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Measure for Product Space of Real Normed Spaces[1]

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Summary. This paper deals with Cartesian product spaces of real linear spaces and Cartesian product spaces of real normed spaces. In principle, both are attributed to the direct product of real linear spaces, but the direct product of normed spaces is needed to deal with the calculus of n-dimensional spaces. We also prove that the Lebesgue measure on tuple-type sets introduced in [6] is σ-finite.

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Introduction

This paper continues the formalization of measure theory [12] in Mizar [9], [13] (compare the encoding of the same area in Isabelle/HOL [10] or Coq [3]) by developing foundational results for measures on n-dimensional normed spaces [2]. The authors have previously formalized several theorems concerning the integrability of continuous functions of two and three real variables [7], [8],

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as well as the consistency of two constructions of higher-dimensional spaces – product-type and tuple-type – within simple real spaces [6]. In Mizar, however, continuity for multivariable functions is defined over normed spaces [11], and measures defined on plain product sets cannot be directly used for integration [14]. There are two approaches to address this issue: (1) constructing new measures for each normed space, which is inefficient, or (2) extracting the carrier (point set) of the normed space and linking it to existing measures on plain sets. This paper adopts the second approach and aims to establish the necessary preservation of operations between the original normed space and its plain counterpart, thereby enabling integration over normed spaces.

The outline of the paper is as follows: Section 1 introduces measures on product sets generated from sequences of real normed spaces, showing that operations on points in the product space reduce to operations on the component spaces. It also formalizes conversions between product-type and tuple-type spaces and proves the compatibility of operations across these representations.

Section 2 examines the relationship between normed product spaces and plain real product spaces, demonstrating that key operations are preserved. In particular, it proves that products of closed intervals are compact in the normed space setting. Section 3 explores the topological structure of normed product spaces and their correspondence with plain real spaces [1]. It establishes conditions under which continuous functions on normed spaces are measurable when viewed as equivalent functions defined over plain product sets. The latter part of the section also clarifies the relationship with tuple-type normed spaces and the measures introduced in [6].

1. Product Space of Real Linear Spaces

Now we state the propositions:

(1) Let us consider a real linear space sequence G, elements v, w, s of $\prod G$, and elements v_1, w_1, s_1 of $\prod \overline{G}$. Suppose $v = v_1$ and $w = w_1$ and $s = s_1$. Then $s = v + w$ if and only if for every element i of $\operatorname{dom} \overline{G}$ and for every vectors v_2, w_2 of $G(i)$ such that $v_2 = v_1(i)$ and $w_2 = w_1(i)$ holds $s_1(i) = v_2 + w_2$.

(2) Let us consider a real linear space sequence G, an element r of $\mathbb{R}$, elements v, w of $\prod G$, and elements v_1, w_1 of $\prod \overline{G}$. Suppose $v = v_1$ and $w = w_1$. Then $w = r \cdot v$ if and only if for every element i of $\operatorname{dom} \overline{G}$ and for every vector v_2 of $G(i)$ such that $v_2 = v_1(i)$ holds $w_1(i) = r \cdot v_2$.

Let n be a non zero natural number and p be an n-element finite sequence. The functor $\mathrm{PtProdFS}(p)$ yielding an n-element finite sequence is defined by

(Def. 1) $it(1) = p(1)$ and for every non zero natural number i such that $i < n$ holds $it(i+1) = \langle it(i), p(i+1)\rangle$.

The functor PtCarProd(p) yielding a set is defined by the term

(Def. 2) (PtProdFS(p))(n).

Now we state the propositions:

(3) Let us consider non zero natural numbers n, m, an n-element finite sequence p, and an m-element finite sequence q. Suppose $m \leqslant n$ and $q = p{\restriction}m$. Then PtProdFS(q) = PtProdFS(p)${\restriction}m$.
PROOF: Define $\mathcal{P}$[natural number, object, object] $\equiv$ $\$_3 = \langle \$_2, p(\$_1+1)\rangle$. For every natural number k such that $1 \leqslant k < m$ holds $\mathcal{P}[k$, (PtProdFS(q)) (k), (PtProdFS(q))($k+1$)]. For every natural number k such that $1 \leqslant k < m$ holds $\mathcal{P}[k$, (PtProdFS(p)${\restriction}m$)(k), (PtProdFS(p)${\restriction}m$)($k+1$)]. . □

(4) Let us consider a non zero natural number n, an $(n+1)$-element finite sequence p, and an n-element finite sequence q. Suppose $q = p{\restriction}n$. Then PtCarProd(p) = $\langle$ PtCarProd(q), $p(n+1)\rangle$. The theorem is a consequence of (3).

(5) Let us consider a non zero natural number n, and n-element finite sequences p_1, p_2. Then PtCarProd(p_1) = PtCarProd(p_2) if and only if $p_1 = p_2$.
PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every $\$_1$-element finite sequences p_1, p_2, PtCarProd(p_1) = PtCarProd(p_2) iff $p_1 = p_2$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(6) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, and an object x. Then $x \in \prod_{\mathrm{FS}} X$ if and only if there exists an n-element finite sequence f such that $f \in \prod X$ and x = PtCarProd(f).
PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence X for every object x, $x \in \prod_{\mathrm{FS}} X$ iff there exists a $\$_1$-element finite sequence f such that $f \in \prod X$ and x = PtCarProd(f). $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [5, (6)]. For every non zero natural number n, $\mathcal{P}[n]$. □

Let n be a non empty natural number. Let us note that there exists a real normed space sequence which is n-element.

Let m, n be non zero natural numbers and X be an n-element real normed space sequence. Assume $m \leqslant n$. The functor ElmFin(X, m) yielding a real normed space is defined by the term

(Def. 3) $X(m)$.

