

Research paper

Determining the parameters of the parabolic approximation of the sawlog based on Nilson's sawlog volume formula

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Abstract. Although parabolic approximations of sawlogs have become widespread, in current specialized literature, there are no parameter values of such approximations or dependencies for their calculation that could be used to analyze the influence of log dimensions on the volume yield of edged boards. The developed technique for determining the parameters of a truncated paraboloid, which approximates a sawlog, is based on Nilson's formula for estimating its volume. Based on this, the equation of the generatrix of the truncated paraboloid as a function of the top diameter and the length of the sawlog (considering the species) and the formula for calculating its taper (in particular, total and main) are derived. The error of approximating sawlogs with a paraboloid was determined by comparing the taper estimate with literature data on the taper of Scots pine (*Pinus sylvestris* L.), Norway spruce (*Picea abies* (L.) H. Karst.), and common oak (*Quercus robur* L.) sawlogs in Ukraine, Sweden, and Finland. This error ranges from -36% to -9%. Using the method of parametric identification of taper distributions based on literature data on their statistical characteristics (unbounded metalog distribution with four parameters and normal distribution were used), the probability that a sawlog has a taper smaller than its estimate was calculated. These probabilities range from 20% to 36%. The third quartiles of the sawlogs' taper distributions are, respectively, between 200% and 150% of the taper estimate. Therefore, the obtained formulas for determining the parameters of the approximating paraboloid led to a rather conservative estimate of the taper (however, acceptable for use in log sawing planning), which confirms their suitability for analyzing the influence of log dimensions on the yield of edged boards.

Key words: sawlog, paraboloid, generatrix, taper, Scots pine, Norway spruce, common oak.

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Introduction

The studies (Baltrušaitis & Pranckevičienė, 2001; Brandstetter & Campean, 2019) showed a significant effect of the sawlogs' taper on the volumetric yield of timber. Therefore, when analyzing the influence of log dimensions on the volumetric yield of boards, a mathematical model describing the sawlogs' shape is needed. For this, both the results of a computer scan of a representative sample of sawlogs or stems from a certain region can be used (for example, the Swedish Pine Stem Bank is a database of about 200 Scots pine trees from all over Sweden), and sawlog approximations by regular geometric bodies (used in commercial software for optimizing patterns for sawing logs of given dimensions). Although a

considerable amount of research has been devoted to approximations of stem profiles, there are few published results of studies on a stems' taper or a sawlogs' taper.

The article (Bilous *et al.*, 2021) presents the results of a study of the taper of Scots pine and common oak sawlogs, which are the most common in Ukraine. Since the purpose of the study was to ensure Ukraine's transition from the previously used measurement of the log volume by the top diameter to the measurement by the mid-point diameter (as adopted in the European Union (EU)), the taper of the sawlog was calculated from the top end to the log middle (mid-log taper). Based on the stems' measurements, their profiles' approximations were determined, and their bucking into sawlogs was modelled. Using the

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regression analysis method, it was determined that the average value of the mid-log taper is $0.845 \text{ cm}\cdot\text{m}^{-1}$ for pine and $1.19 \text{ cm}\cdot\text{m}^{-1}$ for oak, and the regression dependencies of the mid-log taper on the mid-point diameter of the sawlog are given.

In the study (Larsen, 2017), a simple approximation of the shape of the stem, designed to determine its volume, was carried out. The stem is divided into three sections, the lower two are truncated cones, and the upper one is a cone. For 34 tree species common in southern United States, the average values of the taper of the two lower sections are given.

Publications (Chiorescu & Grönlund, 2001; Chiorescu & Grönlund, 2004a; Chiorescu et al., 2003; Chiorescu & Grönlund, 2004b; Pyörälä et al., 2019) contain statistical characteristics of dimensions and tapers of sawlogs obtained during the improvement of log sawing technologies.

One of the traditional ways of representing the shape of sawlogs is the phenomenological model, which is defined by Kunze's equation and describes a cylinder, a paraboloid, a cone, and a neiloid with integer power exponent from 0 to 3 (Inoue et al., 2021):

$$r^2(x) = a x^b, \quad (1)$$

where r is the stem radius, m; a is the coefficient; x is the position from the tip of the geometric body, m; b is the power exponent.

