



Approximation properties of a discrete operator ¹

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Dedicated to professor Vijay Gupta on the occasion of his 60th birthday

Abstract

In the present research, we estimate some approximation properties for the generalization of a sequence of positive linear operators considered by Holhoş [6] with the following operators available in literature as special examples: Szász operator, Ismail-May operator associated with $x(1+x)^2$. We estimate its recurrence relation, moments and verify the same for its particular cases. Finally, we graphically compare the convergence of this new general operator for a continuous function, examine the convergence for each instance, and come to a conclusion regarding which scenario provides the best approximation.

2010 Mathematics Subject Classification: 41A25, 41A36.

Key words and phrases: Szász operator, Ismail-May operator, Convergence Estimates, Voronovskaya Theorem.

1 Introduction

The approximation by linear positive operators is active area of research we mention some recent work [3], [4] and [5] etc. We begin with a straightforward application of Lagrange's expansion theorem (see [9, p. 348]) given by:

$$(1) \quad e^{\alpha z} = \sum_{k=0}^{\infty} \frac{\alpha(\beta k + \alpha)^{k-1}}{k!} \cdot z^k e^{-\beta k z},$$

¹Received 6 October, 2022

Accepted for publication (in revised form) 20 November, 2022

which holds true for $|\beta z| < 1$.

Using eq. (1), Holhos [6] has considered the following operator:

$$\begin{aligned}
 (S_{\alpha,\beta}f)(x) &= \sum_{k=0}^{\infty} s_{\alpha,k,\beta}(x) f\left(\frac{k}{\alpha}\right) \\
 (2) \qquad &= \sum_{k=0}^{\infty} \frac{\alpha(\beta k + \alpha)^{k-1}}{k!} \left(\frac{x}{\beta x + 1}\right)^k e^{-(\beta k + \alpha)\frac{x}{\beta x + 1}} f\left(\frac{k}{\alpha}\right),
 \end{aligned}$$

where $\alpha > 0$, $\beta \geq 0$, $k \in \mathbb{N} \cup \{0\}$, $x \in [0, \infty)$ and $f \in C[0, \infty)$ (the space of all continuous functions on $[0, \infty)$). Our interest in studying more approximation properties and error estimation of the concerned operator was sparked by Holhos' groundbreaking work in his publication [6].

1. For $\alpha = n$, $\beta = 0$ and $x \geq 0$, the reduced operator is given by:

$$\begin{aligned}
 (S_{n,0}f)(x) &:= (S_n f)(x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right) \\
 &= e^{-xn} \sum_{k=0}^{\infty} \frac{(xn)^k}{k!} f\left(\frac{k}{n}\right),
 \end{aligned}$$

which is the well known Szász-Mirakyan operator.

2. For $\alpha = n$, $\beta = 1$ and $x > 0$, the reduced operator is given by:

$$\begin{aligned}
 (S_{n,1}f)(x) &:= (M_n f)(x) = \sum_{k=0}^{\infty} m_{n,k}(x) f\left(\frac{k}{n}\right) \\
 &= e^{\frac{-xn}{1+x}} \sum_{k=0}^{\infty} \frac{n(k+n)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{\frac{-kx}{1+x}} f\left(\frac{k}{n}\right),
 \end{aligned}$$

which is the Ismail-May exponential-type operator connected with the polynomial $p(x) = x(1+x)^2$ (see [7, 3.14]).

Lemma 1 Denoting $T_{\alpha,\beta,m}(x) = (S_{\alpha,\beta}e_m)(x)$, $e_m = t^m$; $m \in \mathbb{N} \cup \{0\}$, then the following relation holds:

$$T_{\alpha,\beta,m+1}(x) = \frac{x(\beta x + 1)^2}{\alpha} T'_{\alpha,\beta,m}(x) + x T_{\alpha,\beta,m}(x).$$

