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Various fractional-order type operators and some of their implications to certain normalized functions analytic in the open unit disc[1]

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Abstract

In this scientific research note, certain necessary-basic information in relation to some types of the operators specified by fractional calculus of arbitrary order will be firstly introduced, various argument properties of certain analytic functions specified by those fractional-order type operators will be then determined, and a number of special consequences of those extensive properties will be also pointed out.

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1 Introduction and main information

As it is known, in terms of science and technology, the familiar fractional calculations of arbitrary order are a very comprehensive scientific research field in terms of both theory and applications. Moreover, most particularly, those fractional-order calculations, *which* have a special place in the science of mathematics, are a different branch of science that is frequently taken into account by mathematical scientists and maintains its continuity. As various basic sources associating with both the

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mentioned fractional calculations of arbitrary order and related operators specified by certain types of the fractional-order calculus, one can also focus on the references given in [2], [13], [16] and [18].

The universal notations like:

$$\mathbb{C} \ , \ \mathbb{R} \ , \ \mathbb{Z} \ , \ \mathbb{N} \text{ and } \mathbb{U}$$

are well known symbols, *which* have been frequently used by mathematicians. As it is well-known, those are also denoted the set of complex numbers, the set of real numbers, the set of integers, the set of natural numbers and the open unit disc of the complex plane, *i.e.,* the set:

$$\mathbb{U} := \Big\{ w \, : \, w \in \mathbb{C} \text{ and } |w| < 1 \Big\},$$

respectively.

Moreover, let any function with the complex variable z like $\eta(z)$ be of the series expansion in the following forms:

$$\eta(z) = z + c_{k+1} z^{k+1} + c_{k+2} z^{k+2} + \cdots \quad \big(c_{k+1} \in \mathbb{C} \ ; \ k \in \mathbb{N} \big), \tag{1}$$

which are (normalized) analytic in the domain $\mathbb{U}$ and also let a complex function like $\rho(z)$ be of the series expansion in the following forms:

$$\rho(z) = e + e_k z^k + e_{k+1} z^{k+1} + \cdots \quad \big(e \in \mathbb{C} - \{0\} \ ; \ e_k \in \mathbb{C} \ ; \ k \in \mathbb{N} \big), \tag{2}$$

which are analytic in $\mathbb{U}$.

Next, for any function like $\phi := \phi(z)$ analytic in any simple-connected region of the complex plane, let the operator:

$$\mathbf{D}_z^{\omega}[\phi] \equiv \mathbf{D}_z^{\omega}[\phi(z)]$$

define in the differ-integral form:

$$\mathbf{D}_z^{\omega}[\phi] = \begin{cases} \frac{1}{\Gamma(1-\omega)} \frac{d}{dz} \int_0^z \frac{\phi(q)}{(z-q)^{\omega}} \, dq & if \ \ 0 \leq \omega < 1 \\ \frac{d^s}{dz^s} \Big(\mathbf{D}_z^{\omega - s}[\phi] \Big) & if \ \ 0 \leq \omega - s < 1 \end{cases}, \tag{3}$$

where $s \in \mathbb{N}$ and the multiplicity of $(z-q)^{-\omega}$ above is removed requiring $log(z-q)$ to be real *when* $z - q > 0$.

For any analytic function $\phi := \phi(z)$, the mentioned operator $\mathbf{D}_z^{\omega}[\phi]$ is also known as the fractional operator of order ω $(0 \leq \omega < 1)$ in the mathematical literature. Moreover, by making use of the this integral-type operator, for that function $\phi := \phi(z)$, by the notation $\mathbf{D}_z^{s+\omega}[\phi]$, the fractional derivative operator of order $s + \omega$ is also defined as

$$\mathbf{D}_z^{s+\omega}[\phi] = \frac{d^s}{dz^s} \Big(\mathbf{D}_z^{\omega}[\phi] \Big) \quad \big(0 \leq \omega < 1; s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \big) \, . \tag{4}$$

In the same time, we also note here that, for any analytic function like $\phi := \phi(z)$, the operator:

$$\mathbf{D}_z^{s+\omega}[\phi] \equiv \mathbf{D}_z^{s+\omega}[\phi(z)]$$

is also encountered as the Srivastava-Owa fractional derivative operator of order $s+\omega$ $(0 \leq \omega < 1; s \in \mathbb{N}_0)$ in the mathematical literature. For more information associating with the details of both differ-integral operators define by (3) together with (4), one may refer to the main works in [13], [16] and [18], and, for various examples, it can be also checked the results in the (recent) studies presented by [1], [7] and [10].