As noted in Scaling Manual (2011), sawlogs approach truncated paraboloid, cone, and neiloid shapes.

Carron (1968) states that a truncated paraboloid can be used to describe the following types of logs:

- the part of the stem above the butt-swell, extending into the living crown for some distance (common in most coniferous trees and young broad-leaved species with a single, continuous from ground to tip);
- the part of the stem above the butt-swell, reaching up to the crown break (typical in most broad-leaved and some coniferous species);
- the sawlog, sourced from any part of the main stem of a tree.

West (2009) concludes from earlier studies of the stem shapes, according to which the main part of the tree stem (above the butt-swell and at least to the lower part of the crown) has the shape of a paraboloid. Since this part of the stem is most often used for timber, Smalian's and Huber's formulas have become widely used, giving an unbiased estimate of volume if the sawlog is paraboloid in shape.

It is worth noting that Huber's formula is used to determine the volume of sawlogs in the EU and Ukraine, and Smalian's formula is the official rule for measuring sawlogs in the province of British Columbia (Canada) (Scaling Manual, 2011).

As stated in Scaling Manual (2011), "Normally, the difference between the top and butt diameters of typical logs is less than 30%, and Smalian's formula is an accurate reflection of the true log volume".

In Estonia, Nilson's sawlog volume formula was developed (Jänes, 2001). The structure of Nilson's formula is similar to the phenomenological formulas of multiple regression for determining the stem volume (West, 2009).

Thus, the truncated paraboloid of rotation is a common approximation of the shape of sawlogs. Still, there is no information in current specialized literature about the values of the parameters of corresponding paraboloids or the relationship between their length and diameters of the ends. Therefore, there is a need to determine the parameters of such a parabolic approximation for modelling the sawing of logs.

The article aims to develop a technique for approximating the shape of a sawlog by a truncated paraboloid depending on the species, diameter, and length of the log.

Materials and Methods

Theoretical basis

Before sawing the logs according to the patterns developed for sawlogs of certain size, they are sorted by the top diameter. This applies both to the physical sorting of logs using appropriate equipment, and to their virtual sorting by choosing a certain pattern of sawing the log depending on its top diameter. Therefore, we will determine the parameters of the approximation of the shape of the sawlog depending on its top diameter.

To determine the dependence of the parameters of the paraboloid, which approximates the shape of a sawlog, on its dimensions and tree species, we will use Nilson's formula developed for Estonian forests (Jänes, 2001):

$$V_{\text{Log}}(d_{\text{cm}}, L_{\text{dm}}) = \frac{d_{\text{cm}}^2 L_{\text{dm}} (a_1 + a_2 L_{\text{dm}}) + a_3 L_{\text{dm}}^2}{10000}, \quad (2)$$

where V_{Log} is the log volume, m^3 ; d_{cm} is the top diameter of the log, cm; L_{dm} is the log length, dm; a_1 , a_2 , and a_3 are coefficients.

On the other hand, the sawlog

approximation by a truncated paraboloid is widely recognized. Thus, as a combination of these methodological approaches, the following technique of constructing an approximating paraboloid is proposed:

- for a certain tree species, the diameter of the top end of the log and its length, we calculate the volume of the log using Nilson's formula;
- we approximate the log with a truncated paraboloid using the same values of the length of the log, the diameter of the top end, and the volume;
- based on the above parameters of the truncated paraboloid, we calculate its diameter of the butt end;
- we determine the equation of the generatrix of the truncated paraboloid.

For convenience of further mathematical transformations, we introduce the measurement of linear dimensions in meters in (2):

$$V_{Log}(d, L) = d^2 L (A_1 + A_2 L) + A_3 L^2, \quad (3)$$

$$A_1 = 10 a_1, \quad (4)$$

$$A_2 = 100 a_2, \quad (5)$$

$$A_3 = \frac{a_3}{100}, \quad (6)$$

where d is the top diameter, m; L is the log length, m; A_1 , A_2 , and A_3 are coefficients.

