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Higher Order Partial Differentiable Functions[1]

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Summary. This article formalizes higher-order partial differentiable functions in Mizar: it introduces the concept of partial differentiation for functions between real normed spaces and develops the theory of partial derivatives of arbitrary order. Key results include properties of partial derivatives, their relationship with total derivatives, and criteria for the continuity of higher-order partial derivatives.

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Introduction

This article extends the theory of partial differentiation in the general context of real normed spaces [5], focusing on higher-order derivatives, using the Mizar system [3], [4].

In Section 1 we define the class of all continuously differentiable functions $\mathcal{C}^n(X,Y)$ – it contains functions that not only have a n^{th} derivative but also this derivative is a continuous function from X into Y [7], [8]. To allow for automation in Mizar, it is equipped with the structure of a real normed space; we prove that such functions behave as bounded linear operators (Th. 11) [1]. Next, in Sect. 2 we formalize the concept of partial derivatives (up to a given order) for functions between real normed spaces [10] and prove their fundamental properties (Sect. 3).

Section 4 is devoted to the connection between partial and total derivatives: the final theorem establishes the fundamental property that the differentiability

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(and continuity) of higher-order function on an open region is equivalent to the continuity of all its partial derivatives. The formalization, which is a continuation of [6], provides a foundation for advanced multivariable calculus and functional analysis within the Mizar Mathematical Library.

1. Vector Space of Continuously Differentiable Functions

Now we state the propositions:

(1) Let us consider real normed spaces X, Y, a partial function f from X to Y, and a subset Z of X. Then $f \restriction Z$ is differentiable on Z if and only if f is differentiable on Z.

(2) Let us consider real normed spaces X, Y, a point s of X, a finite sequence w of elements of Y, and a finite sequence f of elements of the real normed space of bounded linear operators from X into Y. Suppose $\operatorname{dom} w = \operatorname{dom} f$ and for every natural number i such that $i \in \operatorname{dom} w$ holds $w(i) = (f_{/i})(s)$. Then $\sum w = (\sum f)(s)$.

Proof: Set $\mathcal{L}$ = the real normed space of bounded linear operators from X into Y. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence w of elements of Y for every finite sequence f of elements of $\mathcal{L}$ such that $\operatorname{len} w = \$_1$ and $\operatorname{dom} w = \operatorname{dom} f$ and for every natural number i such that $i \in \operatorname{dom} w$ holds $w(i) = (f_{/i})(s)$ holds $\sum w = (\sum f)(s)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

Let X be a real normed space. One can verify that there exists a subset of X which is non empty and open. Now we state the proposition:

(3) Let us consider real normed spaces X, Y, a non empty subset Z of X, vectors v, u of RealVectSpace(Z, Y), partial functions f, g from X to Y, and a real number a. Suppose $\operatorname{dom} f = Z$ and $\operatorname{dom} g = Z$ and $v = f$ and $u = g$. Then

(i) $v + u = f + g$, and

(ii) $a \cdot v = a \cdot f$.

Let X, Y be real normed spaces, Z be a non empty, open subset of X, and n be a natural number. The functor $\mathcal{C}^n(Z, Y)$ yielding a subset of RealVectSpace(Z, Y) is defined by the term

(Def. 1) $\{f$, where f is a partial function from X to Y : $\operatorname{dom} f = Z$ and f is differentiable n times on Z and $\operatorname{diff}_Z(f, n)$ is continuous on $Z\}$.

Now we state the proposition:

(4) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, a natural number n, and a partial function f from X to Y. Then

$f \in \mathcal{C}^n(Z, Y)$ if and only if $\operatorname{dom} f = Z$ and f is differentiable n times on Z and $\operatorname{diff}_Z(f, n)$ is continuous on Z.

Let X, Y be real normed spaces, Z be a non empty, open subset of X, and n be a natural number. Observe that $\mathcal{C}^n(Z, Y)$ is non empty and functional.

Let us consider real normed spaces X, Y, a non empty, open subset Z of X, and a non empty natural number n. Now we state the propositions:

(5) $\mathcal{C}^n(Z, Y)$ is linearly closed.