Let n be a non zero natural number. The functor $\prod_{\mathrm{FinS}} X$ yielding an n-element real normed space sequence is defined by

(Def. 4) $it(1) = X(1)$ and for every non zero natural number i such that $i < n$ there exist real normed spaces X_1, X_2 such that $X_1 = it(i)$ and $X_2 = X(i+1)$ and $it(i+1) = X_1 \times X_2$.

The functor $\prod_{\rm FS} X$ yielding a real normed space is defined by the term

(Def. 5) $(\prod_{\rm FinS} X)(n)$.

Let us note that $\prod_{\rm FS} X$ is non empty. Now we state the proposition:

(7) Let us consider non zero natural numbers m, n, and an n-element real normed space sequence X. If $m \leqslant n$, then $X{\restriction}m$ is an m-element real normed space sequence.
PROOF: Set $X_1 = X{\restriction}m$. For every set y such that $y \in \operatorname{rng} X_1$ holds y is a real normed space. □

Let m, n be non zero natural numbers and X be an n-element real normed space sequence. Assume $m \leqslant n$. The functor $\mathrm{SubFin}(X, m)$ yielding an m-element real normed space sequence is defined by the term

(Def. 6) $X{\restriction}m$.

Now we state the propositions:

(8) Let us consider non zero natural numbers i, j, k, n, and an n-element real normed space sequence X. Suppose $i \leqslant j \leqslant k \leqslant n$.
Then $(\prod_{\rm FinS} \mathrm{SubFin}(X, j))(i) = (\prod_{\rm FinS} \mathrm{SubFin}(X, k))(i)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant j$,
then $(\prod_{\rm FinS} \mathrm{SubFin}(X, j))(\$_1) = (\prod_{\rm FinS} \mathrm{SubFin}(X, k))(\$_1)$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number m, $\mathcal{P}[m]$. □

(9) Let us consider non zero natural numbers m, n, and an n-element real normed space sequence X. Suppose $m \leqslant n$. Then $(\prod_{\rm FinS} X)(m) = (\prod_{\rm FinS} \mathrm{SubFin}(X, m))(m)$. The theorem is a consequence of (8).

(10) Let us consider a non zero natural number n, and an $(n+1)$-element real normed space sequence X. Then $\prod_{\rm FS} X = \prod_{\rm FS} \mathrm{SubFin}(X, n) \times \mathrm{ElmFin}(X, n+1)$. The theorem is a consequence of (9).

(11) Let us consider non zero natural numbers k, m, n, and an n-element real normed space sequence X. Suppose $k \leqslant m \leqslant n$. Then $\mathrm{SubFin}(X, k) = \mathrm{SubFin}(\mathrm{SubFin}(X, m), k)$.

(12) Let us consider a non zero natural number n, and an n-element real normed space sequence X. Then there exists a non-empty, n-element finite sequence C_1 such that

(i) $\prod_{\rm FS} C_1 =$ the carrier of $\prod_{\rm FS} X$, and

(ii) $C_1 = \overline{X}$, and

(iii) for every non zero natural number i such that $i \leqslant n$ holds $C_1(i) =$ the carrier of $\mathrm{ElmFin}(X, i)$.

PROOF: Consider C_1 being a non-empty, n-element finite sequence such that $\prod_{\mathrm{FS}} C_1 =$ the carrier of $\prod_{\mathrm{FS}} X$ and for every non zero natural number i such that $i \leqslant n$ holds $C_1(i) =$ the carrier of $\mathrm{ElmFin}(X, i)$. For every natural number k such that $k \in \mathrm{dom}\, C_1$ holds $C_1(k) = \overline{X}(k)$. □

(13) Let us consider non zero natural numbers k, m, n, and an n-element real normed space sequence X. Suppose $k \leqslant m \leqslant n$. Then $\mathrm{ElmFin}(X, k) = \mathrm{ElmFin}(\mathrm{SubFin}(X, m), k)$.

(14) Let us consider non zero natural numbers m, n, and an n-element real normed space sequence X. Suppose $m < n$. Then $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, m+1) = \prod_{\mathrm{FS}} \mathrm{SubFin}(X, m) \times \mathrm{ElmFin}(X, m+1)$. The theorem is a consequence of (13), (10), and (11).

(15) Let us consider non zero natural numbers m, n, and an $(n+1)$-element real normed space sequence X. Suppose $m \leqslant n$. Then $\prod_{\mathrm{FS}} \mathrm{SubFin}(X, m) = \prod_{\mathrm{FS}} \mathrm{SubFin}(\mathrm{SubFin}(X, n), m)$. The theorem is a consequence of (11).

(16) Let us consider a non zero natural number n, an n-element real normed space sequence X, and an object x. Then x is a point of $\prod_{\mathrm{FS}} X$ if and only if there exists an n-element finite sequence f such that $x = \mathrm{PtCarProd}(f)$ and for every non zero natural number i such that $i \leqslant n$ there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every $\$_1$-element real normed space sequence X for every object x, x is a point of $\prod_{\mathrm{FS}} X$ iff there exists a $\$_1$-element finite sequence f such that $x = \mathrm{PtCarProd}(f)$ and for every non zero natural number i such that $i \leqslant \$_1$ there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(17) Let us consider a non zero natural number n, an n-element real normed space sequence X, a point x of $\prod_{\mathrm{FS}} X$, and an n-element finite sequence f. Suppose $x = \mathrm{PtCarProd}(f)$. Let us consider a non zero natural number i. Suppose $i \leqslant n$. Then there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$. The theorem is a consequence of (16) and (5).

(18) Let us consider a non zero natural number n, an n-element real normed space sequence X, points x, y of $\prod_{\mathrm{FS}} X$, and n-element finite sequences f, g. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$. Then there exists an n-element finite sequence p' such that $x + y = \mathrm{PtCarProd}(p')$.