To determine the volume of the sawlog approximation, we use the formula for a truncated paraboloid volume, which corresponds to Smalian's formula:

$$V(d, D, L) = \frac{\pi}{8} (d^2 + D^2) L, \quad (7)$$

where V is the volume of the truncated paraboloid, m³; D is the butt diameter, m.

We equate the volume of a sawlog (3) to the volume of a truncated paraboloid (7) with the same values of length, top diameter, and a certain (unknown at this stage) butt diameter:

$$V_{Log}(d, L) = V(d, D, L), \quad (8)$$

$$d^2 L (A_1 + A_2 L) + A_3 L^2 = \frac{\pi}{8} (d^2 + D^2) L. \quad (9)$$

Solving the equation (9) relative to the butt diameter, we get:

$$D = \sqrt{\frac{8}{\pi} (d^2 (A_1 + A_2 L) + A_3 L) - d^2}. \quad (10)$$

We derive the equation of the generatrix of the truncated paraboloid, which approximates the sawlog. We define the coordinate system so that the X-axis coincides with the axis of the truncated paraboloid and is directed at its increase. The origin of the coordinate system is located at the intersection of the top end of the paraboloid with its axis.

So, we will search for the equation of the generatrix of the truncated paraboloid of rotation in the form of a parabola:

$$r(x) = a\sqrt{c + x}, \quad (11)$$

where r is the log radius, m; a is the coefficient; c is the displacement of the vertex of a parabola relative to the top end of the paraboloid, m.

To determine the parameters of the paraboloid, which are included in the generatrix equation (11), we create a system of equations, equating the radii of the ends of the truncated paraboloid (11) and the radii of the ends of the sawlog:

$$\begin{cases} r(0) = \frac{d}{2} \\ r(L) = \frac{D}{2} \end{cases} \quad (12)$$

Solving the system of equations (12) considering (10) and (11), we obtain:

$$a = \sqrt{\frac{d^2 (A_1 + A_2 L - \pi/4) + A_3 L}{\pi L/2}}, \quad (13)$$

$$c = \frac{(\pi/8) d^2 L}{d^2 (A_1 + A_2 L - \pi/4) + A_3 L}. \quad (14)$$

This makes it possible to denote the equation of the generatrix of approximating the truncated paraboloid in terms of the length of the sawlog, the top diameter, and the parameters of Nilson's sawlog volume formula:

$$r(d, L, x) = \sqrt{\frac{d^2}{4} + \frac{d^2 (A_1 + A_2 L - \pi/4) + A_3 L}{\pi L/2}} x. \quad (15)$$

To further compare the obtained parabolic approximation of the shape of the sawlog (15) with the results of other studies, we will determine the truncated paraboloid's taper according to the rules applied for the sawlogs. We start by defining the total taper of the approximating paraboloid:

$$s_T(d, L) = \frac{D(d, L) - d}{L}, \quad (16)$$

$$s_T(d, L) = \frac{\sqrt{\frac{8}{\pi}(d^2(A_1 + A_2L) + A_3L) - d^2 - d}}{L}, \quad (17)$$

where S_T is the total taper, $\text{m}\cdot\text{m}^{-1}$.

Now we will find the mid-point diameter and determine the taper between the top and mid-point diameter of the log (mid-log taper):

$$D_M(d, L) = \sqrt{\frac{4V_{Log}(d, L)}{\pi L}}, \quad (18)$$

$$D_M(d, L) = \frac{2}{\sqrt{\pi}} \sqrt{d^2(A_1 + A_2L) + A_3L}, \quad (19)$$

$$s_M(d, L) = \frac{D_M(d, L) - d}{L/2}, \quad (20)$$

$$s_M(d, L) = \frac{\frac{4}{\sqrt{\pi}} \sqrt{d^2(A_1 + A_2L) + A_3L} - 2d}{L}, \quad (21)$$

where D_M is the mid-point diameter, m; S_M is the mid-log taper, $\text{m}\cdot\text{m}^{-1}$.