PROOF: Set $W = \mathcal{C}^n(Z, Y)$. For every vectors v, u of $\mathrm{RealVectSpace}(Z, Y)$ such that v, $u \in W$ holds $v + u \in W$. For every real number a and for every vector v of $\mathrm{RealVectSpace}(Z, Y)$ such that $v \in W$ holds $a \cdot v \in W$ by [2, (25),(24)]. □

(6) $\langle \mathcal{C}^n(Z, Y), \mathrm{Zero}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Add}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Mult}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y))\rangle$ is a subspace of $\mathrm{RealVectSpace}(Z, Y)$.

Let X, Y be real normed spaces, Z be a non empty, open subset of X, and n be a non empty natural number. One can verify that $\langle \mathcal{C}^n(Z, Y), \mathrm{Zero}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Add}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Mult}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y))\rangle$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

The functor $\mathcal{C}^n_{\mathrm{RLS}}(Z, Y)$ yielding a strict real linear space is defined by the term

(Def. 2) $\langle \mathcal{C}^n(Z, Y), \mathrm{Zero}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Add}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y)), \mathrm{Mult}(\mathcal{C}^n(Z, Y), \mathrm{RealVectSpace}(Z, Y))\rangle$.

Observe that $\mathcal{C}^n_{\mathrm{RLS}}(Z, Y)$ is constituted of functions. Now we state the proposition:

(7) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, a non empty natural number n, vectors v, u of $\mathcal{C}^n_{\mathrm{RLS}}(Z, Y)$, partial functions f, g from X to Y, and a real number a. Suppose $\operatorname{dom} f = Z$ and $\operatorname{dom} g = Z$ and $v = f$ and $u = g$. Then

(i) $v + u = f + g$, and

(ii) $a \cdot v = a \cdot f$.

The theorem is a consequence of (5) and (3).

Let X, Y be real normed spaces, Z be a non empty, open subset of X, n be a non empty natural number, f be an element of $\mathcal{C}^n_{\mathrm{RLS}}(Z, Y)$, and v be an element of Z. Let us observe that the functor $f(v)$ yields a vector of Y. Now we state the propositions:

(8) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, a non empty natural number n, and vectors f, g, h of $\mathcal{C}^n_{\rm RLS}(Z,Y)$. Then $h = f+g$ if and only if for every element x of Z, $h(x) = f(x)+g(x)$. The theorem is a consequence of (5).

(9) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, a non empty natural number n, vectors f, h of $\mathcal{C}^n_{\rm RLS}(Z,Y)$, and a real number a. Then $h = a \cdot f$ if and only if for every element x of Z, $h(x) = a \cdot f(x)$. The theorem is a consequence of (5).

(10) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, and a non empty natural number n. Then $0_{\mathcal{C}^n_{\rm RLS}(Z,Y)} = Z \longmapsto 0_Y$. The theorem is a consequence of (5).

(11) Let us consider real normed spaces X, Y, a non empty, open subset Z of X, a non empty natural number n, and a point x of X. Suppose $x \in Z$. Let us consider a finite sequence W_0 of elements of the carrier of $\mathcal{C}^n_{\rm RLS}(Z,$ the real normed space of bounded linear operators from X into $Y)$, and a finite sequence W_2 of elements of the real normed space of bounded linear operators from X into Y.

Suppose $\operatorname{dom} W_2 = \operatorname{dom} W_0$ and for every natural number i such that $i \in \operatorname{dom} W_2$ there exists a partial function W_1 from X to the real normed space of bounded linear operators from X into Y such that $W_1 = W_0(i)$ and $W_{2/i} = W_{1/x}$. Then $\sum W_2 = (\sum W_0)(x)$.