PROOF: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every $\$_1$-element real normed space sequence X for every points x, y of $\prod_{\mathrm{FS}} X$ for every $\$_1$-element finite sequences f, g such that $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ there exists a $\$_1$-element finite sequence p' such that $x + y = \mathrm{PtCarProd}(p')$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(19) Let us consider a non zero natural number n, an n-element real normed space sequence X, points x, y, z of $\prod_{\mathrm{FS}} X$, and n-element finite sequences f, g, h. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ and $x + y = \mathrm{PtCarProd}(h)$. Let us consider a non zero natural number i. Suppose $i \leqslant n$. Then there exist points $\bar{p}$, $\hat{p}$ of $\mathrm{ElmFin}(X, i)$ such that

(i) $\bar{p} = f(i)$, and

(ii) $\hat{p} = g(i)$, and

(iii) $\bar{p} + \hat{p} = h(i)$.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every $\$_1$-element real normed space sequence X for every points x, y of $\prod_{\mathrm{FS}} X$ for every $\$_1$-element finite sequences f, g, h such that $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ and $x + y = \mathrm{PtCarProd}(h)$ for every non zero natural number i such that $i \leqslant \$_1$ there exist points $\bar{p}$, $\hat{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$ and $\hat{p} = g(i)$ and $\bar{p} + \hat{p} = h(i)$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(20) Let us consider a non zero natural number n, an n-element real normed space sequence X, a real number r, a point x of $\prod_{\mathrm{FS}} X$, and an n-element finite sequence f. Suppose $x = \mathrm{PtCarProd}(f)$. Then there exists an n-element finite sequence f'' such that $r \cdot x = \mathrm{PtCarProd}(f'')$.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every $\$_1$-element real normed space sequence X for every real number r for every point x of $\prod_{\mathrm{FS}} X$ for every $\$_1$-element finite sequence f such that $x = \mathrm{PtCarProd}(f)$ there exists a $\$_1$-element finite sequence f'' such that $r \cdot x = \mathrm{PtCarProd}(f'')$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(21) Let us consider a non zero natural number n, an n-element real normed space sequence X, a real number r, a point x of $\prod_{\mathrm{FS}} X$, and n-element finite sequences f, f''. Suppose $x = \mathrm{PtCarProd}(f)$ and $r \cdot x = \mathrm{PtCarProd}(f'')$. Let us consider a non zero natural number i. Suppose $i \leqslant n$. Then there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that

(i) $\bar{p} = f(i)$, and

(ii) $r \cdot \bar{p} = f''(i)$.

Proof: Define $\mathcal{P}[\text{non zero natural number}] \equiv$ for every $\$_1$-element real normed space sequence X for every real number r for every point x of $\prod_{\rm FS} X$ for every $\$_1$-element finite sequences f, f'' such that $x = \mathrm{PtCarProd}(f)$ and $r \cdot x = \mathrm{PtCarProd}(f'')$ for every non zero natural number i such that $i \leqslant \$_1$ there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$ and $r \cdot \bar{p} = f''(i)$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. □

(22) Let us consider a non zero natural number n, an n-element real normed space sequence X, a point x of $\prod_{\rm FS} X$, and an n-element finite sequence f. Suppose $x = \mathrm{PtCarProd}(f)$. Then there exists an n-element finite sequence f'' such that $-x = \mathrm{PtCarProd}(f'')$. The theorem is a consequence of (20).

(23) Let us consider a non zero natural number n, an n-element real normed space sequence X, a point x of $\prod_{\rm FS} X$, and n-element finite sequences f, p_3. Suppose $x = \mathrm{PtCarProd}(f)$ and $-x = \mathrm{PtCarProd}(p_3)$. Let us consider a non zero natural number i. Suppose $i \leqslant n$. Then there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that

(i) $\bar{p} = f(i)$, and

(ii) $-\bar{p} = p_3(i)$.

The theorem is a consequence of (21).

(24) Let us consider a non zero natural number n, an n-element real normed space sequence X, points x, y of $\prod_{\rm FS} X$, and n-element finite sequences f, g. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$. Then there exists an n-element finite sequence f such that $x - y = \mathrm{PtCarProd}(f)$. The theorem is a consequence of (22) and (18).

(25) Let us consider a non zero natural number n, an n-element real normed space sequence X, points x, y of $\prod_{\rm FS} X$, and n-element finite sequences f, g, p'. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ and $x - y = \mathrm{PtCarProd}(p')$. Let us consider a non zero natural number i. Suppose $i \leqslant n$. Then there exist points $\bar{p}$, $\hat{p}$ of $\mathrm{ElmFin}(X, i)$ such that

(i) $\bar{p} = f(i)$, and

(ii) $\hat{p} = g(i)$, and

(iii) $\bar{p} - \hat{p} = p'(i)$.

The theorem is a consequence of (22), (17), (23), and (19).

(26) Let us consider a non zero natural number n, an n-element real normed space sequence X, a point x of $\prod_{\rm FS} X$, and an n-element finite sequence f. Suppose $x = \mathrm{PtCarProd}(f)$. Then there exists an element n_1 of $\mathcal{R}^n$ such that

(i) for every non zero natural number i such that $i \leqslant n$ there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$ and $n_1(i) = \|\bar{p}\|$, and

(ii) $\|x\| = |n_1|$.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every $\$_1$-element real normed space sequence X for every point x of $\prod_{\rm FS} X$ for every $\$_1$-element finite sequence f such that $x = \mathrm{PtCarProd}(f)$ there exists an element n_1 of $\mathcal{R}^{\$_1}$ such that for every non zero natural number i such that $i \leqslant \$_1$ there exists a point $\bar{p}$ of $\mathrm{ElmFin}(X, i)$ such that $\bar{p} = f(i)$ and $n_1(i) = \|\bar{p}\|$ and $\|x\| = |n_1|$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

Let n be a non zero natural number and X be an n-element real normed space sequence. The functor $\mathrm{CarProd}(X)$ yielding a function from $\prod_{\rm FS} X$ into $\prod X$ is defined by

(Def. 7) there exists a non-empty, n-element finite sequence C_1 such that $C_1 = \overline{X}$ and $\prod_{\rm FS} C_1 =$ the carrier of $\prod_{\rm FS} X$ and *it* $= \mathrm{CarProd}(C_1)$ and *it* is bijective.