Proof. On differentiating the basis of the operator (2), we have

$$\begin{aligned}
s'_{\alpha,k,\beta}(x) &= \frac{\alpha(\beta k + \alpha)^{k-1} e^{-(\beta k + \alpha) \frac{x}{\beta x + 1}}}{k!} \left[\frac{kx^{k-1}}{(\beta x + 1)^{k+1}} - \frac{(\beta k + \alpha) \cdot x^k}{(\beta x + 1)^{k+2}} \right] \\
&= \frac{(\beta k + \alpha) \cdot e^{-(\beta k + \alpha) \frac{x}{\beta x + 1}}}{(\beta x + 1)^2} \left[\frac{(\alpha + \beta - \beta)(\alpha + \beta + \beta(k-1))^{k-2}}{(k-1)!} \right. \\
&\quad \cdot \left. \left(\frac{x}{\beta x + 1} \right)^{k-1} - \frac{\alpha(\beta k + \alpha)^{k-1}}{k!} \left(\frac{x}{\beta x + 1} \right)^k \right] \\
&= \frac{\beta k + \alpha}{(\beta x + 1)^2} [s_{\alpha+\beta,k-1,\beta}(x) - s_{\alpha,k,\beta}(x)] \\
&\quad - \frac{\beta(\beta k + \alpha)^{k-1}}{(\beta x + 1)^2(k-1)!} \cdot e^{-(\beta k + \alpha) \frac{x}{\beta x + 1}} \left(\frac{x}{\beta x + 1} \right)^{k-1} \\
&= \frac{\beta k + \alpha}{(\beta x + 1)^2} [s_{\alpha+\beta,k-1,\beta}(x) - s_{\alpha,k,\beta}(x)] - \frac{\beta k}{x(\beta x + 1)} \cdot s_{\alpha,k,\beta}(x),
\end{aligned}$$

which can be rewritten in the form

$$(3) \quad s'_{\alpha,k,\beta}(x) = \left(\frac{k - \alpha x}{x(\beta x + 1)^2} \right) s_{\alpha,k,\beta}(x).$$

The required recurrence relation is an easy consequence of the equality (3).

Remark 1 For $\alpha = n$ and $\beta = 0$, the relation in Lemma 1 reduced to the relation for Szász operator given by:

$$s'_{n,k}(x) = n [s_{n,k-1}(x) - s_{n,k}(x)].$$

For $\alpha = n$ and $\beta = 1$, the relation in Lemma 1 reduced to the relation for Ismail-May operator given by:

$$m'_{n,k}(x) = \frac{n+k}{(1+x)^2} [m_{n+1,k-1}(x) - m_{n,k}(x)] - \frac{k}{x(x+1)} \cdot m_{n,k}(x).$$

Lemma 2 Few moments for the operator (2) are given by:

$$\begin{aligned}
T_{\alpha,\beta,0}(x) &= 1 \\
T_{\alpha,\beta,1}(x) &= x \\
T_{\alpha,\beta,2}(x) &= x^2 + \frac{x(\beta x + 1)^2}{\alpha} \\
T_{\alpha,\beta,3}(x) &= x^3 + \frac{x(\beta x + 1)^2}{\alpha^2} [3\beta^2 x^2 + 4\beta x + 3\alpha x + 1] \\
T_{\alpha,\beta,4}(x) &= x^4 + \frac{x(\beta x + 1)^2}{\alpha^3} \\
&\quad \left[(15\beta^2 x^2 + 10\beta x + 1)(\beta x + 1)^2 + \alpha x(15\beta x + 7)(\beta x + 1) + 6\alpha^2 x^2 \right].
\end{aligned}$$

The proof follows using the relation indicated in Lemma 1.

Lemma 3 *If we denote $\mu_{\alpha,\beta}^m(x) = (S_{\alpha,\beta}(t-x)^m)(x)$, $m \in \mathbb{N} \cup \{0\}$, then we have:*