PROOF: Set $G_1 =$ the real normed space of bounded linear operators from X into Y. Set $X_1 = \mathcal{C}^n_{\rm RLS}(Z, G_1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequence W_0 of elements of the carrier of X_1 for every finite sequence W_2 of elements of G_1 such that $\operatorname{len} W_2 = \$_1$ and $\operatorname{dom} W_2 = \operatorname{dom} W_0$ and for every natural number i such that $i \in \operatorname{dom} W_2$ there exists a partial function W_1 from X to G_1 such that $W_1 = W_0(i)$ and $W_{2/i} = W_{1/x}$ holds $\sum W_2 = (\sum W_0)(x)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

2. Higher-Order Partial Differentiable Functions

Let S be a real normed space sequence, T be a real normed space, and I be a non empty finite sequence of elements of $\operatorname{dom} S$. The functor $\operatorname{PdiffSP}(S,T,I)$ yielding a real normed space sequence is defined by

(Def. 3) $\operatorname{dom} it = \operatorname{dom} I$ and $it(1) =$ the real normed space of bounded linear operators from $S(I(1)(\in \operatorname{dom} S))$ into T and for every natural number i such that $1 \leqslant i < \operatorname{len} I$ holds $it(i+1) =$ the real normed space of bounded linear operators from $S(I(i+1)(\in \operatorname{dom} S))$ into $\operatorname{NormSp}_{\mathbb{R}}(it(i))$.

Let i be a natural number. Assume $1 \leqslant i \leqslant \operatorname{len} I$. The functor $\operatorname{PdiffSP}(S, T, I, i)$ yielding a real normed space is defined by the term

(Def. 4) $(\operatorname{PdiffSP}(S, T, I))(i)$.

Now we state the proposition:

(12) Let us consider a real normed space sequence S, a real normed space T, and a non empty finite sequence I of elements of $\operatorname{dom} S$. Then

(i) $\operatorname{PdiffSP}(S, T, I, 1) =$ the real normed space of bounded linear operators from $S(I(1)(\in \operatorname{dom} S))$ into T, and

(ii) for every natural number i such that $1 \leqslant i < \operatorname{len} I$ holds $\operatorname{PdiffSP}(S, T, I, i+1) =$ the real normed space of bounded linear operators from $S(I(i+1)(\in \operatorname{dom} S))$ into $\operatorname{PdiffSP}(S, T, I, i)$.

Let S be a real normed space sequence, T be a real normed space, and I be a non empty finite sequence of elements of $\operatorname{dom} S$. The functor $\operatorname{PDiffSP}(S, T, I)$ yielding a real normed space is defined by the term

(Def. 5) $(\operatorname{PdiffSP}(S, T, I))(\operatorname{len} I)$.

Let Z be a subset of $\prod S$ and f be a partial function from $\prod S$ to T. The functor $\operatorname{PartDiffSeq}(f, Z, I)$ yielding a finite sequence is defined by

(Def. 6) $\operatorname{dom} it = \operatorname{dom} I$ and $it(1) = f{\restriction}^{I(1)} Z$ and for every natural number i such that $1 \leqslant i < \operatorname{len} I$ there exists a partial function g from $\prod S$ to $\operatorname{PdiffSP}(S, T, I, i)$ such that $it(i) = g$ and $it(i+1) = g{\restriction}^{I(i+1)} Z$.

Let i be a natural number. Assume $1 \leqslant i \leqslant \operatorname{len} I$. The functor $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, i)$ yielding a partial function from $\prod S$ to $\operatorname{PdiffSP}(S, T, I, i)$ is defined by the term

(Def. 7) $(\operatorname{PartDiffSeq}(f, Z, I))(i)$.

Now we state the proposition:

(13) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence I of elements of $\operatorname{dom} S$, and a partial function f from $\prod S$ to T. Then

(i) $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, 1) = f{\restriction}^{I(1)} Z$, and

(ii) for every natural number i such that $1 \leqslant i < \operatorname{len} I$ holds $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, i+1) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, i){\restriction}^{I(i+1)} Z$.

Let S be a real normed space sequence, T be a real normed space, Z be a subset of $\prod S$, I be a non empty finite sequence of elements of $\operatorname{dom} S$, and f be a partial function from $\prod S$ to T. We say that f is partially differentiable on Z w.r.t. I if and only if

(Def. 8) f is partially differentiable on Z w.r.t. $I(1)$ and for every natural number i such that $1 \leqslant i < \operatorname{len} I$ holds $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, i)$ is partially differentiable on Z w.r.t. $I(i+1)$.

The functor $f{\restriction}^{I} Z$ yielding a partial function from $\prod S$ to $\operatorname{PdiffSP}(S, T, I, \operatorname{len} I)$ is defined by the term

(Def. 9) $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, I, \operatorname{len} I)$.

Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, an element k of $\operatorname{dom} S$, and a partial function f from $\prod S$ to T. Now we state the propositions:

(14) f is partially differentiable on Z w.r.t. $\langle k \rangle$ if and only if f is partially differentiable on Z w.r.t. k.

(15) $f{\restriction}^{\langle k \rangle} Z = f{\restriction}^{k} Z$.

Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, non empty finite sequences I, G of elements of $\operatorname{dom} S$, and a partial function f from $\prod S$ to T. Now we state the propositions:

(16) (i) for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} G$ holds $\operatorname{PdiffSP}(S, T, G \frown I, n) = \operatorname{PdiffSP}(S, T, G, n)$ and $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, n) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G, n)$, and

(ii) for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} I$ holds $\operatorname{PdiffSP}(S, \operatorname{PdiffSP}(S, T, G, \operatorname{len} G), I, n) = \operatorname{PdiffSP}(S, T, G \frown I, \operatorname{len} G + n)$ and $\operatorname{PDiff}_{\mathrm{Seq}}(f{\restriction}^{G} Z, Z, I, n) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, \operatorname{len} G + n)$.

PROOF: For every natural number i such that $i < \operatorname{len} G$ holds $(G \frown I)(i+1) = G(i+1)$. For every natural number i such that $i < \operatorname{len} I$ holds $(G \frown I)(\operatorname{len} G + (i+1)) = I(i+1)$. Define $\mathcal{U}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} G$, then $\operatorname{PdiffSP}(S, T, G \frown I, \$_1) = \operatorname{PdiffSP}(S, T, G, \$_1)$ and $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, \$_1) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G, \$_1)$. For every natural number k such that $\mathcal{U}[k]$ holds $\mathcal{U}[k+1]$. For every natural number n, $\mathcal{U}[n]$.

Define $\mathcal{V}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} I$, then $\operatorname{PdiffSP}(S, T, G \frown I, \operatorname{len} G + \$_1) = \operatorname{PdiffSP}(S, \operatorname{PdiffSP}(S, T, G, \operatorname{len} G), I, \$_1)$ and $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, \operatorname{len} G + \$_1) = \operatorname{PDiff}_{\mathrm{Seq}}(f{\restriction}^{G} Z, Z, I, \$_1)$. For every natural number k such that $\mathcal{V}[k]$ holds $\mathcal{V}[k+1]$. For every natural number n, $\mathcal{V}[n]$. $\square$

(17) (i) $\operatorname{PdiffSP}(S, T, G \frown I, \operatorname{len} G) = \operatorname{PdiffSP}(S, T, G, \operatorname{len} G)$, and

(ii) $\operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, \operatorname{len} G) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G, \operatorname{len} G)$, and

(iii) $\operatorname{PdiffSP}(S, \operatorname{PdiffSP}(S, T, G, \operatorname{len} G), I, \operatorname{len} I) = \operatorname{PdiffSP}(S, T, G \frown I, \operatorname{len} G + \operatorname{len} I)$, and

(iv) $\operatorname{PDiff}_{\mathrm{Seq}}(f{\restriction}^{G} Z, Z, I, \operatorname{len} I) = \operatorname{PDiff}_{\mathrm{Seq}}(f, Z, G \frown I, \operatorname{len} G + \operatorname{len} I)$.

The theorem is a consequence of (16).

(18) (i) $(f{\restriction}^{G}Z){\restriction}^{I}Z = f{\restriction}^{G^\frown I}Z$, and

(ii) $\mathrm{PdiffSP}(S, \mathrm{PdiffSP}(S, T, G, \mathrm{len}\, G), I, \mathrm{len}\, I) =$
$\mathrm{PdiffSP}(S, T, G \frown I, \mathrm{len}(G \frown I))$.

The theorem is a consequence of (17).

(19) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence G of elements of $\mathrm{dom}\, S$, an element s of $\mathrm{dom}\, S$, and a partial function f from $\prod S$ to T. Then

(i) $\mathrm{PdiffSP}(S, \mathrm{PdiffSP}(S, T, G, \mathrm{len}\, G), \langle s \rangle, 1) =$
$\mathrm{PdiffSP}(S, T, G \frown \langle s \rangle, \mathrm{len}\, G + 1)$, and

(ii) $(f{\restriction}^{G}Z){\restriction}^{s}Z = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, G \frown \langle s \rangle, \mathrm{len}\, G + 1)$.

