal{A})$. We have

1. *for x, $y \in (-1, 0]$, $\mathcal{S}x = \mathcal{S}y = 0$, $\mathcal{T}x = -2x$, $\mathcal{T}y = -2y$ and*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= (|-2x| + |-2y|)^2 \\ &\geq \frac{3}{17}[|-2x|^2 + |-2y|^2] \\ &= a_1 d(\mathcal{S}x, \mathcal{S}y) + a_2[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}y, \mathcal{T}y)] \\ &\quad + a_3[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)] \\ &\quad + a_4[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)] \\ &\quad + a_5[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{S}y)], \end{aligned}$$

2. *for x, $y \in (0, 10)$, $\mathcal{S}x = \frac{x-10}{200}$, $\mathcal{S}y = \frac{y-10}{200}$, $\mathcal{T}x = \mathcal{T}y = 9$ and*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= 324 \\ &\geq \frac{1}{17}\left[\left|\frac{x-10}{200}\right| + \left|\frac{y-10}{200}\right|\right]^2 \\ &\quad + \frac{2}{17}\left[\left(\left|\frac{x-10}{200}\right| + 9\right)^2 + \left(\left|\frac{y-10}{200}\right| + 9\right)^2\right] \\ &\quad + \frac{2}{17}\left[\left(\left|\frac{x-10}{200}\right| + 9\right)^2 + \left(\left|\frac{x-10}{200}\right| + \left|\frac{y-10}{200}\right|\right)^2\right] \\ &= a_1 d(\mathcal{S}x, \mathcal{S}y) + a_2[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}y, \mathcal{T}y)] \\ &\quad + a_3[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)] \\ &\quad + a_4[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)] \\ &\quad + a_5[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{S}y)], \end{aligned}$$

3. for $x \in (-1, 0]$ *and* $y \in (0, 10)$, $\mathcal{S}x = 0$, $\mathcal{S}y = \frac{y-10}{200}$, $\mathcal{T}x = -2x$, $\mathcal{T}y = 9$ *and*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= (|-2x| + 9)^2 \\ &\geq \frac{1}{17}\left(\left|\frac{y-10}{200}\right|\right)^2 + \frac{1}{17}\left[|-2x|^2 + \left(9 + \left|\frac{y-10}{200}\right|\right)^2\right] \\ &\quad + \frac{1}{17}\left[81 + \left(|-2x| + \left|\frac{y-10}{200}\right|\right)^2\right] \\ &\quad + \frac{1}{17}\left[|-2x|^2 + \left(\left|\frac{y-10}{200}\right|\right)^2\right] \\ &\quad + \frac{1}{17}\left[81 + \left(\left|\frac{y-10}{200}\right|\right)^2\right] \\ &= a_1 d(\mathcal{S}x, \mathcal{S}y) + a_2[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}y, \mathcal{T}y)] \\ &\quad + a_3[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)] \\ &\quad + a_4[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)] \\ &\quad + a_5[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{S}y)], \end{aligned}$$

4. for $x \in (0, 10)$ *and* $y \in (-1, 0]$, $\mathcal{S}x = \frac{x-10}{200}$, $\mathcal{S}y = 0$, $\mathcal{T}x = 9$, $\mathcal{T}y = -2y$ *and*

$$\begin{aligned} d(\mathcal{T}x, \mathcal{T}y) &= (9 + |-2y|)^2 \\ &\geq \frac{1}{17}\left(\left|\frac{x-10}{200}\right|\right)^2 + \frac{1}{17}\left[\left(9 + \left|\frac{x-10}{200}\right|\right)^2 + (|-2y|)^2\right] \\ &\quad + \frac{1}{17}\left[\left(\left|\frac{x-10}{200}\right| + |-2y|\right)^2 + 81\right] \\ &\quad + \frac{1}{17}\left[\left(\left|\frac{x-10}{200}\right| + 9\right)^2 + \left(\left|\frac{x-10}{200}\right|\right)^2\right] \\ &\quad + \frac{1}{17}\left[\left(\left|\frac{x-10}{200}\right| + |-2y|\right)^2 + \left(\left|\frac{x-10}{200}\right|\right)^2\right] \\ &= a_1 d(\mathcal{S}x, \mathcal{S}y) + a_2[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}y, \mathcal{T}y)] \\ &\quad + a_3[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}y, \mathcal{T}x)] \\ &\quad + a_4[d(\mathcal{S}x, \mathcal{T}x) + d(\mathcal{S}x, \mathcal{S}y)] \\ &\quad + a_5[d(\mathcal{S}x, \mathcal{T}y) + d(\mathcal{S}x, \mathcal{S}y)], \end{aligned}$$

thereby, all hypotheses of the above theorem are fulfilled; the two maps admit 0 *as the only common fixed point.*

References

[1] M. A. Alghamdi, N. Hussain, P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Inequal. Appl., 2013, 2013:402, 1-25.

[2] I. A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Functional Analysis, Unianowsk Gos. Ped. Inst., vol. 30, 1989, 26-37.

[3] H. Bouhadjera, *Fixed points for occasionally weakly biased mappings of type* $(\mathcal{A})$, Math. Morav., vol. 26, no. 1, 2022, 113-122.

[4] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., vol. 1, 1993, 5-11.

[5] S. G. Matthews, *Metric domains for completness*, Technical report 76, department of computer science, University of Warwick, UK, April 86. Ph.D. Thesis, 1985.

[6] P. K. Mishra, S. Sachdeva, S. K. Banerjee, *Some fixed point theorems in b-metric space*, Turkish Journal of Analysis and Number Theory, vol. 2, no. 1, 2014, 19-22.

[7] R. Shah, *Some new fixed point results in b-metric-like spaces*, Palestine Journal of Mathematics, vol. 11, no. 1, 2022, 378-384.

Hakima Bouhadjera, Ryme Zouaimia
Badji Mokhtar-Annaba University
Faculty of Sciences
Department of Mathematics
Laboratory of Applied Mathematics
P.O. Box 12, 23000 Annaba, Algeria
e-mail: b_hakima2000@yahoo.fr
e-mail: zouaimiaryme@gmail.com

Aya Dari
Chadli Ben Djedid University
Faculty of Sciences and Technology
Department of Mathematics
El-Tarf, Algeria
e-mail: ayadari17@gmail.com