We also calculate the taper measured at 2/3 of the length of the sawlog from its top end, using (15):

$$\begin{aligned} D_{23}(d, L) &= 2r\left(d, L, \frac{2}{3}L\right) = \\ &= 2 \sqrt{\frac{d^2}{4} + \frac{d^2(A_1 + A_2L - \pi/4) + A_3L}{\pi L/2} \cdot \frac{2}{3}L}, \end{aligned} \quad (22)$$

$$s_{23}(d, L) = \frac{D_{23}(d, L) - d}{2L/3}, \quad (23)$$

$$s_{23}(d, L) = \frac{\sqrt{d^2 + \frac{16}{3\pi}(d^2(A_1 + A_2L - \pi/4) + A_3L) - d}}{2L/3}, \quad (24)$$

where D_{23} is the diameter of the sawlog, measured at 2/3 of its length from the top end, m; S_{23} is the taper, measured at 2/3 of the length of the log from the top end, $\text{m}\cdot\text{m}^{-1}$.

So, based on Nilson's formula, the dependencies of the parameters of the paraboloid generatrix (11), which approximates a sawlog, on the tree species, the top diameter and the length of the sawlog (13, 14) were obtained. For comparison with the results of other studies, dependencies for determining the taper of the approximating paraboloid were also obtained (17, 21, 24).

Materials

To evaluate the validity of the developed parabolic sawlog approximation, we will carry out a multiple case study using statistical characteristics of the dimensions and the taper of Scots pine, Norway spruce, and common oak sawlogs from Ukraine, Sweden, and Finland, which are contained in literary sources (Bilous et al., 2021; Chiorescu & Grönlund, 2001; Chiorescu & Grönlund, 2004a; Chiorescu et al., 2003; Chiorescu & Grönlund, 2004b; Pyörälä et al., 2019). Their brief description is given in Table 1.

Table 1. Materials.

Case	Materials	Statistical characteristics
Case 1 (Bilous et al., 2021)	105 Scots pine trees from the Polissya climate zone (Ukraine)	Linear regression dependence of the mid-log taper on the mid-point diameter of the sawlog
Case 2 (Bilous et al., 2021)	149 common oak trees from the Forest-Steppe climate zone (Ukraine)	The same
Case 3 (Chiorescu & Grönlund, 2001)	625 Scots pine sawlogs (the Swedish Pine Stem Bank)	Numerical characteristics of the dimensions and sawlogs' taper (average, standard deviation, and skewness)
Case 4 (Chiorescu et al., 2003)	3000 sawlogs (a mix of Scots pine and Norway spruce) from southern Sweden	The same
Case 5 (Chiorescu & Grönlund, 2004a)	773 Scots pine sawlogs from northern Sweden	The same

Table 1. Continues.

Case	Materials	Statistical characteristics
Case 6 (Chiorescu & Grönlund, 2004)	506 Scots pine sawlogs from southern Sweden	The same
Case 7 (Chiorescu & Grönlund, 2004b)	2665 Norway spruce sawlogs from southern Sweden	The same
Case 8 (Pyörälä <i>et al.</i> , 2019)	42 Scots pine middle sawlogs from southern Finland	Numerical characteristics of the dimensions and taper of sawlogs (mean, standard deviation)
Case 9 (Pyörälä <i>et al.</i> , 2019)	52 Scots pine butt sawlogs from southern Finland	The same

Statistical analysis

We will determine the method of comparing the obtained parabolic approximation of sawlogs with the regression dependence of the mid-log taper of Scots pine and common oak sawlogs on the mid-point diameter (Case 1, Case 2). In the study (Bilous *et al.*, 2021), these dependencies have the following structure:

$$s_M(D_M) = s_0 + kD_M, \quad (25)$$

where s_0 is the intercept of the mid-log taper, $\text{m}\cdot\text{m}^{-1}$; k is the slope of the mid-log taper, m^{-1} .

For the convenience of comparing the taper estimate (21) with the regression dependence (25), we transform it into a function of the top diameter:

$$s_M(d) = \frac{s_0 + k d}{1 - k L/2}. \quad (26)$$

The dimensions of the sawlogs for which we make a comparison will be determined as follows:

- the length of the log is equal to the average value calculated according to the distribution of merchantable logs by length harvested in Ukraine in 2018 (Bilous *et al.*, 2021);
- the upper diameter is calculated as the middle of the interval of top diameters, the share of which was revealed to be 75–80% of the total harvested merchantable wood volume in Ukraine in 2018 (Bilous *et al.*, 2021).