The theorem is a consequence of (17) and (13).

(20) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, non empty finite sequences I, G of elements of $\mathrm{dom}\, S$, and a partial function f from $\prod S$ to T. Then f is partially differentiable on Z w.r.t. $I \frown G$ if and only if f is partially differentiable on Z w.r.t. I and $f{\restriction}^{I}Z$ is partially differentiable on Z w.r.t. G.

Proof: For every natural number i such that $i < \mathrm{len}\, I$ holds $(I \frown G)(i+1) = I(i+1)$. For every natural number i such that $i < \mathrm{len}\, G$ holds $(I \frown G)(\mathrm{len}\, I + (i+1)) = G(i+1)$. For every natural number i such that $1 \leqslant i < \mathrm{len}(I \frown G)$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I \frown G, i)$ is partially differentiable on Z w.r.t. $(I \frown G)(i+1)$. $\square$

3. Properties of Partial Derivatives and Fundamental Operations

Now we state the propositions:

(21) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, partial functions f, g from $\prod S$ to T, and a set i. Suppose Z is open and $i \in \mathrm{dom}\, S$ and f is partially differentiable on Z w.r.t. i and g is partially differentiable on Z w.r.t. i. Then

(i) $f + g$ is partially differentiable on Z w.r.t. i, and

(ii) $(f+g){\restriction}^{i}Z = f{\restriction}^{i}Z + g{\restriction}^{i}Z$.

Proof: For every point x of $\prod S$ such that $x \in Z$ holds $(f+g){\restriction}Z$ is partially differentiable in x w.r.t. i. Set $p_1 = f{\restriction}^{i}Z + g{\restriction}^{i}Z$. For every point x of $\prod S$ such that $x \in Z$ holds $p_{1/x} = \mathrm{partdiff}(f+g, x, i)$. $\square$

(22) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a partial function f from $\prod S$ to T, and a set i. Suppose Z is open and f is differentiable on Z. Then

(i) f is partially differentiable on Z w.r.t. i, and

(ii) for every point x of $\prod S$ such that $x \in Z$ holds $(f {\restriction}^i Z)_{/x} = f'_{\restriction Z/x} \cdot (\mathrm{reproj}(i(\in \mathrm{dom}\, S), 0_{\prod S}))$.

Proof: For every point x of $\prod S$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f, x, i) = f'(x) \cdot (\mathrm{reproj}(i(\in \mathrm{dom}\, S), 0_{\prod S}))$ by [9, (32),(28)]. □

Let us consider a real normed space sequence S and a set s. Now we state the propositions:

(23) $\mathrm{reproj}(s(\in \mathrm{dom}\, S), 0_{\prod S})$ is a Lipschitzian linear operator from $S(s(\in \mathrm{dom}\, S))$ into $\prod S$.

(24) (i) the projection onto $s(\in \mathrm{dom}\, S)$ is a Lipschitzian linear operator from $\prod S$ into $S(s(\in \mathrm{dom}\, S))$, and

(ii) there exists a point L of the real normed space of bounded linear operators from $\prod S$ into $S(s(\in \mathrm{dom}\, S))$ such that $L =$ the projection onto $s(\in \mathrm{dom}\, S)$ and $\|L\| \leqslant 1$.

Proof: Set $L_1 =$ the projection onto $s(\in \mathrm{dom}\, S)$. Set $F = S(s(\in \mathrm{dom}\, S))$. Set $E = \prod S$. Reconsider $L = L_1$ as a point of the real normed space of bounded linear operators from E into F. For every real number x such that $x \in \mathrm{PreNorms}(\mathrm{PartFuncs}(L, E, F))$ holds $x \leqslant 1$. $\mathrm{PreNorms}(\mathrm{PartFuncs}(L, E, F))$ is upper bounded. □

Let us consider a real normed space sequence S, a real normed space G, a non empty subset Z of $\prod S$, a partial function f from $\prod S$ to G, a set s, and a natural number i. Now we state the propositions:

(25) If f is differentiable $i+1$ times on Z, then $f {\restriction}^s Z$ is differentiable i times on Z.

