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Extensions of Languages in Polish Notation[1]

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Summary. This article prepares a collection of basic tools for adding new operators to a language in Polish notation, a parenthesis-free logical system where operators precede their operands. A need for this arose while attempting to formalize some extensions of Roman Suszko's basic non-Fregean logic SCI, namely WB and WH.

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Introduction

In this article, which is continuation of [8], a systematic toolkit for extending formal languages expressed in parenthesis-free Polish notation by adding new symbols for operators and adjusting the corresponding arity-functions is formally developed. This need emerged in the context of formalizing extensions of the basic non-Fregean logic [6], [11] proposed by Suszko [1], [10]. We define in Mizar the notions of "Polish-arity-function" over previously defined "Polish language" and show how one can consistently build larger languages (extensions) that preserve syntactic well-formedness.

In Section 1 we introduce the concept of a Mizar attribute "Polish-arity-like" for function, and then define the conditions under which the arity-function C is a valid extension of another B (so-called "B-extending"). It demonstrates that for any given Polish language T (with arity B), there exist extended Polish languages U (with arity C) such that the set of well-formed formulas under B

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is properly included in the set of well-formed formulas under C. Furthermore, the extended language remains closed and compatible under the extended arity, meaning that new formulas built with newly introduced operators satisfy the syntactic constraints induced by C. In Section 4 we show that for each new symbol p in the extended arity, unary or binary operations on formulas are well-defined, and every formula headed by p admits a unique structural decomposition.

This modular method for enlarging the vocabulary of a logical language allows us to build future formalizations of logics based on Polish notation [7], with the primary aim of non-Fregean identity logics [3], [4], [2] such as SCI (Sentential Calculus with Identity), and support building extended semantic or proof-theoretic systems as in [9] over them expressing WB and WH [5].

1. Preliminaries: Polish Arity Function

From now on a, b denote objects, k, l, m, n denote natural numbers, p, q, r, s denote finite sequences, P denotes a non empty, finite sequence-membered set, S, T denote Polish languages, V denotes a Polish language of T, K denotes a non trivial Polish language, and E denotes a Polish arity-function of K.

Let F be a function. We say that F is Polish-arity-like if and only if

(Def. 1) there exists T such that F is a Polish arity-function of T.

Let us consider T. Note that every Polish arity-function of T is Polish-arity-like.

Let F be a function. Let us observe that F is Polish-arity-like if and only if the condition (Def. 2) is satisfied.

(Def. 2) $\operatorname{dom} F$ is a Polish language and F is natural-valued and has zero.

Note that every function which is Polish-arity-like is also natural-valued and has also zero and there exists a function which is Polish-arity-like.

A Polish-arity-function is a Polish-arity-like function. From now on B denotes a Polish-arity-function.

Let B, C be Polish-arity-functions. We say that C is B-extending if and only if

(Def. 3) $B \subseteq C$.

Let B be a Polish-arity-function. Note that the functor $\operatorname{dom} B$ yields a Polish language. Now we state the proposition:

(1) B is a Polish arity-function of T if and only if $T = \operatorname{dom} B$.

2. The Set of Well Formed Formulas

In the sequel A denotes a Polish arity-function of T.

Let B be a Polish-arity-function. The functor PolishWFFSet(B) yielding a Polish language of $\operatorname{dom} B$ is defined by

(Def. 4) there exists T and A such that $A = B$ and $it = $ Polish-WFF-set(T, A).

Let us consider T and A. Note that A is empty yielding if and only if the condition (Def. 5) is satisfied.

(Def. 5) for every a such that $a \in T$ holds $A(a) = 0$.

Let us consider P. Let us note that P is A-closed if and only if the condition (Def. 6) is satisfied.

(Def. 6) for every p, n, and q such that $p \in T$ and $n = A(p)$ and $q \in P^\frown n$ holds $p^\frown q \in P$.

One can check that there exists a Polish arity-function of T which is empty yielding.

Let us consider V. We say that V is full if and only if

(Def. 7) there exists A such that $V = $ Polish-WFF-set(T, A).

Observe that there exists a Polish language of T which is full.

Let us consider A. Observe that the functor Polish-WFF-set(T, A) yields a full Polish language of T.

3. Language Extensions

Let T, U be Polish languages. We say that U is T-extending if and only if

(Def. 8) $T \subseteq U$.

Let us consider T. Observe that there exists a Polish language which is T-extending.

Let us consider K. Let us observe that every Polish language which is K-extending is also non trivial.