Now we state the propositions:

(27) Let us consider a non zero natural number n, and real normed space sequences X, X_1. Suppose $n \leqslant \operatorname{len} X$ and $X_1 = X \restriction n$. Then $\overline{X_1} = \overline{X} \restriction n$.

(28) Let us consider a non zero natural number n, an n-element real normed space sequence X, and a point x of $\prod_{\rm FS} X$. Then there exists an n-element finite sequence f such that

(i) $x = \mathrm{PtCarProd}(f)$, and

(ii) $(\mathrm{CarProd}(X))(x) = f$.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every $\$_1$-element real normed space sequence X for every point x of $\prod_{\rm FS} X$, there exists a $\$_1$-element finite sequence f such that $x = \mathrm{PtCarProd}(f)$ and $(\mathrm{CarProd}(X))$ $(x) = f$. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

(29) Let us consider a non zero natural number n, an n-element real normed space sequence X, and points x, y of $\prod_{\rm FS} X$. Then $(\mathrm{CarProd}(X))(x+y) = (\mathrm{CarProd}(X))(x) + (\mathrm{CarProd}(X))(y)$. The theorem is a consequence of (28), (18), (19), and (5).

(30) Let us consider a non zero natural number n, an n-element real normed space sequence X, a point x of $\prod_{\rm FS} X$, and an element r of $\mathbb{R}$. Then $(\mathrm{CarProd}(X))(r \cdot x) = r \cdot (\mathrm{CarProd}(X))(x)$. The theorem is a consequence of (28), (20), (21), and (5).

(31) Let us consider a non zero natural number n, an n-element real normed space sequence X, and a point x of $\prod_{\rm FS} X$. Then $\|(\mathrm{CarProd}(X))(x)\| = \|x\|$. The theorem is a consequence of (28) and (26).

2. Product Space of Real Normed Spaces

Let n be a non zero natural number and X be a non empty set. One can check that $\mathrm{Seg}\, n \longmapsto X$ is non-empty and n-element as a finite sequence and $\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$) is n-element, non empty, and real-norm-space-yielding as a finite sequence. Now we state the propositions:

(32) Let us consider a non zero natural number n, and an object u. Then u is a point of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) if and only if there exists an n-element finite sequence f of elements of the real normed space of $\mathbb{R}$ such that $u = \mathrm{PtCarProd}(f)$.

Proof: Set $X = \mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$). Consider f being an n-element finite sequence of elements of the real normed space of $\mathbb{R}$ such that $u = \mathrm{PtCarProd}(f)$. For every non zero natural number i such that $i \leqslant n$ there exists a point q of $\mathrm{ElmFin}(X, i)$ such that $q = f(i)$. $\square$

(33) Let us consider a non zero natural number n, a non-empty, n-element finite sequence D, and an n-element finite sequence f. Then $\mathrm{PtCarProd}(f) \in \prod_{\rm FS} D$ if and only if for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $f(i) \in D(i)$. The theorem is a consequence of (6) and (5).

(34) Let us consider a non zero natural number n, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i) \subseteq \mathbb{R}$. Then $\prod_{\rm FS} D \subseteq \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. The theorem is a consequence of (6).

Let us consider a non zero natural number n. Now we state the propositions:

(35) $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{Q})$ is countable.

Proof: Define $\mathcal{P}$[non zero natural number] $\equiv \prod_{\rm FS}(\mathrm{Seg}\, \$_1 \longmapsto \mathbb{Q})$ is countable. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

(36) $\mathrm{Measure}_{\rm Prod}(\text{L-Meas}(n))$ is σ-finite.

Proof: Set $X = \mathrm{Seg}\, n \longmapsto \mathbb{R}$. Set $S = \text{L-Field}(n)$. Set $m = \text{L-Meas}(n)$. For every natural number i such that $i \in \mathrm{Seg}\, n$ there exists a non empty set $\mathcal{X}$ and there exists a σ-field F_1 of subsets of $\mathcal{X}$ and there exists a σ-measure $\mathcal{M}$ on F_1 such that $\mathcal{X} = X(i)$ and $F_1 = S(i)$ and $\mathcal{M} = m(i)$ and $\mathcal{M}$ is σ-finite. $\square$

(37) ProdMeas($\mathrm{Measure}_{\mathrm{Prod}}$(L-Meas($n$)), L-Meas) is σ-finite. The theorem is a consequence of (36).

(38) (i) $\prod_{\mathrm{FS}}(\mathrm{Seg}(n+1) \longmapsto \mathbb{R}) = \prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \times \mathbb{R}$, and

(ii) $\prod_{\mathrm{FS}}(\mathrm{Seg}(n+1) \longmapsto$ (the real normed space of $\mathbb{R}$)) $= \prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $\times$ (the real normed space of $\mathbb{R}$).

The theorem is a consequence of (10).

(39) Let us consider a non zero natural number n, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is an element of L-Field. Then $\prod_{\mathrm{FS}} D$ is an element of $\prod_{\mathrm{Field}}$ L-Field(n).

Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence D such that for every natural number i such that $i \in \mathrm{Seg}\, \$_1$ holds $D(i)$ is an element of L-Field holds $\prod_{\mathrm{FS}} D$ is an element of $\prod_{\mathrm{Field}}$ L-Field($\$_1$). $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

(40) Let us consider a non zero natural number n. Then $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R}) \in \prod_{\mathrm{Field}}$ L-Field(n). The theorem is a consequence of (39).

(41) Let us consider a non zero natural number n, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is an interval. Then $\prod_{\mathrm{FS}} D$ is an element of $\prod_{\mathrm{Field}}$ L-Field(n).

Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence D such that for every natural number i such that $i \in \mathrm{Seg}\, \$_1$ holds $D(i)$ is an interval holds $\prod_{\mathrm{FS}} D$ is an element of $\prod_{\mathrm{Field}}$ L-Field($\$_1$). $\mathcal{P}[1]$ by [4, (75)], [6, (41)]. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

(42) Let us consider a non zero natural number n. Then the carrier of $\prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) $= \prod_{\mathrm{FS}}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. The theorem is a consequence of (12).

(43) Let us consider a non zero natural number n, and an n-element finite sequence f of elements of $\mathbb{R}$. Then $f \in \prod(\mathrm{Seg}\, n \longmapsto \mathbb{R})$.

Proof: Set $\mathfrak{f} = \mathrm{Seg}\, n \longmapsto \mathbb{R}$. For every object y such that $y \in \mathrm{dom}\, f$ holds $f(y) \in \mathfrak{f}(y)$. $\square$

(44) Let us consider a non zero natural number n, and an n-element finite sequence f of elements of $\mathbb{Q}$. Then $f \in \prod(\mathrm{Seg}\, n \longmapsto \mathbb{Q})$.

Proof: Set $\mathfrak{f} = \mathrm{Seg}\, n \longmapsto \mathbb{Q}$. For every object y such that $y \in \mathrm{dom}\, f$ holds $f(y) \in \mathfrak{f}(y)$. $\square$

Let us consider a non zero natural number n and an object x. Now we state the propositions:

(45) $x \in \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$ if and only if there exists an n-element finite sequence f of elements of $\mathbb{R}$ such that $x = \mathrm{PtCarProd}(f)$. The theorem is a consequence of (6) and (43).

(46) $x \in \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{Q})$ if and only if there exists an n-element finite sequence f of elements of $\mathbb{Q}$ such that $x = \mathrm{PtCarProd}(f)$. The theorem is a consequence of (6) and (44).

(47) Suppose $x \in \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. Then there exists an n-element finite sequence f of elements of $\mathbb{R}$ such that

(i) $f \in \prod(\mathrm{Seg}\, n \longmapsto \mathbb{R})$, and

(ii) $x = \mathrm{PtCarProd}(f)$.

The theorem is a consequence of (45) and (43).

(48) Let us consider a non zero natural number n, and non-empty, n-element finite sequences X, Y. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $X(i) \subseteq Y(i)$. Then $\prod_{\rm FS} X \subseteq \prod_{\rm FS} Y$. The theorem is a consequence of (6).

(49) Let us consider a non zero natural number n. Then $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{Q}) \subseteq \prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto \mathbb{R})$. The theorem is a consequence of (48).

(50) Let us consider a non zero natural number n, a non empty set X, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a subset of X. Then $\prod_{\rm FS} D$ is a subset of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto X)$. The theorem is a consequence of (48).

(51) Let us consider a non zero natural number n, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$. Then there exists a subset E of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that

(i) $E = \prod_{\rm FS} D$, and

(ii) E is compact.

PROOF: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence D such that for every natural number i such that $i \in \mathrm{Seg}\, \$_1$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$ there exists a subset E of $\prod_{\rm FS}(\mathrm{Seg}\, \$_1 \longmapsto$ (the real normed space of $\mathbb{R}$)) such that $E = \prod_{\rm FS} D$ and E is compact. $\mathcal{P}[1]$. For every non zero natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every non zero natural number n, $\mathcal{P}[n]$. $\square$

(52) Let us consider a non zero natural number n, points x, y of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and n-element finite sequences f, g of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$. Then there exists an n-element finite sequence p' of elements of $\mathbb{R}$ such that $x + y = \mathrm{PtCarProd}(p')$. The theorem is a consequence of (18) and (19).

(53) Let us consider a non zero natural number n, points x, y of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and n-element finite sequences f, g, p' of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ and $x + y = \mathrm{PtCarProd}(p')$. Let us consider a natural number i. If $i \in \mathrm{Seg}\, n$, then $p'(i) = f(i) + g(i)$.

PROOF: For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $p'(i) = f(i) + g(i)$. $\square$

(54) Let us consider a non zero natural number n, points x, y of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and n-element finite sequences f, g of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$. Then there exists an n-element finite sequence p' of elements of $\mathbb{R}$ such that $x - y = \mathrm{PtCarProd}(p')$. The theorem is a consequence of (24) and (25).

(55) Let us consider a non zero natural number n, points x, y of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and n-element finite sequences f, g, p' of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$ and $y = \mathrm{PtCarProd}(g)$ and $x - y = \mathrm{PtCarProd}(p')$. Let us consider a natural number i. If $i \in \mathrm{Seg}\, n$, then $p'(i) = f(i) - g(i)$.

PROOF: For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $p'(i) = f(i) - g(i)$. $\square$

(56) Let us consider a non zero natural number n, a point x of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a real number r, and an n-element finite sequence f of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$. Then there exists an n-element finite sequence f'' of elements of $\mathbb{R}$ such that $r \cdot x = \mathrm{PtCarProd}(f'')$. The theorem is a consequence of (20) and (21).

(57) Let us consider a non zero natural number n, a point x of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), a real number r, and n-element finite sequences f, f'' of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$ and $r \cdot x = \mathrm{PtCarProd}(f'')$. Let us consider a natural number i. If $i \in \mathrm{Seg}\, n$, then $f''(i) = r \cdot f(i)$.