Proof: f is partially differentiable on Z w.r.t. s and for every point z of $\prod S$ such that $z \in Z$ holds $(f {\restriction}^s Z)_{/z} = f'_{\restriction Z/z} \cdot (\mathrm{reproj}(s(\in \mathrm{dom}\, S), 0_{\prod S}))$. Set $P_1 = f {\restriction}^s Z$. Reconsider $L_1 = \mathrm{reproj}(s(\in \mathrm{dom}\, S), 0_{\prod S})$ as a Lipschitzian linear operator from $S(s(\in \mathrm{dom}\, S))$ into $\prod S$. Set $B_1 =$ the real normed space of bounded linear operators from $S(s(\in \mathrm{dom}\, S))$ into $\prod S$. Reconsider $M_1 = Z \longmapsto L_1$ as a partial function from $\prod S$ to B_1. For every point z of $\prod S$ such that $z \in Z$ holds $(f {\restriction}^s Z)_{/z} = f'_{\restriction Z/z} \cdot (M_{1/z})$.

Consider B being a Lipschitzian bilinear operator from the real normed space of bounded linear operators from $S(s(\in \mathrm{dom}\, S))$ into $\prod S \times$ the real normed space of bounded linear operators from $\prod S$ into G into the real normed space of bounded linear operators from $S(s(\in \mathrm{dom}\, S))$ into G such that for every point u of the real normed space of bounded linear operators from $S(s(\in \mathrm{dom}\, S))$ into $\prod S$ and for every point v of the real normed space of bounded linear operators from $\prod S$ into G, $B(u,v) = v \cdot u$. Set $w_2 = \langle M_1, f'_{\restriction Z}\rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $\prod S$ to the real normed space of bounded linear operators from $S(s(\in \operatorname{dom} S))$ into G. For every object x_0 such that $x_0 \in \operatorname{dom} P_1$ holds $P_1(x_0) = W(x_0)$. $\square$

(26) Suppose f is differentiable $i+1$ times on Z and $\operatorname{diff}_Z(f, i+1)$ is continuous on Z. Then

(i) $f \restriction^s Z$ is differentiable i times on Z, and

(ii) $\operatorname{diff}_Z(f \restriction^s Z, i)$ is continuous on Z.

PROOF: f is partially differentiable on Z w.r.t. s and for every point z of $\prod S$ such that $z \in Z$ holds $(f \restriction^s Z)_{/z} = f'_{\restriction Z/z} \cdot (\operatorname{reproj}(s(\in \operatorname{dom} S), 0_{\prod S}))$. Set $P_1 = f \restriction^s Z$. Set $E = S(s(\in \operatorname{dom} S))$. Reconsider $L_0 = \operatorname{reproj}(s(\in \operatorname{dom} S), 0_{\prod S})$ as a Lipschitzian linear operator from $S(s(\in \operatorname{dom} S))$ into $\prod S$. Set $B_1 =$ the real normed space of bounded linear operators from E into $\prod S$. Reconsider $L_1 = Z \longmapsto L_0$ as a partial function from $\prod S$ to B_1. For every point z of $\prod S$ such that $z \in Z$ holds $(f \restriction^s Z)_{/z} = f'_{\restriction Z/z} \cdot (L_{1/z})$.

Consider B being a Lipschitzian bilinear operator from the real normed space of bounded linear operators from E into $\prod S \times$ the real normed space of bounded linear operators from $\prod S$ into G into the real normed space of bounded linear operators from E into G such that for every point u of the real normed space of bounded linear operators from E into $\prod S$ and for every point v of the real normed space of bounded linear operators from $\prod S$ into G, $B(u,v) = v \cdot u$. Set $w_2 = \langle L_1, f'_{\restriction Z}\rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $\prod S$ to the real normed space of bounded linear operators from E into G. For every object x_0 such that $x_0 \in \operatorname{dom} P_1$ holds $P_1(x_0) = W(x_0)$. $\square$

Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a partial function f from $\prod S$ to T, a real number a, and a set i. Now we state the propositions:

(27) Suppose Z is open and $i \in \operatorname{dom} S$ and f is partially differentiable on Z w.r.t. i. Then

(i) $a \cdot f$ is partially differentiable on Z w.r.t. i, and

(ii) $(a \cdot f) \restriction^i Z = a \cdot (f \restriction^i Z)$.