Let us consider B and S. We say that S is B-compatible if and only if

(Def. 9) for every s such that $s \in S$ and s is $(\operatorname{dom} B)$-headed there exists n such that $n = B(\operatorname{dom} B\text{-head}(s))$ and $\operatorname{dom} B\text{-tail}(s) \in S^\frown n$.

Let us consider T and A. Let us note that S is A-compatible if and only if the condition (Def. 10) is satisfied.

(Def. 10) for every s such that $s \in S$ and s is T-headed holds $T\text{-tail}(s) \in S^\frown A(T\text{-head}(s))$.

Let us note that Polish-WFF-set(T, A) is A-compatible and there exists a Polish language which is A-closed and A-compatible.

Let us consider B. One can check that there exists a Polish-arity-function which is B-extending and there exists a Polish language which is B-closed and B-compatible.

A Polish-ext-set of B is a B-closed, B-compatible Polish language.

An extension of B is a B-extending Polish-arity-function. From now on C denotes an extension of B.

Let us consider B and C. One can check that every Polish-arity-function which is C-extending is also B-extending and every Polish language which is C-closed is also B-closed and every Polish language which is C-compatible is also B-compatible. In the sequel Z denotes a B-closed Polish language. Now we state the propositions:

(2) If $p \in \operatorname{dom} B$ and $B(p) = 1$ and $q \in Z$, then $p \frown q \in Z$.

(3) If $p \in \operatorname{dom} B$ and $B(p) = 2$ and q, $r \in Z$, then $p \frown (q \frown r) \in Z$.

4. The Operations

Let us consider B, Z, and p. Assume $p \in \operatorname{dom} B$ and $B(p) = 1$. The functor Polish-unOp(B, Z, p) yielding a unary operation on Z is defined by

(Def. 11) for every q such that $q \in Z$ holds $it(q) = p \frown q$.

Assume $p \in \operatorname{dom} B$ and $B(p) = 2$. The functor Polish-binOp(B, Z, p) yielding a binary operation on Z is defined by

(Def. 12) for every q and r such that q, $r \in Z$ holds $it(q, r) = p \frown (q \frown r)$.

From now on J denotes a Polish-ext-set of B.

Let us consider B and J. The functor PolishExtComplement(B, J) yielding a subset of J is defined by the term

(Def. 13) $\{p$, where p is an element of $J : p$ is not $(\operatorname{dom} B)$-headed$\}$.

The functor PolishExtDomain(B, J) yielding a $(\operatorname{dom} B)$-extending Polish language is defined by the term

(Def. 14) $\operatorname{dom} B \cup \text{PolishExtComplement}(B, J)$.

Now we state the proposition:

(4) PolishWFFSet$(B) \subseteq J$. The theorem is a consequence of (1).

Let us consider P and n. The functor $P{\restriction}n$ yielding a subset of P is defined by the term

(Def. 15) $\{p$, where p is an element of $P : \operatorname{len} p \leqslant n\}$.

Now we state the proposition:

(5) T is a full Polish language of T.

5. The Arity and the Head in Extensions

In the sequel V denotes a full Polish language of T, U denotes a T-extending Polish language, and W denotes a full Polish language of U.

Let us consider T and V. The Polish arity V yielding a Polish arity-function of T is defined by

(Def. 16) $V = \text{Polish-WFF-set}(T, it)$.

Let us consider A. Let us note that the Polish arity Polish-WFF-set(T, A) reduces to A.

Let us consider V. Observe that Polish-WFF-set$(T, \text{the Polish arity } V)$ reduces to V.

Let us consider B and J. The functor PolishExtArity(B, J) yielding a Polish arity-function of PolishExtDomain(B, J) is defined by

(Def. 17) $J = \text{Polish-WFF-set}(\text{PolishExtDomain}(B, J), it)$.

Let us consider T.

A formula of T is an element of T. Let us consider B and J. Let F be a formula of J.

The functor PolishExtHead(F) yielding an element of PolishExtDomain(B, J) is defined by the term

(Def. 18) PolishExtDomain(B, J)-head(F).

Now we state the proposition:

(6) If S-head$(p) \in T$, then p is T-headed and S-head$(p) = T$-head(p).

Let us consider B, J, and a formula F of J. Now we state the propositions:

(7) If PolishExtHead$(F) \in \operatorname{dom} B$, then PolishExtHead$(F) = \operatorname{dom} B$-head$(F)$.

(8) If F is $(\operatorname{dom} B)$-headed, then PolishExtHead$(F) = \operatorname{dom} B$-head$(F)$.