$$\begin{aligned}
\mu_{\alpha,\beta}^0(x) &= 1 \\
\mu_{\alpha,\beta}^1(x) &= 0 \\
\mu_{\alpha,\beta}^2(x) &= \frac{x(\beta x + 1)^2}{\alpha} \\
\mu_{\alpha,\beta}^3(x) &= \frac{x(\beta x + 1)^3}{\alpha^2} [3\beta x + 1] \\
\mu_{\alpha,\beta}^4(x) &= \frac{x(\beta x + 1)^4}{\alpha^3} [15\beta^2 x^2 + 10\beta x + 3\alpha x + 1] \\
\mu_{\alpha,\beta}^5(x) &= \frac{x(\beta x + 1)^5}{\alpha^4} [105\beta^3 x^3 + 105\beta^2 x^2 + 25\beta x + 10\alpha x(3\beta x + 1) + 1] \\
\mu_{\alpha,\beta}^6(x) &= \frac{x(\beta x + 1)^6}{\alpha^5} \left[945\beta^4 x^4 + 1260\beta^3 x^3 \right. \\
&\quad \left. + 490\beta^2 x^2 + 5\alpha x(63\beta^2 x^2 + 42\beta x + 5) + 15\alpha^2 x^2 + 56\beta x + 1 \right].
\end{aligned}$$

Further, the following limits hold true:

$$\begin{aligned}
\lim_{\alpha \rightarrow \infty} \alpha \mu_{\alpha,\beta}^2(x) &= x(\beta x + 1)^2 \\
\lim_{\alpha \rightarrow \infty} \frac{\mu_{\alpha,\beta}^6(x)}{\mu_{\alpha,\beta}^2(x)} &= 0.
\end{aligned}$$

Theorem 1 *For $f \in \hat{C}_B[0, \infty)$ (the space of all bounded functions on non-negative real axis which are continuous), $m \geq 1$, $s \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$ and $x \geq 0$, we have*

$$\lim_{\alpha \rightarrow \infty} (S_{m\alpha,\beta} f(\alpha t)) \left(\frac{x}{\alpha} \right) = (S_\alpha f)(x),$$

and

$$\lim_{\alpha \rightarrow \infty} \left(S_{m,\beta} f \left(\frac{t}{\alpha} \right) \right) (\alpha x) = 1.$$

Proof. Following [6, Th. 3], the m.g.f. of the operator $S_{\alpha,\beta}$ is given by:

$$(4) \quad (S_{\alpha,\beta} e^{At})(x) = \exp \left(\frac{-\alpha x}{\beta x + 1} - \frac{\alpha}{\beta} \cdot W \left(\frac{-\beta x}{\beta x + 1} e^{\frac{-\beta x}{\beta x + 1}} e^{\frac{A}{\alpha}} \right) \right),$$

where W is the Lambert function. Obviously for each $m \geq 1$, $s \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$ and $x \geq 0$,

$$\lim_{\alpha \rightarrow \infty} (S_{m\alpha,\beta} e^{is\alpha t}) \left(\frac{x}{\alpha} \right) = e^{-mx} \cdot \lim_{\alpha \rightarrow \infty} \exp \left(-\frac{m\alpha}{\beta} \cdot W \left(\frac{-\beta x}{\alpha + \beta x} \cdot e^{\frac{-\beta x}{\alpha + \beta x}} \cdot e^{\frac{is}{m}} \right) \right).$$

Clearly, as $W(0) = 0$

$$\Rightarrow \lim_{\alpha \rightarrow \infty} W \left(\frac{-\beta x}{\alpha + \beta x} \cdot e^{\frac{-\beta x}{\alpha + \beta x}} \cdot e^{\frac{is}{m}} \right) = 0.$$

Thus

$$\lim_{\alpha \rightarrow \infty} \left(\frac{W \left(\frac{-\beta x}{\alpha + \beta x} \cdot e^{\frac{-\beta x}{\alpha + \beta x}} \cdot e^{\frac{is}{m}} \right)}{\frac{-\beta}{m\alpha}} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using L'Hôpital's Rule with $W'(p) = \frac{W(p)}{p(1+W(p))}$ and let $p = \frac{-\beta x}{\alpha + \beta x} \cdot e^{\frac{-\beta x}{\alpha + \beta x}} \cdot e^{\frac{is}{m}}$, we have

$$\lim_{\alpha \rightarrow \infty} \left(\frac{W(p)}{-\beta/(m\alpha)} \right) = \lim_{\alpha \rightarrow \infty} \frac{\frac{W(p)}{p(1+W(p))} \cdot \frac{e^{\frac{is}{m}} \cdot e^{-\frac{\beta x}{\alpha + \beta x}} \alpha \beta x}{(\alpha + \beta x)^3}}{\beta/(m\alpha^2)},$$

which is again $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ form. Use of L'Hôpital Rule repeatedly provides us the following:

$$\lim_{\alpha \rightarrow \infty} (S_{m\alpha, \beta} e^{is\alpha t}) \left(\frac{x}{\alpha} \right) = e^{(e^{\frac{is}{m}} - 1)mx} = (S_{\alpha} e^{is\alpha t})(x).$$