PROOF: For every point x of $\prod S$ such that $x \in Z$ holds $(a \cdot f) \restriction Z$ is partially differentiable in x w.r.t. i. Set $p_1 = a \cdot (f \restriction^i Z)$. For every point x of $\prod S$ such that $x \in Z$ holds $p_{1/x} = \operatorname{partdiff}(a \cdot f, x, i)$. $\square$

(28) Suppose Z is open and $i \in \operatorname{dom} S$ and f is partially differentiable on Z w.r.t. i. Then

(i) $-f$ is partially differentiable on Z w.r.t. i, and

(ii) $(-f){\restriction^i}Z = -f{\restriction^i}Z$.

The theorem is a consequence of (27).

(29) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, partial functions f, g from $\prod S$ to T, and a set i. Suppose Z is open and $i \in \operatorname{dom} S$ and f is partially differentiable on Z w.r.t. i and g is partially differentiable on Z w.r.t. i. Then

(i) $f - g$ is partially differentiable on Z w.r.t. i, and

(ii) $(f - g){\restriction^i}Z = f{\restriction^i}Z - g{\restriction^i}Z$.

The theorem is a consequence of (28) and (21).

4. Higher-Order Partial and Total Differentiability

Now we state the propositions:

(30) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence I of elements of $\operatorname{dom} S$, and partial functions f, g from $\prod S$ to T. Suppose Z is open and f is partially differentiable on Z w.r.t. I and g is partially differentiable on Z w.r.t. I. Then

(i) $f + g$ is partially differentiable on Z w.r.t. I, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(f + g, Z, I, i) = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i) + \mathrm{PDiff}_{\mathrm{Seq}}(g, Z, I, i)$.

Proof: $f{\restriction^{I(1)}}Z = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, 1)$. $g{\restriction^{I(1)}}Z = \mathrm{PDiff}_{\mathrm{Seq}}(g, Z, I, 1)$. $\mathrm{PdiffSP}(S, T, I, 1) =$ the real normed space of bounded linear operators from $S(I(1)(\in \operatorname{dom} S))$ into T. $\mathrm{PDiff}_{\mathrm{Seq}}(f + g, Z, I, 1) = (f + g){\restriction^{I(1)}}Z$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 < \operatorname{len} I$, then $\mathrm{PDiff}_{\mathrm{Seq}}(f+g, Z, I, \$_1)$ is partially differentiable on Z w.r.t. $I(\$_1+1)$ and $\mathrm{PDiff}_{\mathrm{Seq}}(f+g, Z, I, \$_1 + 1) = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, \$_1 + 1) + \mathrm{PDiff}_{\mathrm{Seq}}(g, Z, I, \$_1 + 1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(f + g, Z, I, i) = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i) + \mathrm{PDiff}_{\mathrm{Seq}}(g, Z, I, i)$. $\square$

(31) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence I of elements of $\operatorname{dom} S$, a partial function f from $\prod S$ to T, and a real number a. Suppose Z is open and f is partially differentiable on Z w.r.t. I. Then

(i) $a \cdot f$ is partially differentiable on Z w.r.t. I, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(a \cdot f, Z, I, i) = a \cdot \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i)$.

Proof: $f{\restriction}^{I(1)}Z = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, 1)$. $\mathrm{PdiffSP}(S, T, I, 1) =$ the real normed space of bounded linear operators from $S(I(1)(\in \operatorname{dom} S))$ into T. $\mathrm{PDiff}_{\mathrm{Seq}}(a \cdot f, Z, I, 1) = (a \cdot f){\restriction}^{I(1)}Z$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 < \operatorname{len} I$, then $\mathrm{PDiff}_{\mathrm{Seq}}(a \cdot f, Z, I, \$_1)$ is partially differentiable on Z w.r.t. $I(\$_1 + 1)$ and $\mathrm{PDiff}_{\mathrm{Seq}}(a \cdot f, Z, I, \$_1 + 1) = a \cdot \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, \$_1 + 1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k, $\mathcal{P}[k]$. For every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(a \cdot f, Z, I, i) = a \cdot \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i)$. □

