

Higher-Order Differentiation and Inverse Function Theorem in Real Normed Spaces¹

Kazuhisa Nakasho 
Yamaguchi University
Yamaguchi, Japan

Yasunari Shidama
Karuizawa Hotch 244-1
Nagano, Japan

Summary. This article extends the formalization of the theory of differentiation in real normed spaces in the Mizar system. The focus is on higher-order derivatives and the inverse function theorem. Additionally, we encode the differentiability of the inversion operator on invertible linear operators.

MSC: 46G05 47A07 68V20

Keywords: higher-order derivative; inverse function theorem; real normed space; vector-valued function

MML identifier: NDIFF13, version: 8.1.14 5.87.1483

INTRODUCTION

This article, using the Mizar system [1], [2], extends the theory of differentiation in real normed spaces [5], focusing on higher-order derivatives and the inverse function theorem [7]. The work presents a comprehensive treatment of higher-order derivatives for vector-valued functions and develops theorems on the composition of differentiable functions and their higher-order derivatives [10], [11]. It provides an analysis of partial derivatives for multivariable vector-valued functions and extends the inverse function theorem to higher-order differentiability, including continuity properties [12]. Additionally, the paper formalizes the differentiability of the inversion operator on invertible linear operators [3].

¹This work was supported by JSPS KAKENHI Grant Number 24K14897.

1. FOUNDATIONS OF DIFFERENTIATION IN REAL NORMED SPACES

From now on E, F, G, S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T , Z denotes a subset of S , and i, n denote natural numbers. Now we state the proposition:

- (1) Let us consider real normed spaces S, T , a partial function f from S to T , a subset Z of S , and a point x of S . Suppose Z is open and $x \in Z$ and $Z \subseteq \text{dom } f$. Then $f|Z$ is differentiable in x if and only if f is differentiable in x .

Let us consider real normed spaces S, T , a partial function f from S to T , and a subset Z of S . Now we state the propositions:

- (2) $f|Z$ is differentiable on Z if and only if f is differentiable on Z .
 (3) If f is differentiable on Z , then $(f|Z)'|_Z = f'|_Z$. The theorem is a consequence of (2).

Let us consider real normed spaces S, T , a partial function f from S to T , and subsets X, Z of S . Now we state the propositions:

- (4) If Z is open and $Z \subseteq X$ and f is differentiable on X , then $f'|_Z = f'|_X|Z$.
 PROOF: For every object x such that $x \in \text{dom}(f'|_X|Z)$ holds $(f'|_X|Z)(x) = f'|_Z(x)$. $\square$
 (5) If Z is open and $Z \subseteq X$ and f is differentiable on X and $f'|_X$ is continuous on X , then $f'|_Z$ is continuous on Z . The theorem is a consequence of (4).

Let us consider real normed spaces S, T , a partial function f from S to T , a subset Z of S , and a natural number k . Now we state the propositions:

- (6) Suppose f is differentiable k times on Z . Then
 (i) $f|Z$ is differentiable k times on Z , and
 (ii) $\text{diff}_Z(f|Z, k) = \text{diff}_Z(f, k)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if f is differentiable $\$1$ times on Z , then $f|Z$ is differentiable $\$1$ times on Z and $\text{diff}_Z(f|Z, \$1) = \text{diff}_Z(f, \$1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k , $\mathcal{P}[k]$. $\square$

- (7) Suppose f is differentiable k times on Z and $\text{diff}_Z(f, k)$ is continuous on Z . Then
 (i) $f|Z$ is differentiable k times on Z , and
 (ii) $\text{diff}_Z(f|Z, k)$ is continuous on Z .

Let us consider real normed spaces S, T , a partial function f from S to T , subsets X, Z of S , and a natural number i . Now we state the propositions:

- (8) Suppose Z is open and $Z \subseteq X$. Then if f is differentiable i times on X , then f is differentiable i times on Z and $\text{diff}_Z(f, i) = \text{diff}_X(f, i)|_Z$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if f is differentiable $\$1$ times on X , then f is differentiable $\$1$ times on Z and $\text{diff}_Z(f, \$1) = \text{diff}_X(f, \$1)|_Z$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (9) Suppose Z is open and $Z \subseteq X$. Then suppose f is differentiable i times on X and $\text{diff}_X(f, i)$ is continuous on X . Then

- (i) f is differentiable i times on Z , and
- (ii) $\text{diff}_Z(f, i)$ is continuous on Z .

The theorem is a consequence of (8).

