



Unique common fixed points in metric-like space¹

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Dedicated to professor Vijay Gupta on the occasion of his 60th birthday

Abstract

In this article, two unique common fixed point theorems for four self-maps on a metric-like space by using the concept of occasionally weakly biased maps of type $(\mathcal{A})$ are proved. Our results represent an improvement and extension of some fixed point findings. We justify our results by giving two appropriate examples.

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1 Introduction

In his thesis [9], Matthews proposed the notion of **metric domains** in order to promote the notion of completeness. Matthews pointed out that there is a one to one correspondence between the class of metric domains and the class of metric spaces.

In his dissertation [5], Hitzler utilised metric domains under the name of **dislocated metrics** and he investigated the topological structure underlying the notion of dislocated metric.

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In order to initiate the fixed point theory in metric-like spaces, Amini-Harandi [2] presented a new generalization of partial metric spaces, called **metric-like spaces**. He also derived some new fixed point results which unify and generalize some well-known results in the literature.

Remark 1 *In fact, the notions of metric domains, metric-like spaces and dislocated metric spaces are exactly the same, and they also named d -metric spaces.*

Now, in 2022, we suggested the concept of **occasionally weakly biased maps of type $(\mathcal{A})$** which generalizes the concepts of weakly compatible and occasionally weakly compatible maps.

2 Preliminaries

Definition 1 ([9]) A **metric domain** is a pair $\langle D, d \rangle$ where D is a non-empty set, and d is a function from $D \times D$ to $\mathbb{R}^+$ such that

1. $\forall x, y \in D \ d(x, y) = 0 \Rightarrow x = y$
2. $\forall x, y \in D \ d(x, y) = d(y, x)$
3. $\forall x, y, z \in D \ d(x, y) \leq d(x, z) + d(z, y)$.

Definition 2 ([2]) A map $\sigma : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$, where $\mathcal{X}$ is a nonempty set, is said to be **metric-like on $\mathcal{X}$** if for any $x, y, z \in \mathcal{X}$, the following three conditions hold true:

- ($\sigma 1$) $\sigma(x, y) = 0 \Rightarrow x = y$
- ($\sigma 2$) $\sigma(x, y) = \sigma(y, x)$
- ($\sigma 3$) $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

The pair $(\mathcal{X}, \sigma)$ is then called a **metric-like space**. Then a metric-like on $\mathcal{X}$ satisfies all of the conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in \mathcal{X}$.

Definition 3 ([3]) Let $\mathcal{S}$ and $\mathcal{T}$ be self-maps of a non-empty set $\mathcal{X}$. The pair $(\mathcal{S}, \mathcal{T})$ is said to be **occasionally weakly $\mathcal{S}$ -biased of type $(\mathcal{A})$** and **occasionally weakly $\mathcal{T}$ -biased of type $(\mathcal{A})$** , respectively, if and only if, there exists a point ζ in $\mathcal{X}$ such that $\mathcal{S}\zeta = \mathcal{T}\zeta$ implies

$$\begin{aligned} d(\mathcal{S}\mathcal{S}\zeta, \mathcal{T}\zeta) &\leq d(\mathcal{T}\mathcal{S}\zeta, \mathcal{S}\zeta), \\ d(\mathcal{T}\mathcal{T}\zeta, \mathcal{S}\zeta) &\leq d(\mathcal{S}\mathcal{T}\zeta, \mathcal{T}\zeta), \end{aligned}$$

respectively.

In this investigation, using our new definition, we would like to present two unique common fixed point theorems in a metric-like space. In our work, we have avoided some conditions required in many papers in the setting of metric, partial metric and metric-like spaces. Consequently, our findings improve and/or extend these results and some references therein.