In the intervals of the top diameters and lengths of the sawlogs mentioned above, we will construct the graphical dependencies of their mid-log taper (21) on the dimensions of the log.

We will calculate the estimate of the mid-log taper of the sawlogs using formula (21) and compare it with the mid-log taper calculated using formula (26), determining the relative error of the estimate.

We determine the method of comparing the obtained parabolic approximation of sawlogs (15) with the numerical characteristics of the dimensions and taper of sawlogs (Case 3 – Case 9). We will compare the taper estimates calculated by formulas (17, 21, 24) at the average values of the top diameter and length of the log with the appropriate average taper values. Such a comparison can be made by approximately considering the functions (17, 21, 24) as linear relative to the top diameter and length of the log in the value ranges of these arguments. Using the properties of functions of random variables (Soubra & Bastidas-Arteaga, 2014) under the specified conditions, we obtain:

$$s(\mathbf{M}[d], \mathbf{M}[L]) \approx \mathbf{M}[s], \quad (27)$$

where $\mathbf{M}$ is the mathematical expectation; s is the taper estimation, $\text{m}\cdot\text{m}^{-1}$.

When using the data (Chiorescu & Grönlund, 2004a) (Case 5), the average top diameter will be calculated using the average mid-point diameter, the average total taper, and the average length of the sawlog.

The above considerations were related to comparing the taper estimate with their average value. We then develop the technique of considering the probability distribution of the taper of sawlogs when comparing data from literary sources (Case 3 – Case 9) and estimates of the taper of logs according to formulas (17, 21, 24).

Considering the sawing of logs with the production of edged boards, we pay attention

to the case when the actual taper of the sawlog is less than its value adopted during the development of the log sawing pattern. In this case, there is a risk that the boards, at least partially located outside the cylindrical part of the log, will not be produced. Therefore, as a measure of the specified production risk, we will calculate the probability that the sawlogs' taper is less than its estimate according to formulas (16, 20, 23). We will calculate this probability as the taper quantile corresponding to estimating the taper of the sawlog.

To do this, we will obtain approximations of the distributions of the sawlogs' taper based on its numerical characteristics. First, we choose an approximate distribution. We have to take into account that the mean value, standard deviation, and skewness

of the taper are known (relevant data are given in Case 3 – Case 7). In this instance, the number of parameters of the approximating distribution, which will have the same values of these numerical characteristics, should be at least three. We will use the metalog distribution, which can have an arbitrary number of parameters and is used to approximate empirical distributions in various fields (Keelin, 2016). Namely, we will use the unbounded metalog distribution of the fourth order, which can be used to approximate the asymmetric unimodal distribution. We will calculate the value of the parameters of the approximating distribution by solving a system of three equations, each expressing one of the numerical characteristics of the approximating distribution through its parameters (Keelin, 2016):

$$\begin{cases} m = \alpha_1 + \frac{\alpha_3}{2} \\ \sigma^2 = \pi^2 \frac{\alpha_2^2}{3} + \frac{\alpha_3^2}{12} + \pi^2 \frac{\alpha_3^2}{36} + \alpha_2 \alpha_4 + \frac{\alpha_4^2}{12} \\ S = \pi^2 \alpha_2^2 \alpha_3 + \pi^2 \frac{\alpha_3^3}{24} + \frac{\alpha_2 \alpha_3 \alpha_4}{2} + \pi^2 \frac{\alpha_2 \alpha_3 \alpha_4}{6} + \frac{\alpha_3 \alpha_4^2}{8} \end{cases}, \quad (28)$$

where m is the average; α_1 – α_4 are the parameters of the metalog distribution; σ is the standard deviation; S is the skewness.