- (10) Let us consider real normed spaces X, Y , a real number a , Lipschitzian linear operators v_1, v_2 from X into Y , and points w_1, w_2 of the real norm space of bounded linear operators from X into Y . Suppose $v_1 = w_1$ and $v_2 = w_2$. Then

- (i) $v_1 + v_2 = w_1 + w_2$, and
- (ii) $a \cdot v_1 = a \cdot w_1$.

PROOF: Reconsider $w_{12} = w_1 + w_2$ as a point of the real norm space of bounded linear operators from X into Y . For every object s such that $s \in \text{dom}(v_1 + v_2)$ holds $(v_1 + v_2)(s) = w_{12}(s)$. Reconsider $w_{12} = a \cdot w_1$ as a point of the real norm space of bounded linear operators from X into Y . For every object s such that $s \in \text{dom}(a \cdot v_1)$ holds $(a \cdot v_1)(s) = w_{12}(s)$. $\square$

- (11) Let us consider real normed spaces X, Y , Lipschitzian linear operators v_1, v_2, v_3 from X into Y , and real numbers a, b . Then

- (i) $v_1 + v_2 = v_2 + v_1$, and
- (ii) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$, and
- (iii) $a \cdot (v_1 + v_2) = a \cdot v_1 + a \cdot v_2$, and
- (iv) $(a + b) \cdot v_1 = a \cdot v_1 + b \cdot v_1$, and
- (v) $a \cdot b \cdot v_1 = a \cdot (b \cdot v_1)$.

The theorem is a consequence of (10).

- (12) Let us consider real normed spaces X, Y, Z , a Lipschitzian linear operator v from X into Y , a Lipschitzian linear operator s from Y into Z , a point p_6 of the real norm space of bounded linear operators from X into Y , and a point p_5 of the real norm space of bounded linear operators from Y into Z . If $v = p_6$ and $s = p_5$, then $s \cdot v = p_5 \cdot p_6$.

- (13) Let us consider real normed spaces X, Y, Z , Lipschitzian linear operators v_1, v_2 from X into Y , Lipschitzian linear operators s_1, s_2 from Y into Z , and a real number a . Then

- (i) $s_1 \cdot (v_1 + v_2) = s_1 \cdot v_1 + s_1 \cdot v_2$, and
- (ii) $(s_1 + s_2) \cdot v_1 = s_1 \cdot v_1 + s_2 \cdot v_1$, and
- (iii) $s_1 \cdot (a \cdot v_1) = a \cdot s_1 \cdot v_1$.

The theorem is a consequence of (10) and (12).

- (14) Let us consider real normed spaces S, T, U , a partial function f_1 from S to T , a partial function f_2 from T to U , and a point x_0 of S . Suppose $x_0 \in \text{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in f_1/x_0 . Then $f_2 \cdot f_1$ is continuous in x_0 .

PROOF: Set $f = f_2 \cdot f_1$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \text{dom } f$ and $\|x_1 - x_0\| < s$ holds $\|f/x_1 - f/x_0\| < r$. $\square$

- (15) Let us consider real normed spaces E, F, G , a subset Z of E , a subset T of F , a partial function u from E to F , and a partial function v from F to G . Suppose $u^\circ Z \subseteq T$ and u is continuous on Z and v is continuous on T . Then $v \cdot u$ is continuous on Z .

PROOF: Set $f = v \cdot u$. For every point x_0 of E and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f/x_1 - f/x_0\| < r$. $\square$

- (16) Let us consider real normed spaces X, Y , a point x of X , a point y of Y , and a point z of $X \times Y$. Suppose $z = \langle x, y \rangle$. Then $\|z\| \leq \|x\| + \|y\|$.
- (17) Let us consider real normed spaces E, F, G , a partial function u from E to F , a Lipschitzian linear operator L from F into G , and a point x of E . Suppose u is differentiable in x . Then

- (i) $L \cdot u$ is differentiable in x , and
- (ii) $(L \cdot u)'(x) = L \cdot u'(x)$.

- (18) Let us consider real normed spaces E, F, G , a subset Z of E , a subset T of F , a partial function u from E to F , and a partial function v from F to G . Suppose $u^\circ Z \subseteq T$ and u is differentiable on Z and v is differentiable on T . Then

- (i) $v \cdot u$ is differentiable on Z , and
- (ii) for every point x of E such that $x \in Z$ holds $(v \cdot u)'_{|Z/x} = v'_{|T/u_x} \cdot (u'_{|Z/x})$.