3 Main Results

Theorem 1 *Let $(\mathcal{X}, d)$ be a metric-like space. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be maps satisfying*

1. *the pairs $(\mathcal{A}, \mathcal{T})$ and $(\mathcal{B}, \mathcal{S})$ are occasionally weakly $\mathcal{T}$ -biased (respectively $\mathcal{S}$ -biased) of type $(\mathcal{A})$ and*

2. $d(\mathcal{A}x, \mathcal{B}y) \leq \varpi \max\{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}$

for all $x, y \in \mathcal{X}$, where $\varpi \in \left[0, \frac{1}{2}\right)$. Then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. By hypotheses, there are two points ι and j in $\mathcal{X}$ such that $\mathcal{A}\iota = \mathcal{T}\iota$ implies $d(\mathcal{T}\mathcal{T}\iota, \mathcal{A}\iota) \leq d(\mathcal{A}\mathcal{T}\iota, \mathcal{T}\iota)$ and $\mathcal{B}j = \mathcal{S}j$ implies $d(\mathcal{S}\mathcal{S}j, \mathcal{B}j) \leq d(\mathcal{B}\mathcal{S}j, \mathcal{S}j)$.

First, we are going to prove that $\mathcal{A}\iota = \mathcal{B}j$. Suppose that $\mathcal{A}\iota \neq \mathcal{B}j$, from inequality (2) we have

$$\begin{aligned}
 d(\mathcal{A}\iota, \mathcal{B}j) &\leq \varpi \max\{d(\mathcal{S}j, \mathcal{A}\iota), d(\mathcal{T}\iota, \mathcal{S}j), d(\mathcal{T}\iota, \mathcal{A}\iota), d(\mathcal{B}j, \mathcal{S}j), d(\mathcal{T}\iota, \mathcal{B}j)\} \\
 &= \varpi \max\{d(\mathcal{B}j, \mathcal{A}\iota), d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{A}\iota), d(\mathcal{B}j, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{B}j)\} \\
 &\leq \varpi \max\{d(\mathcal{B}j, \mathcal{A}\iota), d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{B}j) + d(\mathcal{B}j, \mathcal{A}\iota), \\
 &\quad d(\mathcal{B}j, \mathcal{A}\iota) + d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{B}j)\} \\
 &= 2\varpi d(\mathcal{A}\iota, \mathcal{B}j) \\
 &< d(\mathcal{A}\iota, \mathcal{B}j)
 \end{aligned}$$

which is a contradiction, thus $\mathcal{A}\iota = \mathcal{B}j$.

Now, we assert that $\mathcal{A}\mathcal{A}\iota = \mathcal{A}\iota$. If not, then the use of condition (2) gives

$$\begin{aligned}
 d(\mathcal{A}\mathcal{A}\iota, \mathcal{B}j) &\leq \varpi \max\{d(\mathcal{S}j, \mathcal{A}\mathcal{A}\iota), d(\mathcal{T}\mathcal{A}\iota, \mathcal{S}j), d(\mathcal{T}\mathcal{A}\iota, \mathcal{A}\mathcal{A}\iota), d(\mathcal{B}j, \mathcal{S}j), \\
 &\quad d(\mathcal{T}\mathcal{A}\iota, \mathcal{B}j)\};
 \end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i) &\leq \varpi \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}_i, \mathcal{A}_i), \\
&\quad d(\mathcal{T}\mathcal{A}_i, \mathcal{A}_i)\} \\
&= \varpi \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}_i, \mathcal{A}_i), \\
&\quad d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)\} \\
&\leq \varpi \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i) + d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i) + d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)\} \\
&\leq \varpi \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i), d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i) + d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad 2d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i)\} \\
&= \varpi \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i), 2d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i), 2d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i)\} \\
&= 2\varpi d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i) \\
&< d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i)
\end{aligned}$$

which is a contradiction, therefore $\mathcal{A}\mathcal{A}_i = \mathcal{A}_i$, consequently, $\mathcal{T}\mathcal{A}_i = \mathcal{A}_i$.