The solution of the system of equations (28) will be carried out by the search method, considering the constraints (Keelin, 2016):

$$\alpha_2 > 0, \quad (29)$$

$$\frac{|\alpha_3|}{\alpha_2} < 1.66711. \quad (30)$$

Next, using the obtained parameters of the metalog distribution, we will determine its quantile function (Keelin, 2016):

$$M_4(y) = \alpha_1 + \alpha_2 \ln \frac{y}{1-y} + \alpha_3 (y - 0.5) \ln \frac{y}{1-y} + \alpha_4 (y - 0.5), \quad (31)$$

where M_4 is the quantile function of the metalog distribution of the fourth order; y is the probability.

So, as it follows from (31), the conditions for the monotonicity of the quantile function are, in addition to (29), also the following constraints:

$$\alpha_3 > 0, \alpha_4 > 0. \quad (32, 33)$$

Constraints (32, 33) will be applied additionally to (29, 30) when solving the system of equations (28). We start by considering the variation of the taper of sawlogs' mean sizes as

the same as the variation of the taper of all sawlogs of the given case. Next, we will determine the probability that the sawlog taper is less than its estimate according to formulas (17, 21), solving equations numerically:

$$M_4(Q) = s(M[d], M[L]), \quad (34)$$

where Q is the probability that the sawlog taper is less than its estimate.

Since the variation of the taper of sawlogs of mean sizes is smaller than the variation in the taper of all sawlogs of a given case, the resulting value of this probability is its upper estimate.

In the study (Pyörälä et al., 2019) (Case 8, Case 9), the numerical characteristics of the sawlog taper are given no higher than the second order, so the distribution of the taper will be approximated by the normal distribution. Since in this study, the measurements of the dimensions of sawlogs were carried out twice; we will average their dimensions to assess the taper and average the mean values and variances of the taper when determining the parameters of their normal distribution. The probability that the taper of the sawlog is less than its estimate according to formula (23) is determined by the formula:

$$Q = \Phi(m, \sigma, s_{23}(M[d], M[L])). \quad (35)$$

We will also calculate the first and third quartiles of the approximating distribution, expressing them through the taper estimate. The range obtained in such a way is limited to:

- a relatively small taper (however, edged boards can be made within its limits in the vast majority of cases);
- a relatively large taper (however, edged boards can be produced within its limits relatively rarely)

We will calculate the corresponding quartiles of the taper's distribution according to the following formulas for Case 3 – Case 7 and the equations for Case 8 and Case 9, respectively:

$$s_{0.25} = M_4(0.25), s_{0.75} = M_4(0.75), \quad (36, 37)$$

$$\Phi(m, \sigma, s_{0.25}) = 0.25, \Phi(m, \sigma, s_{0.75}) = 0.75, \quad (38, 39)$$

where $s_{0.25}$, $s_{0.75}$ are, respectively, the first and third quartiles of the approximate distribution of the sawlogs' taper.

For the convenience of comparing the first and third quartiles with the estimate of the sawlogs' taper, we will calculate the ratio of these quartiles and the specified estimate.

Results and Discussion

We compare the results of the parabolic

approximation of sawlogs (15) and the statistical characteristics of their dimensions with (Bilous *et al.*, 2021) (Case 1, Case 2). Namely, we compare the results of calculations of the mid-log taper using the regression dependence on the mid-point diameter (25) and the estimation of their mid-log taper according to the formula (21). After the transformation (26), regression dependencies (25) take the following forms for Scots pine and common oak sawlogs, respectively:

$$s_M(d) = \frac{0.915 - 0.00281 d}{1 + 0.001405 L}, \quad (40)$$

$$s_M(d) = \frac{1.02 + 0.00575 d}{1 - 0.002875 L}. \quad (41)$$

The dimensions of the sawlogs (average values of the top diameter and length), for which we will make a comparison, were calculated according to the above method. The results of calculations of the mid-log taper according to the formulas (40, 41), estimation of the mid-log taper according to the formula (21), and the relative errors of these estimates (assuming that the values calculated by formulas (40, 41) are true). When approximating the common oak sawlog, the values of the coefficients given in (Jänes, 2001) for birch and other broad-leaved trees were used.

Table 2. Average dimensions of sawlogs and estimates of their tapers.