Similarly, we can calculate

$$\lim_{\alpha \rightarrow \infty} (S_{m, \beta} e^{ist/\alpha})(\alpha x) = 1.$$

Following [2, Th. 1.1] and references therein, we get the required result.

Denote the space of all continuous real-valued functions $f(x)$ having finite limit for sufficiently large element x by $\hat{C}^*[0, \infty)$. Further, let us consider the following moduli:

$$\hat{\omega}(f, \varrho) = \sup_{x, t \geq 0} \{ |f(x) - f(t)| : |e^{-x} - e^{-t}| \leq \varrho \}, \quad \varrho \geq 0.$$

Theorem 2 *If f, f'' belongs to $\hat{C}^*(0, \infty)$, then for $x \geq 0$, the following inequality exists:*

$$\begin{aligned} & \left| \alpha [(S_{\alpha, \beta} f)(x) - f(x)] - \frac{x(\beta x + 1)^2}{2} f''(x) \right| \\ & \leq \frac{1}{2} \left[x(\beta x + 1)^2 + \left[\frac{x(\beta x + 1)^4}{\alpha} (15\beta^2 x^2 + 10\beta x + 3\alpha x + 1) \right]^{1/2} \right. \\ & \quad \left. \left[\alpha^2 (S_{\alpha, \beta} (e^{-x} - e^{-t})^4)(x) \right]^{1/2} \right] \hat{\omega} \left(f'', \frac{1}{\sqrt{\alpha}} \right). \end{aligned}$$

Proof. According to Taylor's expansion, there exists δ between t and x such that

$$(5) \quad f(t) = \sum_{k=0}^2 \frac{(t-x)^k}{k!} f^{(k)}(x) + \epsilon(\delta, x)(e_1 - x e_0)^2,$$

where

$$\epsilon(t, x) := \frac{f''(t) - f''(x)}{2}.$$

Application of the operator $S_{\alpha, \beta}$ to (5) and using Lemma 3, we have the following:

$$\begin{aligned} \left| \alpha [(S_{\alpha, \beta} f)(x) - f(x)] - \frac{x(\beta x + 1)^2}{2} f''(x) \right| &\leq \frac{1}{2} |\alpha \mu_{\alpha, \beta}^2(x) - x(\beta x + 1)^2| |f''(x)| \\ &\quad + |\alpha (S_{\alpha, \beta} \epsilon(\delta, x)(e_1 - e_0 x)^2)(x)|. \end{aligned}$$

Using the property of $\hat{\omega}(f, \varrho)$, $\varrho > 0$ given by

$$|f(t) - f(x)| \leq \left(1 + \frac{(e^{-x} - e^{-t})^2}{\varrho^2} \right) \hat{\omega}(f, \varrho),$$

we obtain

$$|\epsilon(\delta, x)| \leq \frac{1}{2} \left(1 + \frac{(e^{-x} - e^{-t})^2}{\varrho^2} \right) \hat{\omega}(f'', \varrho).$$

Hence, applying Cauchy-Schwarz inequality and choosing $\varrho = \frac{1}{\sqrt{\alpha}}$, we get

$$\begin{aligned} &\alpha (S_{\alpha, \beta} |\epsilon(\delta, x)| (e_1 - x e_0)^2)(x) \\ &\leq \frac{1}{2} \hat{\omega} \left(f'', \frac{1}{\sqrt{\alpha}} \right) \left[\alpha \mu_{\alpha, \beta}^2(x) + \sqrt{\alpha^2 (S_{\alpha, \beta} (e^{-t} - e^{-x})^4)(x)} \sqrt{\alpha^2 \mu_{\alpha, \beta}^4(x)} \right]. \end{aligned}$$

Finally, we obtain the required result by employing Lemma 3.

