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# Triangular Fuzzy Set Composed of Two Intersecting Affine Maps

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**Summary.** In this article, various properties of triangular membership functions are formally proven, including the relationship between a triangular membership function composed of two straight lines and a MAX function, as well as a triangular membership function defined by the horizontal coordinates of the triangle's vertices. Furthermore, we formalize defuzzified value of a triangular membership function and the integration of two connected functions.

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## Introduction

This paper, building on our previous work [10], formulates several properties regarding triangular membership functions in fuzzy set theory [19]. Recently, fundamental abstract algebraic properties of fuzzy sets have been formalized by means of the Mizar proof assistant [7] (see, for example, recent developments on triangular norms and conorms [5]). In contrast, our approach is more concrete: as in the encoding [4] of fuzzy numbers [1], [2], we focus on selected properties of affine maps.

The main theorem describes the connection between a triangular membership function composed of two straight lines with a MAX function and a triangular membership function defined from the horizontal coordinates of the triangle's vertices. Moreover, defuzzified value [14] of triangular membership function [3],

[12], [17] (e.g., the notion of centroid [18] is present in Th. 23 and 24), the construction of a trapezoidal membership function using triangular membership function, and the integration of two connected functions are formalized [8]. The usefulness of these notions in engineering applications [13], [16] for fuzzy approximate reasoning is widely studied [9], although the development of the Mizar Mathematical Library as a whole is also an important issue (see a comparative view for formalization of fuzzy and rough sets [6]).

## 1. Preliminaries

Let us consider real numbers $p$, $q$, $s$, $t$. Now we state the propositions:

(1) If $p > 0$ and $p \cdot s < 0$, then $t > q$ iff $\frac{t-q}{p-s} > 0$.

(2) If $p > 0$ and $p \cdot s < 0$, then $t < q$ iff $\frac{t-q}{p-s} < 0$. The theorem is a consequence of (1).

(3) If $p > 0$ and $p \cdot s < 0$, then $\frac{p \cdot t - q \cdot s}{p-s} < 0$ iff $-\frac{q}{p} > -\frac{t}{s}$.

(4) If $p > 0$ and $p \cdot s < 0$, then $\frac{p \cdot t - q \cdot s}{p-s} > 0$ iff $-\frac{q}{p} < -\frac{t}{s}$.

(5) Let us consider real numbers $a$, $b$. Then $a, b \in \mathbb{R} \setminus ]a, b[$.

(6) Let us consider real numbers $a$, $b$, $c$, $d$. Suppose $a < b < c < d$. Then $\mathbb{R} \setminus ]a, d[ \cap [b, c] = \emptyset$.

(7) Let us consider real numbers $p$, $q$, $s$, $t$. Suppose $s \neq p$. Then

  (i) $(\mathrm{AffineMap}(p, q))(\frac{t-q}{p-s}) = (\mathrm{AffineMap}(s, t))(\frac{t-q}{p-s})$, and

  (ii) $(\mathrm{AffineMap}(s, t))(\frac{t-q}{p-s}) = \frac{p \cdot t - q \cdot s}{p-s}$.

(8) Let us consider real numbers $p$, $q$, $s$, $t$. Suppose $p \cdot s < 0$ and $p > 0$ and $\frac{p \cdot t - q \cdot s}{p-s} > 0$. Then $-\frac{q}{p} < \frac{t-q}{p-s} < -\frac{t}{s}$.