Case	Type of taper	Taper (mm·m ⁻¹)	Taper estimate (mm·m ⁻¹)	The relative error of the taper estimate
Case 1	Mid-log taper (inside the bark)	8.5*	7.3	-14%
Case 2	Mid-log taper (inside the bark)	12.0*	10.1	-15%
Case 3	Total taper	9.18**	6.6	-28%
Case 4	Total taper (inside the bark)	9**	7.6	-15%
Case 5	Total taper (inside the bark)	11**	7.0	-36%
Case 6	Mid-log taper (inside the bark)	9**	6.9	-23%
Case 7	Mid-log taper (inside the bark)	9**	7.7	-15%
Case 8	Taper at 2/3 of the length (over the bark)	7.3**	6.5	-9%
Case 9	Taper at 2/3 of the length (over the bark)	8.0**	6.7	-18%

* – calculated using regression dependence

** – is the average value of the taper

Graphs of the dependencies of the mid-log taper (21), constructed for Scots pine and common oak sawlogs, are shown in Figures 1–2.

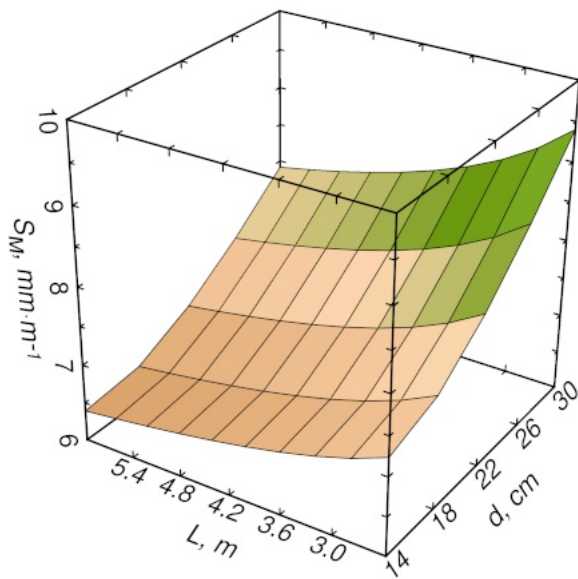


Figure 1. Graph of dependence of the mid-log taper on the top diameter and length of the pine log.

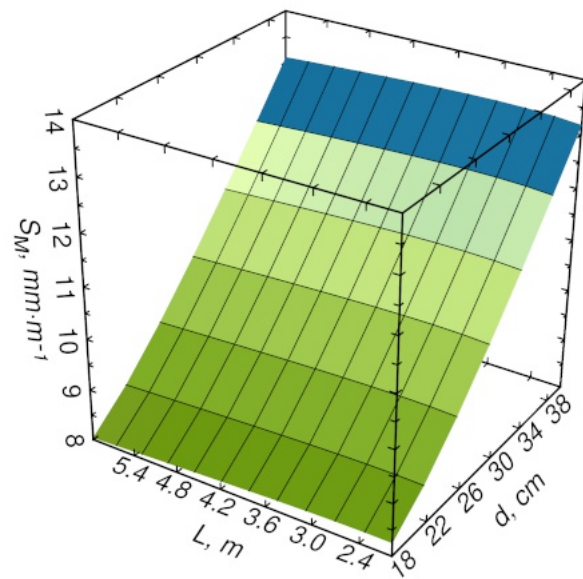


Figure 2. Graph of dependence of the mid-log taper on the top diameter and length of the oak log.

We compare the results of the parabolic approximation and the numerical characteristics of the taper of sawlogs from Case 3 to Case 9. Table 2 shows the taper estimates according to formulas (17, 21, 24) and the relative errors of these estimates. The largest (modulo) relative error of the taper estimate calculated by formulas (17, 21, 24) for Case 1 – Case 9 is -36% and occurs for a mix of Scots pine sawlogs of three classes of diameters (14.2÷14.7 cm, 22.2÷24.3 cm, and 29.9÷54.0 cm)

in approximately the same quantities in northern Sweden (Case 5). The taper estimate's smallest (modulo) relative error is -9% (Case 8).