2 Voronovskaya Estimate

Voronovskaya type theorem is one of the most crucial technique for estimating the rate of pointwise convergence of the operator $S_{\alpha, \beta}$ in approximation by linear positive operators. Here we discuss a quantitative Voronovskaya theorem in weighted space and the corresponding modulus of continuity. All the theory described below in this section has its written version in [1, 8]. Consider the space $\mathcal{B}_2[0, \infty)$ as the space of all functions defined on $[0, \infty)$ satisfying the condition $|f(x)| \leq C_f(1 + x^2)$, where C_f is a positive constant depending on f . By $\mathcal{C}_2[0, \infty)$, we denote the subspace of $C[0, \infty)$ in $\mathcal{B}_2[0, \infty)$. $\mathcal{C}_2^*[0, \infty)$ denotes the subspace of all functions in $\mathcal{C}_2[0, \infty)$ for which $\lim_{x \rightarrow \infty} \frac{f(x)}{1 + x^2}$ is finite. Further assume a subspace $\mathcal{W}$ of $C[0, \infty)$ containing polynomials.

Consider the weighted modulus of continuity for $f \in \mathcal{C}_2[0, \infty)$ given by:

$$\Omega(f, \varrho) = \sup_{\substack{0 \leq h < \varrho \\ x \in [0, \infty)}} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)}.$$

This weighted modulus has the following properties if the function is in the space $\mathcal{C}_2^*[0, \infty)$:

$$1. \lim_{\varrho \rightarrow 0} \Omega(f, \varrho) = 0$$

$$2. \Omega(f, \lambda \varrho) \leq 2(1 + \varrho^2)(1 + \lambda)\Omega(f, \varrho), \quad \lambda > 0.$$

For $f \in \mathcal{C}_2^*[0, \infty)$, if we use definition of weighted modulus, we obtain

$$|f(t) - f(x)| \leq (1 + (t - x)^2)(1 + x^2)\Omega(f, |t - x|).$$

Next, using the second property of weighted modulus Ω on the above inequality for a function in the class $\mathcal{C}_2^*[0, \infty)$, we get

$$(6) \quad |f(t) - f(x)| \leq 2 \left(1 + \frac{|t - x|}{\varrho}\right) (1 + (t - x)^2)(1 + \varrho^2)\Omega(f, \varrho)(1 + x^2).$$

Theorem 3 Let $f \in \mathcal{W}$, $f'' \in \mathcal{C}_2^*[0, \infty)$ and $S_{\alpha, \beta} : \mathcal{W} \rightarrow C[0, \infty)$. Then for $x \in [0, \infty)$, we have

$$\begin{aligned} & \left| (S_{\alpha, \beta} f)(x) - f(x) - \frac{x(\beta x + 1)^2}{2\alpha} f''(x) \right| \\ & \leq 16(1 + x^2) \left(\frac{x(\beta x + 1)^2}{\alpha} \right) \Omega \left(f'', \frac{\beta x + 1}{\alpha} [\mathcal{E}_{\alpha, \beta}(x)]^{1/4} \right), \end{aligned}$$

where

$$\mathcal{E}_{\alpha, \beta}(x) = 945\beta^4 x^4 + 1260\beta^3 x^3 + 490\beta^2 x^2 + 56\beta x + 71\alpha^2 x^2 + 5\alpha x(63\beta^2 x^2 + 42\beta x + 5) + 1.$$

Proof. Applying the operator $S_{\alpha, \beta}$ on (5), we have

$$\begin{aligned} & \left| (S_{\alpha, \beta} f)(x) - f(x) - \frac{x(\beta x + 1)^2}{2\alpha} f''(x) \right| \\ & \leq \left(S_{\alpha, \beta} |\epsilon(\delta, x)| (t - x)^2 \right)(x). \end{aligned}$$