PROOF: For every natural number i such that $i \in \mathrm{Seg}\, n$ holds $f''(i) = r \cdot f(i)$. $\square$

(58) Let us consider a non zero natural number n, a point x of $\prod_{\rm FS}(\mathrm{Seg}\, n \longmapsto$ (the real normed space of $\mathbb{R}$)), and an n-element finite sequence f of elements of $\mathbb{R}$. Suppose $x = \mathrm{PtCarProd}(f)$. Then there exists an element n_1 of $\mathcal{R}^n$ such that

(i) for every natural number i such that $i \in \mathrm{Seg}\, n$ holds $n_1(i) = |f(i)|$, and

(ii) $\|x\| = |n_1|$.

Proof: Set $X = \operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$). Consider n_1 being an element of $\mathcal{R}^n$ such that for every non zero natural number i such that $i \leqslant n$ there exists a point $\bar{p}$ of $\operatorname{ElmFin}(X, i)$ such that $\bar{p} = f(i)$ and $n_1(i) = \|\bar{p}\|$ and $\|x\| = |n_1|$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $n_1(i) = |f(i)|$. $\square$

(59) Let us consider a non zero natural number n, a point x of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), an n-element finite sequence f of elements of $\mathbb{R}$, and a real number r. Suppose $0 \leqslant r$ and $x = \operatorname{PtCarProd}(f)$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $|f(i)| \leqslant \frac{r}{n}$. Then $\|x\| \leqslant r$. The theorem is a consequence of (58).

3. σ-finite Measure on the Real Product

Now we state the propositions:

(60) Let us consider a non zero natural number n, and a set A. Suppose $A \subseteq \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$ and $A \cap \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{Q}) \neq \emptyset$. Then there exists a sequence p of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$ such that $\operatorname{rng} p = A \cap \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{Q})$. The theorem is a consequence of (35).

(61) Let us consider a non zero natural number n, a non-empty, n-element finite sequence D, and an n-element finite sequence f. Then $\operatorname{PtCarProd}(f) \in \prod_{\mathrm{FS}} D$ if and only if for every natural number i such that $i \in \operatorname{Seg} n$ holds $f(i) \in D(i)$. The theorem is a consequence of (6) and (5).

(62) Let us consider a non zero natural number n, a real number r, and a partial function f from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$. Suppose f is continuous on $\operatorname{dom} f$. Then there exists a subset H of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that

(i) $H \cap \operatorname{dom} f = f^{-1}(]-\infty, r[)$, and

(ii) H is open.

Proof: Set $E = \operatorname{dom} f$. Set $S = \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point t of S and there exists a real number s such that $t = \$_1$ and $s = \$_2$ and $0 < s$ and for every object t_1 such that $t_1 \in E \cap \{t_1$, where t_1 is a point of $S : \|t_1 - t\| < s\}$ holds $f(t_1) \in]-\infty, r[$. For every object z such that $z \in f^{-1}(]-\infty, r[)$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[z, y]$. Consider R being a function from $f^{-1}(]-\infty, r[)$ into $\mathbb{R}$ such that for every object x such that $x \in f^{-1}(]-\infty, r[)$ holds $\mathcal{P}[x, R(x)]$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ there exists a point t of S such that $t = \$_1$ and $0 < R(\$_1)$ and $\$_2 = \{t_1$, where t_1 is a point of $S : \|t_1 - t\| < R(\$_1)\}$.

For every object z such that $z \in f^{-1}(]-\infty, r[)$ there exists an object y such that $y \in 2^{\alpha}$ and $\mathcal{Q}[z, y]$, where α is the carrier of S. Consider B being a function from $f^{-1}(]-\infty, r[)$ into $2^{(\text{the carrier of } S)}$ such that for every object x such that $x \in f^{-1}(]-\infty, r[)$ holds $\mathcal{Q}[x, B(x)]$. Set $H = \bigcup \operatorname{rng} B$. For every object z, $z \in H \cap E$ iff $z \in f^{-1}(]-\infty, r[)$. For every point z of S such that $z \in H$ there exists a neighbourhood N of z such that $N \subseteq H$. $\square$

(63) Let us consider a non zero natural number n, a point z of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), and a real number r. Suppose $0 < r$. Then there exists a real number s and there exists an n-element finite sequence x of elements of $\mathbb{R}$ and there exists a non-empty, n-element finite sequence D such that $0 < s < r$ and $z = \operatorname{PtCarProd}(x)$ and for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) =]x(i)-s, x(i)+s[$ and $\prod_{\mathrm{FS}} D \subseteq \operatorname{Ball}(z, r)$. The theorem is a consequence of (32), (34), (45), (42), and (61).

Let us consider a non zero natural number n and a subset A of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). Now we state the propositions:

(64) Suppose A is open and for every n-element finite sequence a of elements of $\mathbb{R}$ such that $\operatorname{PtCarProd}(a) \in A$ there exists a real-membered set R_2 such that R_2 is non empty and upper bounded and $R_2 = \{r$, where r is a real number : $0 < r$ and there exists a non-empty, n-element finite sequence D such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) =]a(i) - r, a(i) + r[$ and $\prod_{\mathrm{FS}} D \subseteq A\}$.