## 2. Triangular Membership Function

Now we state the propositions:

(9) Let us consider a real number $r$, subsets $C$, $U$ of $\mathbb{R}$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Then $r \cdot (f{\restriction}C \mathbin{+\cdot} g{\restriction}U) = r \cdot (f{\restriction}C) \mathbin{+\cdot} r \cdot (g{\restriction}U)$.
Proof: Set $f_1 = r \cdot (f{\restriction}C \mathbin{+\cdot} g{\restriction}U)$. Set $f_2 = r \cdot (f{\restriction}C) \mathbin{+\cdot} r \cdot (g{\restriction}U)$. For every object $x$ such that $x \in \mathrm{dom}\, f_1$ holds $f_1(x) = f_2(x)$. $\square$

(10) Let us consider functions $f$, $g$, $h$ from $\mathbb{R}$ into $\mathbb{R}$, and real numbers $a$, $b$, $c$. Suppose $a \leqslant b \leqslant c$ and $f$ is continuous and $g$ is continuous and $h{\restriction}[a, c] =$

$f\restriction[a,b] +\cdot g\restriction[b,c]$ and $\int\limits_{[a,b]} f(x)dx \neq 0$ and $\int\limits_{[b,c]} g(x)dx \neq 0$ and $f(b) = g(b)$.

Then $\mathrm{centroid}(h,[a,c]) = \dfrac{\int\limits_{[a,b]} (\mathrm{id}_{\mathbb{R}} \cdot f)(x)dx + \int\limits_{[b,c]} (\mathrm{id}_{\mathbb{R}} \cdot g)(x)dx}{\int\limits_{[a,c]} h(x)dx}$.

(11) Let us consider real numbers $a$, $b$, $c$. Suppose $a < b < c$. Then TriangularFS$(a,b,c)\restriction[a,c] = (\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction[a,b]+\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b,c]$.

PROOF: Set $\bar{f} = (\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction[a,b]+\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b,c]$. For every object $x$ such that $x \in \mathrm{dom}(\mathrm{TriangularFS}(a,b,c)\restriction[a,c])$ holds $(\mathrm{TriangularFS}(a,b,c)\restriction[a,c])(x) = \bar{f}(x)$. □

Let us consider real numbers $a$, $b$, $c$, $r$ and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(12) Suppose $a < b < c$. Then $(r \cdot \mathrm{TriangularFS}(a,b,c))\restriction[a,c] = (r\cdot(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})))\restriction[a,b]+\cdot(r\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})))\restriction[b,c]$. The theorem is a consequence of (11) and (9).

(13) Suppose $a < b < c$ and for every real number $x$, $f(x) = (r\cdot\mathrm{TriangularFS}(a, b,c))(x)$. Then $f\restriction[a,c] = (r\cdot(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})))\restriction[a,b]+\cdot(r\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})))\restriction[b,c]$.

PROOF: Set $\bar{f} = (r\cdot(\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a})))\restriction[a,b]+\cdot(r\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b})))\restriction[b,c]$. For every object $x$ such that $x \in \mathrm{dom}(f\restriction[a,c])$ holds $(f\restriction[a,c])(x) = \bar{f}(x)$ by (11), (9), [15, (41)]. □

(14) Let us consider real numbers $p$, $q$, $s$, $t$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $p \cdot s < 0$ and $p > 0$ and $\frac{p\cdot t-q\cdot s}{p-s} > 0$. Then $(\mathrm{AffineMap}(p,q))\restriction[-\frac{q}{p}, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s,t))\restriction[\frac{t-q}{p-s}, -\frac{t}{s}] = \frac{p\cdot t-q\cdot s}{p-s} \cdot (\mathrm{TriangularFS}((-\frac{q}{p}), \frac{t-q}{p-s}, (-\frac{t}{s}))\restriction[-\frac{q}{p}, -\frac{t}{s}])$. The theorem is a consequence of (8) and (13).

(15) Let us consider real numbers $p$, $q$, $s$, $t$, $x$. Suppose $p \cdot s < 0$ and $p > 0$ and $\frac{p\cdot t-q\cdot s}{p-s} > 0$ and $x \notin ]-\frac{q}{p}, -\frac{t}{s}[$. Then $((\mathrm{AffineMap}(p,q))\restriction]-\infty, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s,t))\restriction[\frac{t-q}{p-s}, +\infty[)(x) \leqslant 0$. The theorem is a consequence of (8).