Now, suppose that $\mathcal{B}\mathcal{B}_j \neq \mathcal{B}_j$. Using inequality (2) we obtain

$$\begin{aligned}
d(\mathcal{A}_i, \mathcal{B}\mathcal{B}_j) &\leq \varpi \max\{d(\mathcal{S}\mathcal{B}_j, \mathcal{A}_i), d(\mathcal{T}_i, \mathcal{S}\mathcal{B}_j), d(\mathcal{T}_i, \mathcal{A}_i), d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), \\
&\quad d(\mathcal{T}_i, \mathcal{B}\mathcal{B}_j)\};
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j) &\leq \varpi \max\{d(\mathcal{S}\mathcal{B}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), d(\mathcal{B}_j, \mathcal{B}_j), d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), \\
&\quad d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&= \varpi \max\{d(\mathcal{S}\mathcal{S}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}_j), d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), \\
&\quad d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&\leq \varpi \max\{d(\mathcal{S}\mathcal{S}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j) + d(\mathcal{S}\mathcal{S}_j, \mathcal{B}_j), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j) + d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&\leq \varpi \max\{d(\mathcal{B}\mathcal{S}_j, \mathcal{S}_j), d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j), 2d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j) + d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&= \varpi \max\{d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j), 2d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j), 2d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j), \\
&\quad d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&= 2\varpi d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j) \\
&< d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)
\end{aligned}$$

this contradiction implies that $\mathcal{B}\mathcal{B}_j = \mathcal{B}_j$ and so $\mathcal{S}\mathcal{B}_j = \mathcal{B}_j$; i.e., $\mathcal{B}\mathcal{A}_i = \mathcal{A}_i$ and $\mathcal{S}\mathcal{A}_i = \mathcal{A}_i$. Put $\mathcal{A}_i = \mathcal{S}_i = \mathcal{B}_j = \mathcal{T}_j = \mu$, therefore μ is a common fixed point of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.

Finally, let μ and ν be two distinct common fixed points of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$. Then, $\mu = \mathcal{A}\mu = \mathcal{B}\mu = \mathcal{S}\mu = \mathcal{T}\mu$ and $\nu = \mathcal{A}\nu = \mathcal{B}\nu = \mathcal{S}\nu = \mathcal{T}\nu$. From (2) we have

$$d(\mathcal{A}\nu, \mathcal{B}\mu) \leq \varpi \max \{d(\mathcal{S}\mu, \mathcal{A}\nu), d(\mathcal{T}\nu, \mathcal{S}\mu), d(\mathcal{T}\nu, \mathcal{A}\nu), d(\mathcal{B}\mu, \mathcal{S}\mu), d(\mathcal{T}\nu, \mathcal{B}\mu)\};$$

i.e.,

$$\begin{aligned} d(\nu, \mu) &\leq \varpi \max \{d(\mu, \nu), d(\nu, \mu), d(\nu, \nu), d(\mu, \mu), d(\nu, \mu)\} \\ &\leq \varpi \max \{d(\mu, \nu), d(\nu, \mu), d(\nu, \mu) + d(\mu, \nu), \\ &\quad d(\mu, \nu) + d(\nu, \mu), d(\nu, \mu)\} \\ &= \varpi \max \{d(\mu, \nu), d(\nu, \mu), 2d(\nu, \mu), 2d(\mu, \nu), d(\nu, \mu)\} \\ &= 2\varpi d(\nu, \mu) \\ &< d(\nu, \mu) \end{aligned}$$

which is a contradiction, thus $\nu = \mu$.

Now, we give an example which illustrates our above theorem.

Example 1 Let $\mathcal{X} = (-1, 1)$ with the metric-like $d(x, y) = \max \{|x|, |y|\}$. Define

$$\mathcal{A}x = \begin{cases} 0 & \text{if } x \in (-1, 0] \\ -\frac{1}{10} & \text{if } x \in (0, 1), \end{cases} \quad \mathcal{B}x = \begin{cases} -\frac{x}{9} & \text{if } x \in (-1, 0] \\ -\frac{1}{8} & \text{if } x \in (0, 1), \end{cases}$$

$$\mathcal{T}x = \begin{cases} -\frac{3x}{4} & \text{if } x \in (-1, 0] \\ \frac{1}{2} & \text{if } x \in (0, 1), \end{cases} \quad \mathcal{S}x = \begin{cases} -\frac{x}{2} & \text{if } x \in (-1, 0] \\ \frac{3}{4} & \text{if } x \in (0, 1). \end{cases}$$