Now, more recently, we encounter a new operator known as Tremblay fractional derivative operator in the mathematical literature, *which* was firstly seen in [19]. It is also defined by the help of the fractional-order derivative operator given in (4) (*or*, in (3)). For an analytic function like $\phi := \phi(z)$, if we denote it by the notation:

$$\mathbf{T}_z^{t,\ell}[\phi] \equiv \mathbf{T}_z^{t,\ell}[\phi(z)],$$

it is then defined by

$$\mathbf{T}_z^{t,\ell}[\phi] = \frac{\Gamma(\ell)}{\Gamma(t)} z^{1-\ell}\, \mathbf{D}_z^{t-\ell}\left[z^{t-1}\phi(z)\right], \tag{5}$$

where

$$0 < t \leq 1 \quad , \quad 0 < \ell \leq 1 \quad , \quad 0 \leq t-\ell < 1 \quad \text{and} \quad z \in \mathbb{U}. \tag{6}$$

In particular, under the stated conditions just above, for any function $\phi := \phi(z)$ analytic in the disc $\mathbb{U}$, it can be also received that there are several comprehensive relationships between the mentioned operators:

$$\mathbf{D}_z^{s+\omega}[\phi] \equiv \mathbf{D}_z^{s+\omega}[\phi(z)] \quad \text{and} \quad \mathbf{T}_z^{t,\ell}[\phi] \equiv \mathbf{T}_z^{t,\ell}[\phi(z)].$$

Those extensive relations will be important for our investigation and their possible implications. For both various earlier results and/or also extra information relating to both fractional-order (type) operators, it can be consulted in the references in [4], [5], [8], [9], [11], [14], [17] and [20].

As it has been put emphasis on the abstract of this paper, the primary purpose of this scientific note is firstly to determine various argument properties of the normalized analytic function being of any form given by (1), *which* can be specified by the help of the mentioned operators given by (3)-(5), and then to find out a number of more special forms of those argument results, *which* will be also related to many special assertions in relation with Analytic-Geometric Function Theory (cf., e.g., [3, 6, 15]).

Now, for both the establishment and proofs of the indicated-main results and also a number of their implications, let us move on to the second section.

2 A few results and related implications

The following-important assertion will play an important role in our fundamental results. For its details, the main article given by [12] in the references can be also reviewed.

Lemma 1. Let an analytic function $\Phi(z)$ be of the form given by (2) (with $e := 1$). If there exists a point $w_0 \in \mathbb{U}$ such that

$$\Re e\Big(\Phi(z)\Big) > 0 \quad for \quad |z| < |w_0| < 1 \tag{7}$$

and

$$\Re e\Big(\Phi\big(w_0\big)\Big) = 0 \tag{8}$$

and

$$\Phi\big(w_0\big) \neq 0\,, \tag{9}$$

then

$$\frac{z\Phi'(z)}{\Phi(z)}\bigg|_{z:=w_0} = i\beta\,, \tag{10}$$

where $\beta \in \mathbb{R}^* := \mathbb{R} - 0$ with $|\beta| \geq 1$.

The first-extensive result, *which* also consists of various extensive-geometric-analytic properties of certain types of the (normalized) functions analytic in the open set $\mathbb{U}$ specified by the related operators given by (3)-(5), is contained in the following form (just below).

Theorem 1. Let the mentioned parameters t and ℓ satisfy the restricted conditions given in (6), and also let a function like $\Phi := \Phi(z)$ be of the form given by (1). Then, if the statement:

$$\arg\left(z\frac{d^2}{dz^2}\Big(\mathbf{T}_z^{t,\ell}\big[\Phi(z)\big]\Big) + \frac{d}{dz}\Big(\mathbf{T}_z^{t,\ell}\big[\Phi(z)\big]\Big)\right) \in \Big\{2m\pi, \pi + 2m\pi\Big\} \tag{11}$$

is true, then the statement:

$$\Re e\left(\frac{d}{dz}\Big(\mathbf{T}_z^{t,\ell}\big[\Phi(z)\big]\Big)\right) > \kappa \tag{12}$$

is also true, *where*

$$0 \leq \kappa < \frac{t}{\ell} \quad , \quad m \in \mathbb{Z} \quad \text{and} \quad z \in \mathbb{U}\,. \tag{13}$$

Proof. Under the concerned conditions in (6) and (13), and by means of the related definitions in (3)-(5), for the analytic function $\Phi := \Phi(z)$ having the form in (1), the statement:

$$\mathbf{T}_z^{t,\ell}[\Phi(z)] = \frac{t}{\ell} z + \frac{\Gamma(\ell)}{\Gamma(t)} \sum_{r=k+1}^{\infty} \frac{\Gamma(r+t)}{\Gamma(r+\ell)} c_r z^r \tag{14}$$

can be easily established (as in the similar determinations constituted in the references given by [4], [10] and [11]).