Let us consider real numbers $a$, $b$, $c$, $x$. Now we state the propositions:

(16) Suppose $a < b < c$ and $x \notin ]a,c[$. Then $((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction]-\infty,b] +\cdot(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b,+\infty[)(x) \leqslant 0$. The theorem is a consequence of (15).

(17) If $a < b < c$ and $x \in [a,c]$, then $(\mathrm{TriangularFS}(a,b,c))(x) \geqslant 0$.

(18) Let us consider real numbers $a$, $b$, $c$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c$ and $f = (\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction]-\infty,b] +\cdot$

$(\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b, +\infty[$. Then $\max_+(f) = \mathrm{TriangularFS}(a, b, c)$.
PROOF: For every object $x$ such that $x \in \mathrm{dom}(\mathrm{TriangularFS}(a, b, c))$ holds $(\mathrm{TriangularFS}(a, b, c))(x) = (\max_+(f))(x)$. □

(19) Let us consider real numbers $a$, $b$, $c$, $r$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c$ and $r > 0$ and $f = r\cdot((\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))\restriction]-\infty, b] +\cdot (\mathrm{AffineMap}(-\frac{1}{c-b}, \frac{c}{c-b}))\restriction[b, +\infty[)$. Then $\max_+(f) = r\cdot\mathrm{TriangularFS}(a, b, c)$.
PROOF: For every object $x$ such that $x \in \mathrm{dom}(r \cdot \mathrm{TriangularFS}(a, b, c))$ holds $(r \cdot \mathrm{TriangularFS}(a, b, c))(x) = (\max_+(f))(x)$. □

(20) Let us consider real numbers $p$, $q$, $s$, $t$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $p \cdot s < 0$ and $p > 0$ and $\frac{p\cdot t-q\cdot s}{p-s} > 0$ and for every real number $x$, $f(x) = ((\mathrm{AffineMap}(p, q))\restriction]-\infty, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s, t))\restriction[\frac{t-q}{p-s}, +\infty[)(x)$. Then $\max_+(f) = \frac{p\cdot t-q\cdot s}{p-s} \cdot \mathrm{TriangularFS}((-\frac{q}{p}), \frac{t-q}{p-s}, (-\frac{t}{s}))$.
PROOF: Set $r = \frac{p\cdot t-q\cdot s}{p-s}$. For every object $x$ such that $x \in \mathrm{dom}(\max_+(f))$ holds $(\max_+(f))(x) = (r \cdot \mathrm{TriangularFS}((-\frac{q}{p}), \frac{t-q}{p-s}, (-\frac{t}{s})))(x)$. □

(21) Let us consider real numbers $a$, $b$, $c$, $r$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c$ and for every real number $x$, $f(x) = (r \cdot \mathrm{TriangularFS}(a, b, c))(x)$. Then $\int\limits_{[a,c]} f(x)dx = \dfrac{r \cdot (c-a)}{2}$. The theorem is a consequence of (13).

(22) Let us consider real numbers $a$, $b$, $c$. Suppose $a < b < c$.
Then $\mathrm{integral}\,\mathrm{TriangularFS}(a, b, c)\restriction[a, c] = \frac{c-a}{2}$. The theorem is a consequence of (21).

(23) Let us consider real numbers $a$, $b$, $c$, $r$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $a < b < c$ and $r \neq 0$ and for every real number $x$, $f(x) = (r\cdot\mathrm{TriangularFS}(a, b, c))(x)$. Then $\mathrm{centroid}(f, [a, c]) = \frac{a+b+c}{3}$. The theorem is a consequence of (13), (10), and (21).