First, it is clear to see that $\mathcal{A}$ and $\mathcal{T}$ are occasionally weakly $\mathcal{T}$ -biased of type $(\mathcal{A})$ and $\mathcal{B}$ and $\mathcal{S}$ are occasionally weakly $\mathcal{S}$ -biased of type $(\mathcal{A})$. Take $\varpi = \frac{1}{3}$, we get

1. for $x, y \in (-1, 0]$, we have $\mathcal{A}x = 0$, $\mathcal{B}y = -\frac{y}{9}$, $\mathcal{T}x = -\frac{3x}{4}$, $\mathcal{S}y = -\frac{y}{2}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= -\frac{y}{9} \\ &\leq \frac{1}{3} \max \left\{ -\frac{y}{2}, \max \left\{ -\frac{3x}{4}, -\frac{y}{2} \right\}, -\frac{3x}{4}, \max \left\{ -\frac{3x}{4}, -\frac{y}{9} \right\} \right\} \\ &= \varpi \max \{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), d(\mathcal{B}y, \mathcal{S}y), \\ &\quad d(\mathcal{T}x, \mathcal{B}y)\}, \end{aligned}$$

2. for $x, y \in (0, 1)$, we have $\mathcal{A}x = -\frac{1}{10}$, $\mathcal{B}y = -\frac{1}{8}$, $\mathcal{T}x = \frac{1}{2}$, $\mathcal{S}y = \frac{3}{4}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \frac{1}{8} \\ &\leq \frac{1}{3} \max \left\{ \frac{3}{4}, \frac{1}{2} \right\} \\ &= \frac{1}{4} \\ &= \varpi \max \{ d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), d(\mathcal{B}y, \mathcal{S}y), \\ &\quad d(\mathcal{T}x, \mathcal{B}y) \}, \end{aligned}$$

3. for $x \in (-1, 0]$, $y \in (0, 1)$, we have $\mathcal{A}x = 0$, $\mathcal{B}y = -\frac{1}{8}$, $\mathcal{T}x = -\frac{3x}{4}$, $\mathcal{S}y = \frac{3}{4}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \frac{1}{8} \\ &\leq \frac{1}{3} \max \left\{ \frac{3}{4}, -\frac{3x}{4}, \max \left\{ -\frac{3x}{4}, \frac{1}{8} \right\} \right\} \\ &= \frac{1}{4} \\ &= \varpi \max \{ d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), d(\mathcal{B}y, \mathcal{S}y), \\ &\quad d(\mathcal{T}x, \mathcal{B}y) \}, \end{aligned}$$

4. for $x \in (0, 1)$, $y \in (-1, 0]$, we have $\mathcal{A}x = -\frac{1}{10}$, $\mathcal{B}y = -\frac{y}{9}$, $\mathcal{T}x = \frac{1}{2}$, $\mathcal{S}y = -\frac{y}{2}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \max \left\{ \frac{1}{10}, -\frac{y}{9} \right\} \\ &\leq \frac{1}{3} \max \left\{ \max \left\{ -\frac{y}{2}, \frac{1}{10} \right\}, \frac{1}{2}, -\frac{y}{2} \right\} \\ &= \frac{1}{6} \\ &= \varpi \max \{ d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), d(\mathcal{B}y, \mathcal{S}y), \\ &\quad d(\mathcal{T}x, \mathcal{B}y) \}, \end{aligned}$$

so, all hypotheses of Theorem 1 are satisfied and 0 is the unique common fixed point of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$.