We estimate $(S_{\alpha, \beta} |\epsilon(\delta, x)| (t - x)^2)(x)$ to claim the proof of the theorem. According to [8], let $\Omega(f, \cdot)$ be as defined above, we are familiar with inequality (6) for a function in the space $\mathcal{C}_2^*[0, \infty)$. By utilizing the equality (5) and the inequality (6) for f'' and further the inequality $|\delta - x| \leq |t - x|$, we get

$$\begin{aligned} |\epsilon(\delta, x)| &= \frac{1}{2} |f''(\delta) - f''(x)| \\ &\leq \left(1 + \frac{|t - x|}{\varrho}\right) (1 + \varrho^2) \Omega(f'', \varrho) (1 + (t - x)^2) (1 + x^2) \\ &\leq \begin{cases} 2\Omega(f'', \varrho) (1 + \varrho^2)^2 (1 + x^2), & |t - x| < \varrho \\ 2\Omega(f'', \varrho) \frac{(t - x)^4}{\varrho^4} (1 + \varrho^2)^2 (1 + x^2), & |t - x| \geq \varrho. \end{cases} \end{aligned}$$

If we choose $\varrho < 1$, then we have

$$(7) \quad |\epsilon(\delta, x)| \leq 8\Omega(f'', \varrho) \cdot (1 + x^2) \left(1 + \frac{(t - x)^4}{\varrho^4}\right).$$

Finally using (7) (with the assumption that $\varrho < 1$), we get

$$\begin{aligned} (S_{\alpha,\beta} |\epsilon(\delta, x)| (t-x)^2)(x) &\leq 8 \Omega(f'', \varrho) \cdot (1+x^2) \left(\mu_{\alpha,\beta}^2(x) + \frac{1}{\varrho^4} \mu_{\alpha,\beta}^6(x) \right) \\ &= 8 \Omega(f'', \varrho) \cdot (1+x^2) \mu_{\alpha,\beta}^2(x) \left(\frac{1}{\varrho^4} \frac{\mu_{\alpha,\beta}^6(x)}{\mu_{\alpha,\beta}^2(x)} + 1 \right). \end{aligned}$$

Using Lemma 3, we can choose $\varrho = \left(\frac{\mu_{\alpha,\beta}^6(x)}{\mu_{\alpha,\beta}^2(x)} \right)^{\frac{1}{4}}$ which implies

$$\begin{aligned} &\left| (S_{\alpha,\beta} f)(x) - f(x) - \frac{x(\beta x + 1)^2}{2\alpha} f''(x) \right| \\ &\leq 16 \Omega \left(f'', \frac{\beta x + 1}{\alpha} (\mathcal{E}_{\alpha,\beta}(x))^{1/4} \right) (1+x^2) \left(\frac{x(\beta x + 1)^2}{\alpha} \right), \end{aligned}$$

where

$$\mathcal{E}_{\alpha,\beta}(x) = 945\beta^4 x^4 + 1260\beta^3 x^3 + 490\beta^2 x^2 + 56\beta x + 71\alpha^2 x^2 + 5\alpha x(63\beta^2 x^2 + 42\beta x + 5) + 1,$$

as required.

3 Graphical Representation

In this present section, we shall examine the convergence of the considered operator $S_{\alpha,\beta} f$ for a continuous function $f(x) = x^3 - 5x^2 + 7x + 3$ on the unbounded interval $[0, \infty)$ and compare the outcome of the assumed function for various positive and non-negative values of α and β respectively.