(24) Let us consider real numbers $p$, $q$, $s$, $t$, and functions $f$, $F$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $p\cdot s < 0$ and $p > 0$ and $\frac{p\cdot t-q\cdot s}{p-s} > 0$ and $f = (\mathrm{AffineMap}(p, q))\restriction]-\infty, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s, t))\restriction[\frac{t-q}{p-s}, +\infty[$ and $F = \max_+(f)$.
Then $\mathrm{centroid}(F, [-\frac{q}{p}, -\frac{t}{s}]) = \frac{-\frac{q}{p}+\frac{t-q}{p-s}+-\frac{t}{s}}{3}$.
PROOF: $-\frac{q}{p} < \frac{t-q}{p-s} < -\frac{t}{s}$. For every real number $x$, $F(x) = (\frac{p\cdot t-q\cdot s}{p-s} \cdot \mathrm{TriangularFS}((-\frac{q}{p}), \frac{t-q}{p-s}, (-\frac{t}{s})))(x)$. □

Let us consider a non empty, closed interval subset $A$ of $\mathbb{R}$, real numbers $p$, $q$, $r$, $s$, $t$, $u$, and a function $f$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(25) Suppose $p \cdot s < 0$ and $p > 0$ and for every real number $x$, $f(x) = \max(r, \min(u, ((\mathrm{AffineMap}(p, q))\restriction]-\infty, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s, t))\restriction[\frac{t-q}{p-s}, +\infty[)(x)))$. Then $f$ is Lipschitzian.

PROOF: Set $F = (\mathrm{AffineMap}(p, q)){\restriction}]{-\infty}, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s, t))$ ${\restriction}[\frac{t-q}{p-s}, +\infty[$. Consider $r_2$ being a real number such that $0 < r_2$ and for every real numbers $x_1$, $x_2$ such that $x_1$, $x_2 \in \mathrm{dom}\, F$ holds $|F(x_1) - F(x_2)| \leqslant r_2 \cdot |x_1 - x_2|$. There exists a real number $r_1$ such that $0 < r_1$ and for every real numbers $x_1$, $x_2$ such that $x_1$, $x_2 \in \mathrm{dom}\, f$ holds $|f(x_1) - f(x_2)| \leqslant r_1 \cdot |x_1 - x_2|$. $\square$

(26) Suppose $p \cdot s < 0$ and $p > 0$ and for every real number $x$, $f(x) = \max(r, \min(u, ((\mathrm{AffineMap}(p, q)){\restriction}]{-\infty}, \frac{t-q}{p-s}] +\cdot (\mathrm{AffineMap}(s, t))$ ${\restriction}[\frac{t-q}{p-s}, +\infty[)(x)))$. Then

  (i) $f$ is integrable on $A$, and

  (ii) $f{\restriction}A$ is bounded.

The theorem is a consequence of (25).

## 3. TRAPEZOIDAL MEMBERSHIP FUNCTION

Let us consider real numbers $a$, $b$, $c$, $d$ and an object $x$. Now we state the propositions:

(27) If $a < b < c < d$, then if $x \in [a, b]$, then $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(x)$.
PROOF: For every object $x$ such that $x \in [a, b]$ holds $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(\frac{1}{b-a}, -\frac{a}{b-a}))(x)$. $\square$

(28) If $a < b < c < d$, then if $x \in [b, c]$, then $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(0, 1))(x)$.
PROOF: For every object $x$ such that $x \in [b, c]$ holds $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(0, 1))(x)$. $\square$

(29) If $a < b < c < d$, then if $x \in [c, d]$, then $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c}))(x)$.
PROOF: For every object $x$ such that $x \in [c, d]$ holds $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = (\mathrm{AffineMap}(-\frac{1}{d-c}, \frac{d}{d-c}))(x)$. $\square$

(30) Let us consider real numbers $a$, $b$, $c$, $d$. Suppose $a < b < c < d$. Let us consider a real number $x$.
If $x \notin [a, d]$, then $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = 0$.
PROOF: For every real number $x$ such that $x \notin [a, d]$ holds $(\mathrm{TrapezoidalFS}(a, b, c, d))(x) = 0$. $\square$

(31) Let us consider real numbers $a$, $b$, $c$, $d$. Suppose $a < b < c < d$. Then $\mathrm{TriangularFS}(a, b, c) + \mathrm{TriangularFS}(b, c, d) = \mathrm{TrapezoidalFS}(a, b, c, d)$.
PROOF: For every object $x$ such that $x \in \mathrm{dom}(\mathrm{TrapezoidalFS}(a, b, c, d))$ holds $(\mathrm{TriangularFS}(a, b, c) + \mathrm{TriangularFS}(b, c, d))(x) =$

(TrapezoidalFS$(a, b, c, d)$)$(x)$ by [11, (48), (47)], (29), [11, (46)]. □

## 4. Related Properties of the Integral of Two Connected Functions

From now on $A$ denotes a non empty, closed interval subset of $\mathbb{R}$.