Remark 2 Note that $\mathcal{A}\mathcal{X} = \left\{ -\frac{1}{10}, 0 \right\} \not\subseteq \mathcal{S}\mathcal{X} = \left[0, \frac{1}{2} \right) \cup \left\{ \frac{3}{4} \right\}$ and $\mathcal{B}\mathcal{X} = \left[0, \frac{1}{9} \right) \cup \left\{ -\frac{1}{8} \right\} \not\subseteq \mathcal{T}\mathcal{X} = \left[0, \frac{3}{4} \right)$.

Now, we will extend constant ϖ of the above theorem to a function.

Theorem 2 *Let $(\mathcal{X}, d)$ be a metric-like space. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}, \mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be maps satisfying the following inequality*

$$(1) \quad d(\mathcal{A}x, \mathcal{B}y) \leq \varpi(d(\mathcal{T}x, \mathcal{S}y)) \max \{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), \\ d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}$$

for all $x, y \in \mathcal{X}$, where $\varpi : [0, +\infty) \rightarrow [0, 1)$ is a non-decreasing function such that $\varpi(t) < \frac{1}{2}$ for $t > 0$. If the pairs $(\mathcal{A}, \mathcal{T})$ and $(\mathcal{B}, \mathcal{S})$ are occasionally weakly $\mathcal{T}$ -biased (respectively $\mathcal{S}$ -biased) of type $(\mathcal{A})$, then, $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Again, by hypotheses, there are two points ι and j in $\mathcal{X}$ such that $\mathcal{A}\iota = \mathcal{T}\iota$ implies $d(\mathcal{T}\mathcal{T}\iota, \mathcal{A}\iota) \leq d(\mathcal{A}\mathcal{T}\iota, \mathcal{T}\iota)$ and $\mathcal{B}j = \mathcal{S}j$ implies $d(\mathcal{S}\mathcal{S}j, \mathcal{B}j) \leq d(\mathcal{B}\mathcal{S}j, \mathcal{S}j)$.

First, we are going to prove that $\mathcal{A}\iota = \mathcal{B}j$. Suppose that $\mathcal{A}\iota \neq \mathcal{B}j$, from inequality (1) we have

$$\begin{aligned} d(\mathcal{A}\iota, \mathcal{B}j) &\leq \varpi(d(\mathcal{T}\iota, \mathcal{S}j)) \max \{d(\mathcal{S}j, \mathcal{A}\iota), d(\mathcal{T}\iota, \mathcal{S}j), d(\mathcal{T}\iota, \mathcal{A}\iota), d(\mathcal{B}j, \mathcal{S}j), \\ &\quad d(\mathcal{T}\iota, \mathcal{B}j)\} \\ &= \varpi(d(\mathcal{A}\iota, \mathcal{B}j)) \max \{d(\mathcal{B}j, \mathcal{A}\iota), d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{A}\iota), d(\mathcal{B}j, \mathcal{B}j), \\ &\quad d(\mathcal{A}\iota, \mathcal{B}j)\} \\ &\leq \varpi(d(\mathcal{A}\iota, \mathcal{B}j)) \max \{d(\mathcal{B}j, \mathcal{A}\iota), d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{B}j) + d(\mathcal{B}j, \mathcal{A}\iota), \\ &\quad d(\mathcal{B}j, \mathcal{A}\iota) + d(\mathcal{A}\iota, \mathcal{B}j), d(\mathcal{A}\iota, \mathcal{B}j)\} \\ &= 2\varpi(d(\mathcal{A}\iota, \mathcal{B}j))d(\mathcal{A}\iota, \mathcal{B}j) \\ &< d(\mathcal{A}\iota, \mathcal{B}j) \end{aligned}$$

which is a contradiction, thus $\mathcal{A}\iota = \mathcal{B}j$.