For Case 3 – Case 7, the parameters of the approximation of the log taper distribution by the unbounded metalog distribution with four parameters are determined. The values of the distribution parameters and their corresponding numerical characteristics, calculated according to formulas (28), are given in Table 3.

Table 3. Parameters of the approximation of the distribution of the sawlogs' taper by the metalog distribution and numerical characteristics of the distribution.

Case	α_1	α_2	α_3	α_4	m (mm·m ⁻¹)	σ (mm·m ⁻¹)	S
Case 3	9.16	0.044	0.0360	10.129	9.18	3.00	1.07
Case 4	8.99	2.139	0.0151	0.438	9.00	4.00	0.70
Case 5	10.99	2.636	0.0103	0.792	11.00	5.00	0.73
Case 6	8.99	2.408	0.0171	0.480	9.00	4.50	1.00
Case 7	8.99	2.084	0.0133	0.795	9.00	4.00	0.60

For Case 8 and Case 9, the distribution of the sawlogs' taper is approximated by a normal distribution, the parameters of which are determined by averaging the mean values and variances of the taper of two series of measurements. The values of the distribution parameters are given in Table 4.

Table 4. Parameters of the approximation of the distribution of the sawlogs' taper by a normal distribution.

Case	m (mm·m ⁻¹)	σ (mm·m ⁻¹)
Case 8	7.3	1.6
Case 9	8.0	2.4

Based on the obtained approximations of the distributions of the sawlogs' taper according to formulas (34, 35), the probability that the sawlog has a taper smaller than its estimate (17, 21, 24) was calculated. These probabilities are shown in Table 5. Table 5 also contains the first and third quartiles of the sawlogs' taper distributions expressed by the taper estimate. Thus, the taper estimates according to formulas (17, 21, 24) are quite conservative.

Table 5. Probability value that a sawlog has a taper less than its estimate and the first and third quartiles of the taper distribution.

Case	Q	$s_{0.25}$	$s_{0.75}$
Case 3	25%	6.6 mm·m ⁻¹ (100% s_T)	11.8 mm·m ⁻¹ (179% s_T)
Case 4	35%	6.5 mm·m ⁻¹ (86% s_T)	11.5 mm·m ⁻¹ (150% s_T)
Case 5	20%	7.9 mm·m ⁻¹ (112% s_T)	14.1 mm·m ⁻¹ (200% s_T)
Case 6	31%	6.2 mm·m ⁻¹ (90% s_M)	11.8 mm·m ⁻¹ (170% s_M)
Case 7	36%	6.5 mm·m ⁻¹ (85% s_M)	11.5 mm·m ⁻¹ (150% s_M)
Case 8	34%	6,3 mm·m ⁻¹ (94% s_{23})	8.4 mm·m ⁻¹ (126% s_{23})
Case 9	27%	6,4 mm·m ⁻¹ (98% s_{23})	9.6 mm·m ⁻¹ (146% s_{23})

However, according to the authors, they are close to the rational taper values, which are reasonable to use when developing sawing patterns for sawlogs (due to the use of larger taper values (for example, medians), although will allow the development of sawing patterns with a higher yield of timber, will lead to a decrease in the probability of successful implementation of such patterns).

The reasons for the differences in the estimates of the sawlogs' taper calculated according to formulas (17, 21, 24) from the statistical data are likely:

- the difference between tree genotypes and forest vegetation conditions in the forests of Estonia, for which Nilson's sawlog volume formula was developed, and the countries where experimental research was conducted (Ukraine, Sweden, and Finland);
- a decrease in the thickness of the bark towards the top end of the log (therefore, the measurement of the diameter over the bark leads to an increase in the taper compared to the measurement inside the bark);

- the difference in the logs' shape from the parabolic one (since the logs in the form of a cone or neiloid will have a larger total taper with the same volumes, top diameters, and lengths);
- the approximate nature of the formula (26) due to the nonlinearity of functions (17, 21, 24).

Conclusions

Approximation of the sawlog shape by a truncated paraboloid, developed based on Nilson's sawlog volume formula, considers the tree species, the top diameter, and the length of the log. The results of the statistical analysis indicate the validity of this approximation for tree species and countries where the research was carried out.

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