From now on M denotes a Polish-ext-set of C, e denotes an element of $\operatorname{dom} C$, and F, G, H denote formulae of M. Now we state the propositions:

(9) If $C(e) = 1$ and PolishExtHead$(F) = e$, then there exists G such that $F = (\text{Polish-unOp}(C, M, e))(G)$. The theorem is a consequence of (6).

(10) If $C(e) = 1$, then PolishExtHead$((\text{Polish-unOp}(C, M, e))(G)) = e$. The theorem is a consequence of (8).

(11) If $C(e) = 2$ and PolishExtHead$(F) = e$, then there exists G and there exists H such that $F = (\text{Polish-binOp}(C, M, e))(G, H)$. The theorem is a consequence of (6).

(12) If $C(e) = 2$, then PolishExtHead$((\text{Polish-binOp}(C, M, e))(G, H)) = e$. The theorem is a consequence of (8).

Let us consider T, U, V, and W. Now we state the propositions:

(13) the Polish arity $V \subseteq$ the Polish arity W if and only if $V \subseteq W$.

PROOF: If the Polish arity $V \subseteq$ the Polish arity W, then $V \subseteq W$. Set $A =$ the Polish arity V. Set $B =$ the Polish arity W. For every a such that $a \in \operatorname{dom} A$ holds $A(a) = B(a)$. □

(14) $V \subseteq W$ if and only if W is (the Polish arity V)-closed.

PROOF: Set $A =$ the Polish arity V. If $V \subseteq W$, then W is A-closed. □

Let us consider T, U, V, and W. One can check that W is V-extending if and only if the condition (Def. 19) is satisfied.

(Def. 19) the Polish arity $V \subseteq$ the Polish arity W.

Let us observe that there exists a full Polish language of U which is V-extending.

An extension of V is a Polish-ext-set of the Polish arity V. One can verify that every extension of V is (the Polish arity V)-closed and (the Polish arity V)-compatible.

6. The Operations in Extensions Revisited

In the sequel Q denotes an extension of V. Now we state the proposition:

(15) $V \subseteq Q$.

Let us consider T, V, and Q. The functor PolishExtComplement(Q) yielding a subset of Q is defined by the term

(Def. 20) PolishExtComplement(the Polish arity V, Q).

The functor PolishExtDomain(Q) yielding a T-extending Polish language is defined by the term

(Def. 21) PolishExtDomain(the Polish arity V, Q).

The functor PolishExtArity(Q) yielding a Polish arity-function of PolishExt-Domain(Q) is defined by the term

(Def. 22) PolishExtArity(the Polish arity V, Q).

Let F be a formula of Q. The functor PolishExtHead(F) yielding an element of PolishExtDomain(Q) is defined by the term

(Def. 23) PolishExtDomain(Q)-head(F).

Let us consider T, V, Q, and a formula F of Q. Now we state the propositions:

(16) If PolishExtHead(F) $\in T$, then PolishExtHead(F) $= T$-head(F).

(17) If F is T-headed, then PolishExtHead(F) $= T$-head(F).

From now on M denotes an extension of Polish-WFF-set(K, E), e denotes an element of K, and F, G, H denote formulae of M.

Let us consider T, V, and Q. Let t be an element of T. Assume (the Polish arity $V)(t) = 1$. The functor Polish-unOp(T, Q, t) yielding a unary operation on Q is defined by

(Def. 24) for every p such that $p \in Q$ holds $it(p) = t \frown p$.

Assume (the Polish arity $V)(t) = 2$. The functor Polish-binOp(T, Q, t) yielding a binary operation on Q is defined by

(Def. 25) for every p and q such that $p, q \in Q$ holds $it(p, q) = t \frown (p \frown q)$.

Now we state the propositions:

(18) If $E(e) = 1$ and PolishExtHead$(F) = e$, then there exists G such that $F =$ (Polish-unOp$(K, M, e))(G)$. The theorem is a consequence of (6).

(19) If $E(e) = 1$, then PolishExtHead((Polish-unOp$(K, M, e))(G)) = e$. The theorem is a consequence of (17).

(20) If $E(e) = 2$ and PolishExtHead$(F) = e$, then there exists G and there exists H such that $F =$ (Polish-binOp$(K, M, e))(G, H)$. The theorem is a consequence of (6).

(21) If $E(e) = 2$, then PolishExtHead((Polish-binOp$(K, M, e))(G, H)) = e$. The theorem is a consequence of (17).

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