(32) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence I of elements of $\operatorname{dom} S$, and a partial function f from $\prod S$ to T. Suppose Z is open and f is partially differentiable on Z w.r.t. I. Then

(i) $-f$ is partially differentiable on Z w.r.t. I, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds
$\mathrm{PDiff}_{\mathrm{Seq}}(-f, Z, I, i) = -\mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i)$.

The theorem is a consequence of (31).

(33) Let us consider a real normed space sequence S, a real normed space T, a subset Z of $\prod S$, a non empty finite sequence I of elements of $\operatorname{dom} S$, and partial functions f, g from $\prod S$ to T. Suppose Z is open and f is partially differentiable on Z w.r.t. I and g is partially differentiable on Z w.r.t. I. Then

(i) $f - g$ is partially differentiable on Z w.r.t. I, and

(ii) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $\mathrm{PDiff}_{\mathrm{Seq}}(f - g, Z, I, i) = \mathrm{PDiff}_{\mathrm{Seq}}(f, Z, I, i) - \mathrm{PDiff}_{\mathrm{Seq}}(g, Z, I, i)$.

The theorem is a consequence of (32) and (30).

Let S be a real normed space sequence, T be a real normed space, Z be a subset of $\prod S$, f be a partial function from $\prod S$ to T, and k be a natural number. We say that f is partial differentiable up to order k on Z if and only if

(Def. 10) for every non empty finite sequence I of elements of $\operatorname{dom} S$ such that $\operatorname{len} I \leqslant k$ holds f is partially differentiable on Z w.r.t. I.

Let us consider a real normed space sequence G, a non empty subset X of $\prod G$, a natural number n, a real normed space F, a partial function f from $\prod G$ to F, and a non empty finite sequence I of elements of $\operatorname{dom} G$. Now we state the propositions:

(34) If X is open, then if f is differentiable n times on X, then if $\operatorname{len} I \leqslant n$, then f is partially differentiable on X w.r.t. I.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed space F for every partial function f from $\prod G$ to F such that f is differentiable $\$_1$ times on X for every non empty finite sequence I of elements of $\operatorname{dom} G$ such that $\operatorname{len} I \leqslant \$_1$ holds f is partially differentiable on X w.r.t. I. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(35) Suppose X is open. Then suppose f is differentiable n times on X and $\operatorname{diff}_X(f, n)$ is continuous on X. Then if $\operatorname{len} I \leqslant n$, then f is partially differentiable on X w.r.t. I and $f{\restriction}^I X$ is continuous on X.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed space F for every partial function f from $\prod G$ to F such that f is differentiable $\$_1$ times on X and $\operatorname{diff}_X(f, \$_1)$ is continuous on X for every non empty finite sequence I of elements of $\operatorname{dom} G$ such that $\operatorname{len} I \leqslant \$_1$ holds f is partially differentiable on X w.r.t. I and $f{\restriction}^I X$ is continuous on X. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

(36) Let us consider a real normed space sequence G, a non empty, open subset X of $\prod G$, a real normed space F, a partial function f from $\prod G$ to F, and a natural number n. Suppose $1 \leqslant n$. Then f is differentiable n times on X and $\operatorname{diff}_X(f, n)$ is continuous on X if and only if for every non empty finite sequence I of elements of $\operatorname{dom} G$ such that $\operatorname{len} I \leqslant n$ holds f is partially differentiable on X w.r.t. I and $f{\restriction}^I X$ is continuous on X.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed space F for every partial function f from $\prod G$ to F such that $1 \leqslant \$_1$ and for every non empty finite sequence I of elements of $\operatorname{dom} G$ such that $\operatorname{len} I \leqslant \$_1$ holds f is partially differentiable on X w.r.t. I and $f{\restriction}^I X$ is continuous on X holds f is differentiable $\$_1$ times on X and $\operatorname{diff}_X(f, \$_1)$ is continuous on X. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$. $\square$

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