Accordingly, for the pending proof, a special function needs to be constructed, *which* also satisfies the hypotheses of Lemma 1. For it, by means of expression in (14), the relevant function can be easily created with the help of the relationships given by the following terms:

$$\begin{aligned} &\frac{d}{dz}\left(\mathbf{T}_z^{t,\ell}[\Phi(z)]\right) \\ &= \frac{t}{\ell} + \frac{\Gamma(\ell)}{\Gamma(t)} \sum_{r=k+1}^{\infty} \frac{\Gamma(r+t)}{\Gamma(r+\ell)} r c_r z^{r-1} \\ &= \kappa + \left(\frac{t}{\ell} - \kappa\right) Q(z)\,, \end{aligned} \tag{15}$$

of course, for some suitable values of the mentioned parameters given by the restricted conditions in (6) and (13). Clearly, when centering upon the implicit form of the function $Q(z)$ arranged in (15), it is easily observed that it is of the suitable form for Lemma 1. Namely, the function $Q(z)$ is both an analytic function in the disc $\mathbb{U}$ and has any form in (2) (with $e := 1$). Therefore, the following assertion:

$$z\frac{d^2}{dz^2}\left(\mathbf{T}_z^{t,\ell}[\Phi(z)]\right) = \left(\frac{t}{\ell} - \kappa\right) z \frac{d}{dz}\Big(Q(z)\Big) \tag{16}$$

can be then identified by the help of the information given by (15).

Additionally, specially, by combining of the results determined by (15) and (16), the relationships:

$$\begin{aligned} &z\frac{d^2}{dz^2}\left(\mathbf{T}_z^{t,\ell}[\Phi(z)]\right) + \frac{d}{dz}\left(\mathbf{T}_z^{t,\ell}[\Phi(z)]\right) \\ &= \left(\frac{t}{\ell} - \kappa\right) z\frac{d}{dz}\Big(Q(z)\Big) + \kappa + \left(\frac{t}{\ell} - \kappa\right) Q(z) \\ &= \kappa + \left(\frac{t}{\ell} - \kappa\right) Q(z) + \left(\frac{t}{\ell} - \kappa\right) zQ'(z) \end{aligned} \tag{17}$$

can be also designated under the mentioned conditions presented in (6) and (13).

We now assume that there exists a point w_0, which belongs to the complex set $\mathbb{U}$, satisfying the condition:

$$\Re e\Big(Q\big(w_0\big)\Big) = 0\,,\ \textit{or},\ \textit{equivalently},\ Q\big(w_0\big) = i\alpha\,,$$

where $\alpha \in \mathbb{R}^*$ and $w_0 \in \mathbb{U}$. Then, by making use of the assertion presented by (10) of Lemma 1 and also by considering the following relationships:

$$Q\big(w_0\big) = i\alpha \quad \text{and} \quad w_0Q'\big(w_0\big) = i\beta\Big(\alpha + \frac{1}{\alpha}\Big)Q\big(w_0\big),$$

the assertions given by (17) immediately follows that

$$\begin{aligned}
&\arg\left\{w_0\frac{d^2}{dz^2}\Big(\mathbf{T}_z^{t,\ell}\big[\Phi\big(w_0\big)\big]\Big) + \frac{d}{dz}\Big(\mathbf{T}_z^{t,\ell}\big[\big(w_0\big)\big]\Big)\right\} \\
&\qquad = \arg\left\{\Big(\frac{t}{\ell} - \kappa\Big)w_0Q'\big(w_0\big) + \kappa + \Big(\frac{t}{\ell} - \kappa\Big)Q\big(w_0\big)\right\} \\
&\qquad = \cdots \qquad (18) \\
&\qquad = \arg\left\{\kappa - \alpha\beta\Big(\frac{t}{\ell} - \kappa\Big) + \Big(\frac{\tau}{\mu} - \kappa\Big)i\alpha\right\} \\
&\qquad \in \begin{cases}
\Big(2m\pi, \frac{\pi}{2} + 2m\pi\Big] & \textit{when}\ \ \Omega \geq 0\ \&\ \alpha > 0 \\
\Big[-\frac{\pi}{2} + 2m\pi, 2m\pi\Big) & \textit{when}\ \ \Omega \geq 0\ \&\ \alpha < 0 \\
\Big[\frac{\pi}{2} + 2m\pi, \pi + 2m\pi\Big) & \textit{when}\ \ \Omega \leq 0\ \&\ \alpha > 0 \\
\Big(-\pi + 2m\pi, -\frac{\pi}{2} + 2m\pi\Big] & \textit{when}\ \ \Omega \leq 0\ \&\ \alpha < 0
\end{cases}
\end{aligned}$$

are then obtained by considering the values of the related parameters particularized in (6), (9), (10) and (13), *where*

$$\Omega := \Omega(\kappa, t, \ell; \alpha, \beta) = \kappa - \alpha\beta\left(\frac{t}{\ell} - \kappa\right).$$