Then there exists a function F from A into $\mathbb{R}$ such that for every n-element finite sequence a of elements of $\mathbb{R}$ such that $\operatorname{PtCarProd}(a) \in A$ there exists a real-membered set R_2 such that R_2 is non empty and upper bounded and $R_2 = \{r$, where r is a real number : $0 < r$ and there exists a non-empty, n-element finite sequence D such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) =]a(i)-r, a(i)+r[$ and $\prod_{\mathrm{FS}} D \subseteq A\}$ and $F(\operatorname{PtCarProd}(a)) = \frac{\sup R_2}{2}$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists an n-element finite sequence a of elements of $\mathbb{R}$ and there exists a real-membered set R_2 such that $\$_1 = \operatorname{PtCarProd}(a)$ and R_2 is non empty and upper bounded and $R_2 = \{r$, where r is a real number : $0 < r$ and there exists a non-empty, n-element finite sequence D such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) =]a(i) - r, a(i) + r[$ and $\prod_{\mathrm{FS}} D \subseteq A\}$ and $\$_2 = \frac{\sup R_2}{2}$. For every object x such that $x \in A$ there exists an object y such that $y \in \mathbb{R}$ and $\mathcal{P}[x, y]$. Consider F being a function from A into $\mathbb{R}$ such that for every object x such that $x \in A$ holds $\mathcal{P}[x, F(x)]$. For every n-element finite sequence a of elements of $\mathbb{R}$ such that $\operatorname{PtCarProd}(a) \in A$ there exists a real-membered set R_2 such that R_2 is non empty and upper bounded and $R_2 = \{r$, where r is a real number : $0 < r$ and there exists a non-empty, n-element finite sequence D such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) =]a(i)-r, a(i)+r[$ and $\prod_{\mathrm{FS}} D \subseteq A\}$ and $F(\mathrm{PtCarProd}(a)) = \frac{\sup R_2}{2}$. $\square$

(65) If A is open, then $A \in \prod_{\mathrm{Field}} \text{L-Field}(n)$. The theorem is a consequence of (42), (63), (5), (45), (33), (40), (64), (46), (49), (60), (48), (39), and (61).

(66) Let us consider a non zero natural number n, a partial function f from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to the real normed space of $\mathbb{R}$, a partial function g from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$ to $\mathbb{R}$, and an element E of $\prod_{\mathrm{Field}} \text{L-Field}(n)$. Suppose $f = g$ and f is continuous on E and $\operatorname{dom} f = E$. Then g is E-measurable. The theorem is a consequence of (62) and (65).

(67) Let us consider a non zero natural number n, and an object f. Then f is an n-element finite sequence of elements of $\mathbb{R}$ if and only if $f \in \prod(\operatorname{Seg} n \longmapsto \mathbb{R})$. The theorem is a consequence of (43).

(68) Let us consider a non zero natural number n, and an object x. Then $x \in \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$ if and only if there exists an n-element finite sequence f of elements of $\mathbb{R}$ such that $f \in \prod(\operatorname{Seg} n \longmapsto \mathbb{R})$ and $x = \mathrm{PtCarProd}(f)$. The theorem is a consequence of (45) and (67).

Let us consider a non zero natural number n. Now we state the propositions:

(69) $\overline{\operatorname{Seg} n \longmapsto \alpha} = \operatorname{Seg} n \longmapsto \mathbb{R}$, where α is the real normed space of $\mathbb{R}$.

(70) $\mathrm{CarProd}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) $= \mathrm{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R})$. The theorem is a consequence of (69).

(71) Let us consider a non zero natural number n, and a non-empty, n-element finite sequence D. Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) \subseteq \mathbb{R}$. Then $\prod D = (\mathrm{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^\circ(\prod_{\mathrm{FS}} D)$.

Proof: Set $I = \mathrm{CarProd}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). $\prod_{\mathrm{FS}} D \subseteq \prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$. $\prod_{\mathrm{FS}} D \subseteq$ the carrier of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). For every object x, $x \in \prod D$ iff $x \in I^\circ(\prod_{\mathrm{FS}} D)$. $\square$

(72) Let us consider a non zero natural number n. Then there exists a linear operator I from $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) into $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that

(i) $I = \mathrm{CarProd}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), and

(ii) I is one-to-one, onto, and isometric.

The theorem is a consequence of (29), (30), and (31).

(73) Let us consider a non zero natural number n, and a sequence s of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). Then s is convergent if and only if (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)))$\cdot s$ is convergent. The theorem is a consequence of (72).

(74) Let us consider a non zero natural number n,a subset X of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), and a subset Y of $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). Suppose $Y =$ (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)))$^{\circ}X$. Then

(i) X is closed iff Y is closed, and

(ii) X is open iff Y is open, and

(iii) X is compact iff Y is compact.

The theorem is a consequence of (72).

(75) Let us consider a non zero natural number n, a real normed space Z, a point y of $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), a point x of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), and a partial function f from $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to Z. Suppose $y =$ (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)))(x) and $y \in \operatorname{dom} f$. Then f is continuous in y if and only if $f \cdot$ (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$))) is continuous in x. The theorem is a consequence of (72).

(76) Let us consider a non zero natural number n, a real normed space Z, a subset X of $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), a subset Y of $\prod_{\mathrm{FS}}(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)), and a partial function f from $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) to Z. Suppose $Y =$ (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)))$^{-1}(X)$. Then f is continuous on X if and only if $f \cdot$ (CarProd($\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$))) is continuous on Y. The theorem is a consequence of (72).

Let us consider a non zero natural number n and a non-empty, n-element finite sequence D. Now we state the propositions:

(77) Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is a closed interval subset of $\mathbb{R}$. Then there exists a subset E of $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)) such that

(i) $E = \prod D$, and

(ii) E is compact.

The theorem is a consequence of (51), (71), (70), and (74).

(78) Suppose for every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is an element of L-Field. Then $\prod D$ is an element of XL-Field(n). The theorem is a consequence of (39) and (71).

(79) Let us consider a non zero natural number n, and a subset Y of $\prod(\operatorname{Seg} n \longmapsto$ (the real normed space of $\mathbb{R}$)). If Y is open, then $Y \in$ XL-Field(n). The theorem is a consequence of (72), (74), (65), and (70).