Now, we assert that $\mathcal{A}\mathcal{A}\iota = \mathcal{A}\iota$. If not, then the use of condition (1) gives

$$d(\mathcal{A}\mathcal{A}\iota, \mathcal{B}j) \leq \varpi(d(\mathcal{T}\mathcal{A}\iota, \mathcal{S}j)) \max \{d(\mathcal{S}j, \mathcal{A}\mathcal{A}\iota), d(\mathcal{T}\mathcal{A}\iota, \mathcal{S}j), d(\mathcal{T}\mathcal{A}\iota, \mathcal{A}\mathcal{A}\iota), \\ d(\mathcal{B}j, \mathcal{S}j), d(\mathcal{T}\mathcal{A}\iota, \mathcal{B}j)\};$$

i.e.,

$$\begin{aligned}
d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i) &\leq \varpi(d(\mathcal{T}\mathcal{A}_i, \mathcal{A}_i)) \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad d(\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{A}_i, \mathcal{A}_i)\} \\
&= \varpi(d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)) \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad d(\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)\} \\
&\leq \varpi(d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)) \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i) + d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i) + d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i), d(\mathcal{T}\mathcal{T}_i, \mathcal{A}_i)\} \\
&\leq \varpi(d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i)) \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i) + d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), \\
&\quad 2d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), d(\mathcal{A}\mathcal{T}_i, \mathcal{T}_i)\} \\
&= \varpi(d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i)) \max\{d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i), 2d(\mathcal{A}_i, \mathcal{A}\mathcal{A}_i)\} \\
&= 2\varpi(d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i))d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i) \\
&< d(\mathcal{A}\mathcal{A}_i, \mathcal{A}_i)
\end{aligned}$$

which is a contradiction, therefore $\mathcal{A}\mathcal{A}_i = \mathcal{A}_i$, consequently, $\mathcal{T}\mathcal{A}_i = \mathcal{A}_i$.

Now, suppose that $\mathcal{B}\mathcal{B}_j \neq \mathcal{B}_j$. Using inequality (1) we obtain

$$\begin{aligned}
d(\mathcal{A}_i, \mathcal{B}\mathcal{B}_j) &\leq \varpi(d(\mathcal{T}_i, \mathcal{S}\mathcal{B}_j)) \max\{d(\mathcal{S}\mathcal{B}_j, \mathcal{A}_i), d(\mathcal{T}_i, \mathcal{S}\mathcal{B}_j), d(\mathcal{T}_i, \mathcal{A}_i), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), d(\mathcal{T}_i, \mathcal{B}\mathcal{B}_j)\};
\end{aligned}$$

i.e.,

$$\begin{aligned}
d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j) &\leq \varpi(d(\mathcal{B}_j, \mathcal{S}\mathcal{B}_j)) \max\{d(\mathcal{S}\mathcal{B}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), d(\mathcal{B}_j, \mathcal{B}_j), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{B}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&= \varpi(d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j)) \max\{d(\mathcal{S}\mathcal{S}_j, \mathcal{B}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}_j), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&\leq \varpi(d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j)) \max\{d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j) + d(\mathcal{S}\mathcal{S}_j, \mathcal{B}_j), \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j) + d(\mathcal{B}_j, \mathcal{S}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&\leq \varpi(d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j)) \max\{d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j), 2d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j), d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j) \\
&\quad d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j) + d(\mathcal{S}_j, \mathcal{B}\mathcal{S}_j)\} \\
&= \varpi(d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)) \max\{d(\mathcal{B}\mathcal{B}_j, \mathcal{B}_j), 2d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)\} \\
&= 2\varpi(d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j))d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j) \\
&< d(\mathcal{B}_j, \mathcal{B}\mathcal{B}_j)
\end{aligned}$$

this contradiction implies that $\mathcal{B}\mathcal{B}_j = \mathcal{B}_j$ and so $\mathcal{S}\mathcal{B}_j = \mathcal{B}_j$; i.e., $\mathcal{B}\mathcal{A}_i = \mathcal{A}_i$ and $\mathcal{S}\mathcal{A}_i = \mathcal{A}_i$. Put $\mathcal{A}_i = \mathcal{S}_i = \mathcal{B}_j = \mathcal{T}_j = \mu$, therefore μ is a common fixed point of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$.