Let us consider real numbers $a$, $b$, $c$ and functions $f$, $g$ from $[a, c]$ into $\mathbb{R}$. Now we state the propositions:

(32) If $a \leqslant b \leqslant c$, then $f \restriction [a, b] +\cdot g \restriction [b, c]$ is a function from $[a, c]$ into $\mathbb{R}$.

(33) Suppose $a \leqslant b \leqslant c$ and $f \restriction [a, c]$ is bounded and $g \restriction [a, c]$ is bounded. Then $(f \restriction [a, b] +\cdot g \restriction [b, c]) \restriction [a, c]$ is bounded.

Proof: Set $h = f \restriction [a, b] +\cdot g \restriction [b, c]$. There exists a real number $r$ such that for every set $y$ such that $y \in \mathrm{dom}(h \restriction [a, c])$ holds $|(h \restriction [a, c])(y)| < r$. □

(34) Let us consider a real number $a$, and functions $f$, $g$ from $\mathbb{R}$ into $\mathbb{R}$. Suppose $f \restriction A$ is bounded and $g \restriction A$ is bounded and $a \in A$. Then

(i) $(f \restriction ]{-\infty}, a[ +\cdot g \restriction [a, +\infty[) \restriction [\inf A, a]$ is bounded, and

(ii) $(f \restriction ]{-\infty}, a[ +\cdot g \restriction [a, +\infty[) \restriction [a, \sup A]$ is bounded.

Proof: Set $F = f \restriction ]{-\infty}, a[ +\cdot g \restriction [a, +\infty[$. Set $L_A = [\inf A, a]$. Set $R_A = [a, \sup A]$. There exists a real number $r$ such that for every set $y$ such that $y \in \mathrm{dom}(F \restriction L_A)$ holds $|(F \restriction L_A)(y)| < r$. There exists a real number $r$ such that for every set $y$ such that $y \in \mathrm{dom}(F \restriction R_A)$ holds $|(F \restriction R_A)(y)| < r$. □

Let us consider a real number $a$ and functions $f$, $g$, $h$ from $\mathbb{R}$ into $\mathbb{R}$. Now we state the propositions:

(35) Suppose $f \restriction A$ is bounded and $f$ is integrable on $A$ and $g \restriction A$ is bounded and $g$ is integrable on $A$ and $a \in A$ and $h = f \restriction ]{-\infty}, a] +\cdot g \restriction [a, +\infty[$ and $f(a) = g(a)$. Then

(i) $h$ is integrable on $[\inf A, a]$, and

(ii) $h$ is integrable on $[a, \sup A]$.

Proof: For every object $x$ such that $x \in \mathrm{dom}(f \restriction [\inf A, a])$ holds $(f \restriction [\inf A, a])(x) = (h \restriction [\inf A, a])(x)$. For every object $x$ such that $x \in \mathrm{dom}(g \restriction [a, \sup A])$ holds $(g \restriction [a, \sup A])(x) = (h \restriction [a, \sup A])(x)$. □

(36) Suppose $f \restriction A$ is bounded and $f$ is integrable on $A$ and $g \restriction A$ is bounded and $g$ is integrable on $A$ and $a \in A$ and $h = f \restriction ]{-\infty}, a] +\cdot g \restriction [a, +\infty[$ and $f(a) = g(a)$. Then $\int\limits_{A} h(x)dx = \int\limits_{[\inf A, a]} f(x)dx + \int\limits_{[a, \sup A]} g(x)dx$.

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