But, the assertion given in (18) is a contradiction with the hypothesis of Theorem 1, *which* is the assertion given by (11). Thus, in view of the assertion given in (7), for all $z \in \mathbb{U}$, the statement stated by (15) easily arrives at the following inequality:

$$\Re e\left(\frac{\frac{d}{dz}\Big(\mathbf{T}_z^{t,\ell}\big[\Phi(z)\big]\Big) - \kappa}{\frac{t}{\ell} - \kappa}\right) > 0 \qquad \big(z \in \mathbb{U}\big),$$

which also requires to the mentioned assertion given by (12). Thereby, the desired proof is completed.

The second-extensive result, *whose* proof is quite similar to the proof of Theorem 1, is contained in the following theorem.

Theorem 2. Let the mentioned parameters t, ℓ and κ ensure the conditions created in (6) and (13), and also let a function like $\Phi := \Phi(z)$ possess the form given by (1). Then, if the proposition:

$$\arg\left(\frac{d}{dz}\left(\mathbf{T}_z^{t,\ell}\left[\Phi(z)\right]\right) - \frac{\mathbf{T}_z^{t,\ell}\left[\Phi(z)\right]}{z}\right) \neq \pi + 2m\pi \tag{19}$$

holds true, then the proposition:

$$\Re e\left(\frac{\mathbf{T}_z^{t,\ell}\left[\Phi(z)\right]}{z}\right) > \kappa \tag{20}$$

holds true, *where* $m \in \mathbb{Z}$ and $z \in \mathbb{U}$.

Proof. Under the conditions specified as in (6) and (13), and in the light of definitions in (3)-(5), and also by using the elementary result in (14), for the normalized analytic function $\Phi(z)$ being of the form in (1), if one describes an analytic function like $Q(z)$, which will be both $Q(0) = 1$ and analytic in $\mathbb{U}$, in the implicit form given by

$$\begin{aligned}\mathbf{T}_z^{t,\ell}[\Phi(z)] &= z\left(\frac{t}{\ell} + \frac{\Gamma(\ell)}{\Gamma(t)}\sum_{r=k+1}^{\infty}\frac{\Gamma(r+t)}{\Gamma(r+\ell)}c_r z^{r-1}\right)\\ &= z\left[\kappa + \left(\frac{t}{\ell} - \kappa\right)Q(z)\right],\end{aligned}$$

and then follows all steps taken in consideration the proof of Theorem 1, the desired proof can be easily ended. Therefore, since all considered steps will require repetition, the details of the pending proof are omitted here.

In this last part, we would like to present certain information about both various implications of our research and certain suggestions for the interested researchers. By focusing on two comprehensive theorems determined in this chapter, so many specific results can be also reorganized (*or* revealed). For those, it will be sufficient to choose the concerned parameters used in the theorems appropriately. In particular, the mentioned parameters situated in the statements in (11), (12), (19) and (20) play important roles. In addition, since some of those results include various analytic-geometric properties of the normalized analytic functions given the form in (1), they will also be important for the related researchers. Nevertheless, as an example, we would like to compose only one of those specific results for you. For the details of those indicated-special functions, see [3], [5], [6] and [15].

As only one of all possible implications which can be also revealed by the help of our main results, by taking $t := 1$ and $\ell := 1$ in Theorem 2, the following-special consequence dealing with Analytic-Geometric Function Theory (cf., e.g., [3,6,15]) can be easily constituted, *which* is just below.

Proposition 1. Let a function $\Phi(z)$ be of the form given by (1). Then, the following assertion is satisfied:

$$\arg\left(\Phi'(z) - \frac{\Phi(z)}{z}\right) \neq \pi + 2m\pi \quad \Rightarrow \quad \Re e\left(\frac{\Phi(z)}{z}\right) > \kappa\,,$$

where $0 \leq \kappa < 1$, $m \in \mathbb{Z}$ and $z \in \mathbb{U}$.

3 Conclusions

Clearly, this scientific note basically consists of two sections. In the first section, certain basic information and some definitions have been firstly presented and various fractional-order type operators have been then introduced. In the second part, some possible effects of the related operators on some normalized analytical functions have been next determined and then some special consequences of those effects have been also highlighted. In fact, it has been created as an example proposition relating to particular conclusions. At the same time, some specific recommendations have been made for interested researchers regarding other possible implications of this research note.

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