Finally, let μ and ν be two distinct common fixed points of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$. Then, $\mu = \mathcal{A}\mu = \mathcal{B}\mu = \mathcal{S}\mu = \mathcal{T}\mu$ and $\nu = \mathcal{A}\nu = \mathcal{B}\nu = \mathcal{S}\nu = \mathcal{T}\nu$. From (1) we

have

$$d(\mathcal{A}\nu, \mathcal{B}\mu) \leq \varpi(d(\mathcal{T}\nu, \mathcal{S}\mu)) \max\{d(\mathcal{S}\mu, \mathcal{A}\nu), d(\mathcal{T}\nu, \mathcal{S}\mu), d(\mathcal{T}\nu, \mathcal{A}\nu), d(\mathcal{B}\mu, \mathcal{S}\mu), d(\mathcal{T}\nu, \mathcal{B}\mu)\};$$

i.e.,

$$\begin{aligned} d(\nu, \mu) &\leq \varpi(d(\nu, \mu)) \max\{d(\mu, \nu), d(\nu, \mu), d(\nu, \nu), d(\mu, \mu), d(\nu, \mu)\} \\ &\leq \varpi(d(\nu, \mu)) \max\{d(\mu, \nu), d(\nu, \mu), d(\nu, \mu) + d(\mu, \nu), \\ &\quad d(\mu, \nu) + d(\nu, \mu), d(\nu, \mu)\} \\ &= \varpi(d(\nu, \mu)) \max\{d(\mu, \nu), d(\nu, \mu), 2d(\nu, \mu), 2d(\mu, \nu), d(\nu, \mu)\} \\ &= 2\varpi(d(\nu, \mu))d(\nu, \mu) \\ &< d(\nu, \mu) \end{aligned}$$

which is a contradiction, thus $\nu = \mu$.

Again, we give an example which illustrates our above theorem.

Example 2 Let $\mathcal{X} = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with the metric-like $d(x, y) = |x| + |y|$. Define

$$\begin{aligned} \mathcal{A}x &= \begin{cases} 0 & \text{if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{1000} & \text{if } x \in \left(0, \frac{\pi}{2}\right) \end{cases}, & \mathcal{B}x &= \begin{cases} 0 & \text{if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{2000} & \text{if } x \in \left(0, \frac{\pi}{2}\right) \end{cases}, \\ \mathcal{T}x &= \begin{cases} x & \text{if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{4} & \text{if } x \in \left(0, \frac{\pi}{2}\right) \end{cases}, & \mathcal{S}x &= \begin{cases} x & \text{if } x \in \left(-\frac{\pi}{2}, 0\right] \\ \frac{\pi}{3} & \text{if } x \in \left(0, \frac{\pi}{2}\right) \end{cases}. \end{aligned}$$

First, it is clear to see that $\mathcal{A}$ and $\mathcal{T}$ are occasionally weakly $\mathcal{T}$ -biased of type $(\mathcal{A})$ and $\mathcal{B}$ and $\mathcal{S}$ are occasionally weakly $\mathcal{S}$ -biased of type $(\mathcal{A})$. Take $\varpi(t) = \frac{|\sin t|}{3}$, we get

1. for $x, y \in \left(-\frac{\pi}{2}, 0\right]$, we have $\mathcal{A}x = 0$, $\mathcal{B}y = 0$, $\mathcal{T}x = x$, $\mathcal{S}y = y$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= 0 \\ &\leq \frac{1}{3} |\sin(|x| + |y|)| \max\{|y|, |x| + |y|, |x|\} \\ &= \frac{1}{3} (|x| + |y|) |\sin(|x| + |y|)| \\ &= \varpi(d(\mathcal{T}x, \mathcal{S}y)) \max\{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), \\ &\quad d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}, \end{aligned}$$