(80) Let us consider non empty sets X, Y, a σ-field S of subsets of X, a function T from X into Y, and a set sequence F of S. Suppose T is bijective. Then $({}^\circ T\restriction S) \cdot F$ is a set sequence of $\mathrm{Field}_{\mathrm{Copy}}(T, S)$.
Proof: Set $H = {}^\circ T\restriction S$. $\operatorname{rng} F \subseteq S$. Reconsider $H_1 = H \cdot F$ as a function from $\mathbb{N}$ into $\mathrm{Field}_{\mathrm{Copy}}(T, S)$. For every natural number n, $H_1(n) \in \mathrm{Field}_{\mathrm{Copy}}(T, S)$. $\square$

(81) Let us consider non empty sets X, Y, a function T from X into Y, a σ-field S of subsets of X, and a σ-measure M on S. Suppose T is bijective and M is σ-finite. Then $\mathrm{Measure}_{\mathrm{Copy}}(T, M)$ is σ-finite.
Proof: Consider E being a set sequence of S such that for every natural number n, $M(E(n)) < +\infty$ and $\bigcup E = X$. Set $M_1 = \mathrm{Measure}_{\mathrm{Copy}}(T, M)$. Set $S_1 = \mathrm{Field}_{\mathrm{Copy}}(T, S)$. Reconsider $E_1 = ({}^\circ T\restriction S) \cdot E$ as a set sequence of S_1. For every natural number n, $M_1(E_1(n)) < +\infty$. For every object z, $z \in \bigcup E_1$ iff $z \in Y$. $\square$

(82) Let us consider a non zero natural number n. Then XL-Meas(n) is σ-finite. The theorem is a consequence of (81) and (36).

Let us consider a non zero natural number n and a non-empty, n-element finite sequence D. Now we state the propositions:

(83) If $\operatorname{rng} D \subseteq$ L-Field, then $\prod_{\mathrm{FS}} D \in \prod_{\mathrm{Field}}$ L-Field(n).
Proof: Define $\mathcal{Q}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence D such that $\operatorname{rng} D \subseteq$ L-Field holds $\prod_{\mathrm{FS}} D \in \prod_{\mathrm{Field}}$ L-Field$(\$_1)$. $\mathcal{Q}[1]$. For every non zero natural number n such that $\mathcal{Q}[n]$ holds $\mathcal{Q}[n+1]$. For every non zero natural number n, $\mathcal{Q}[n]$. $\square$

(84) Suppose $\operatorname{rng} D \subseteq$ L-Field and for every natural number i such that $i \in \operatorname{Seg} n$ holds (L-Meas)$(D(i)) \in \mathbb{R}$. Then there exists an n-element finite sequence E of elements of $\mathbb{R}$ such that

(i) for every natural number i such that $i \in \operatorname{Seg} n$ holds $E(i) =$ (L-Meas) $(D(i))$, and

(ii) $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(n)))(\prod_{\mathrm{FS}} D) = \prod E$.

Proof: Define $\mathcal{Q}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence D such that $\operatorname{rng} D \subseteq$ L-Field and for every natural number i such that $i \in \operatorname{Seg} \$_1$ holds (L-Meas)$(D(i)) \in \mathbb{R}$ there exists a $\$_1$-element finite sequence E of elements of $\mathbb{R}$ such that for every natural number i such that $i \in \operatorname{Seg} \$_1$ holds $E(i) =$ (L-Meas)$(D(i))$ and $(\mathrm{Measure}_{\mathrm{Prod}}(\text{L-Meas}(\$_1)))(\prod_{\mathrm{FS}} D) = \prod E$. $\mathcal{Q}[1]$. For every non zero natural number n such that $\mathcal{Q}[n]$ holds $\mathcal{Q}[n+1]$. For every non zero natural number n, $\mathcal{Q}[n]$. $\square$

(85) Suppose $\operatorname{rng} D \subseteq$ L-Field and for every natural number i such that $i \in \operatorname{Seg} n$ holds $(\text{L-Meas})(D(i)) \in \mathbb{R}$. Then there exists an n-element finite sequence E of elements of $\mathbb{R}$ such that

(i) for every natural number i such that $i \in \operatorname{Seg} n$ holds $E(i) = (\text{L-Meas})(D(i))$, and

(ii) $(\text{XL-Meas}(n))(\prod D) = \prod E$.

Proof: For every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i) \subseteq \mathbb{R}$. $\prod D = (\operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R}))^{\circ}(\prod_{\text{FS}} D)$. Consider E being an n-element finite sequence of elements of $\mathbb{R}$ such that for every natural number i such that $i \in \operatorname{Seg} n$ holds $E(i) = (\text{L-Meas})(D(i))$ and $(\text{Measure}_{\text{Prod}}(\text{L-Meas}(n)))(\prod_{\text{FS}} D) = \prod E$. Set $M = \text{Measure}_{\text{Prod}}(\text{L-Meas}(n))$. Set $S = \prod_{\text{Field}} \text{L-Field}(n)$. Set $f = \operatorname{CarProd}(\operatorname{Seg} n \longmapsto \mathbb{R})$. Set $M_1 = \text{Measure}_{\text{Copy}}(f, M)$. $\prod_{\text{FS}} D \subseteq \prod_{\text{FS}}(\operatorname{Seg} n \longmapsto \mathbb{R})$. For every natural number i such that $i \in \operatorname{Seg} n$ holds $D(i)$ is an element of L-Field. $\prod_{\text{FS}} D$ is an element of $\prod_{\text{Field}} \text{L-Field}(n)$. Consider t being an element of S such that $\prod D = f^{\circ}t$ and $M_1(\prod D) = M(t)$. $\square$

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