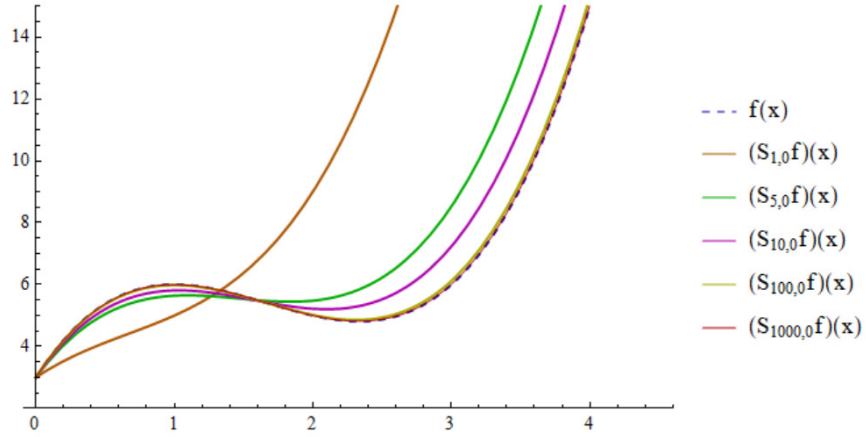


Figure 1: Convergence of the operator $(S_{\alpha,0}f)(x)$ for $f(x) = x^3 - 5x^2 + 7x + 3$

In Fig. 1, we set $\beta = 0$ and consider different positive values of α . We come to the conclusion that as α increases, the approximation gets considerably better and the operator converges to the supposed continuous function more quickly.

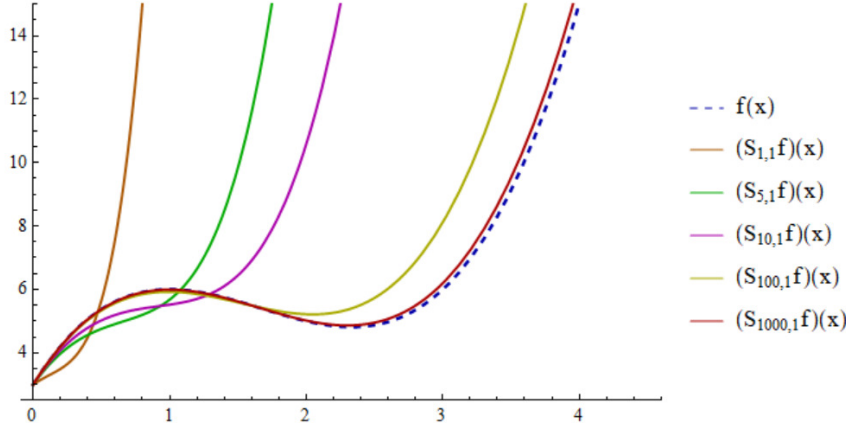


Figure 2: Convergence of the operator $(S_{\alpha,1}f)(x)$ for $f(x) = x^3 - 5x^2 + 7x + 3$

In Fig. 2, we assumed $\beta = 1$ and used several positive values of α . In a similar manner as above, we draw the conclusion that increasing α improves approximation performance i.e. the operator converges to the assumed continuous function more rapidly as α increases in value.

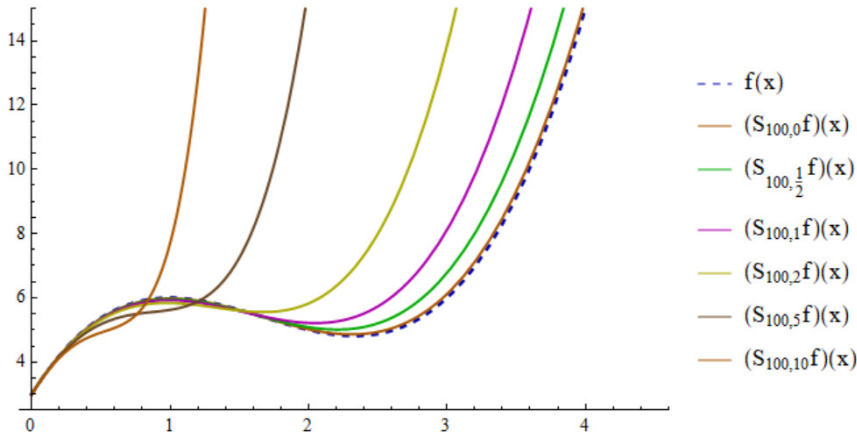


Figure 3: Convergence of the operator $(S_{100,\beta}f)(x)$ for $f(x) = x^3 - 5x^2 + 7x + 3$

In Fig. 3, various non-negative values of β were utilized with α set to 100. It has been seen that the operator converges to the alleged continuous function faster i.e. the approximation performs noticeably better when β is lowered to zero.

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