2. for $x, y \in \left(0, \frac{\pi}{2}\right)$, we have $\mathcal{A}x = \frac{\pi}{1000}$, $\mathcal{B}y = \frac{\pi}{2000}$, $\mathcal{T}x = \frac{\pi}{4}$, $\mathcal{S}y = \frac{\pi}{3}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \frac{3\pi}{2000} \\ &\leq \frac{1}{3} \left| \sin \left(\frac{7\pi}{12} \right) \right| \max \left\{ \frac{1003\pi}{3000}, \frac{7\pi}{12}, \frac{251\pi}{1000}, \frac{2003\pi}{6000}, \frac{501\pi}{2000} \right\} \\ &= \frac{7\pi}{36} \left| \sin \left(\frac{7\pi}{12} \right) \right| \\ &= \varpi(d(\mathcal{T}x, \mathcal{S}y)) \max \{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), \\ &\quad d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}, \end{aligned}$$

3. for $x \in \left(-\frac{\pi}{2}, 0\right]$, $y \in \left(0, \frac{\pi}{2}\right)$ we have $\mathcal{A}x = 0$, $\mathcal{B}y = \frac{\pi}{2000}$, $\mathcal{T}x = x$, $\mathcal{S}y = \frac{\pi}{3}$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \frac{\pi}{2000} \\ &\leq \frac{1}{3} \left| \sin \left(|x| + \frac{\pi}{3} \right) \right| \max \left\{ \frac{\pi}{3}, |x| + \frac{\pi}{3}, |x|, \frac{2003\pi}{6000}, |x| + \frac{\pi}{2000} \right\} \\ &= \varpi(d(\mathcal{T}x, \mathcal{S}y)) \max \{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), \\ &\quad d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}, \end{aligned}$$

4. for $x \in \left(0, \frac{\pi}{2}\right)$, $y \in \left(-\frac{\pi}{2}, 0\right]$, we have $\mathcal{A}x = \frac{\pi}{1000}$, $\mathcal{B}y = 0$, $\mathcal{T}x = \frac{\pi}{4}$, $\mathcal{S}y = y$ and

$$\begin{aligned} d(\mathcal{A}x, \mathcal{B}y) &= \frac{\pi}{1000} \\ &\leq \frac{1}{3} \sin \left(|y| + \frac{\pi}{4} \right) \max \left\{ |y| + \frac{\pi}{1000}, |y| + \frac{\pi}{4}, \frac{1004\pi}{4000}, |y|, \frac{\pi}{4} \right\} \\ &= \varpi(d(\mathcal{T}x, \mathcal{S}y)) \max \{d(\mathcal{S}y, \mathcal{A}x), d(\mathcal{T}x, \mathcal{S}y), d(\mathcal{T}x, \mathcal{A}x), \\ &\quad d(\mathcal{B}y, \mathcal{S}y), d(\mathcal{T}x, \mathcal{B}y)\}, \end{aligned}$$

so, all hypotheses of Theorem 2 are satisfied and 0 is the unique common fixed point of maps $\mathcal{A}$, $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{S}$.

Remark 3 Note that $\mathcal{A}\mathcal{X} = \left\{0, \frac{\pi}{1000}\right\} \not\subseteq \mathcal{S}\mathcal{X} = \left(-\frac{\pi}{2}, 0\right] \cup \left\{\frac{\pi}{3}\right\}$ and $\mathcal{B}\mathcal{X} = \left\{0, \frac{\pi}{2000}\right\} \not\subseteq \mathcal{T}\mathcal{X} = \left(-\frac{\pi}{2}, 0\right] \cup \left\{\frac{\pi}{4}\right\}$.

4 Conclusion

To summarize, we have established two theorems in a metric-like space which improve and extend some results on the existence and uniqueness of fixed and common fixed points in metric as well as partial and dislocated metric spaces; among them,

we can cite Theorem 3.6 of [1], Theorem 1.1, Theorem 1.4 and Corollary 1.7 of [6], Theorem 1 and Theorem 3 of [10], Theorem 2 and Theorem 4 of [11], Theorem 2.1 of [4], Theorem 2.1 of [12], Theorem 2.2 and Theorem 2.7 of [7], Theorem 3.1 of [8] and so on. Further, we justified our results with suitable examples which clearly show the advantage of our present results over the cited papers and may be other given results. Therefore, our study is an improvement and extension of the cited articles and other results in the fixed point theory's literature.

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