PROOF: For every point x of E such that $x \in Z$ holds $v \cdot u$ is differentiable in x and $(v \cdot u)'(x) = v'(u/x) \cdot u'(x)$. For every point x of E such that $x \in Z$ holds $(v \cdot u)'_{|Z/x} = v'_{|T/u/x} \cdot (u'_{|Z/x})$. $\square$

(19) Let us consider real normed spaces F , G , and a Lipschitzian linear operator L from F into G . Then

- (i) L is differentiable on Ω_F , and
- (ii) $L'_{|\Omega_F}$ is continuous on Ω_F , and
- (iii) for every point x of F , $L'_{|\Omega_F/x} = L$.

(20) Let us consider real normed spaces E , F , G , a partial function u from E to F , a subset Z of E , and a Lipschitzian linear operator L from F into G . Suppose u is differentiable on Z . Then

- (i) $L \cdot u$ is differentiable on Z , and
- (ii) for every point x of E such that $x \in Z$ holds $(L \cdot u)'_{|Z/x} = L \cdot (u'_{|Z/x})$.

The theorem is a consequence of (19) and (18).

Let E , F , G be real normed spaces and L be a Lipschitzian linear operator from F into G . The functor $\text{LTRN}(L, E)$ yielding a function is defined by

(Def. 1) $\text{dom } it = \mathbb{N}$ and $it(0) = L$ and for every natural number i , there exists a Lipschitzian linear operator K from $\text{diff}_{\text{SP}}(E^{(i+1)}, F)$ into $\text{diff}_{\text{SP}}(E^{(i+1)}, G)$ and there exists a Lipschitzian linear operator M from $\text{diff}_{\text{SP}}(E^i, F)$ into $\text{diff}_{\text{SP}}(E^i, G)$ such that $it(i+1) = K$ and $it(i) \in (\text{the real norm space of bounded linear operators from } \text{diff}_{\text{SP}}(E^i, F) \text{ into } \text{diff}_{\text{SP}}(E^i, G)) = M$ and for every Lipschitzian linear operator V from E into $\text{diff}_{\text{SP}}(E^i, F)$, $K(V) = M \cdot V$.

Let i be a natural number. The functor $\text{LTRN}(i, L, E)$ yielding a Lipschitzian linear operator from $\text{diff}_{\text{SP}}(E^i, F)$ into $\text{diff}_{\text{SP}}(E^i, G)$ is defined by the term

(Def. 2) $(\text{LTRN}(L, E))(i)$.

2. HIGHER-ORDER DIFFERENTIATION AND FUNCTION COMPOSITION

Now we state the propositions:

(21) Let us consider real normed spaces E , F , G , and a Lipschitzian linear operator L from F into G . Then

- (i) $\text{LTRN}(0, L, E) = L$, and
- (ii) for every natural number i and for every Lipschitzian linear operator V from E into $\text{diff}_{\text{SP}}(E^i, F)$, $(\text{LTRN}(i+1, L, E))(V) = (\text{LTRN}(i, L, E)) \cdot V$.

- (22) Let us consider real normed spaces E, F, G , a subset Z of E , a subset T of F , a partial function u from E to F , and a partial function v from F to G . Suppose $u^\circ Z \subseteq T$ and u is differentiable on Z and $u'_{|Z}$ is continuous on Z and v is differentiable on T and $v'_{|T}$ is continuous on T . Then

- (i) $v \cdot u$ is differentiable on Z , and
- (ii) $(v \cdot u)'_{|Z}$ is continuous on Z .

PROOF: $v \cdot u$ is differentiable on Z and for every point x of E such that $x \in Z$ holds $(v \cdot u)'_{|Z/x} = v'_{|T/u_x} \cdot (u'_{|Z/x})$. Set $f = (v \cdot u)'_{|Z}$. For every point x_0 of E and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of E such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. $\square$

- (23) Let us consider real normed spaces E, F, G , a subset Z of E , a partial function u from E to F , and a Lipschitzian linear operator L from F into G . Suppose u is differentiable on Z and $u'_{|Z}$ is continuous on Z . Then

- (i) $L \cdot u$ is differentiable on Z , and
- (ii) $(L \cdot u)'_{|Z}$ is continuous on Z .

The theorem is a consequence of (19) and (22).

Let us consider real normed spaces E, F, G , a subset Z of E , a partial function u from E to F , a Lipschitzian linear operator L from F into G , and a natural number i . Now we state the propositions:

- (24) Suppose u is differentiable i times on Z . Then

- (i) $L \cdot u$ is differentiable i times on Z , and
- (ii) $\text{diff}_Z(L \cdot u, i) = (\text{LTRN}(i, L, E)) \cdot \text{diff}_Z(u, i)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if u is differentiable $\$1$ times on Z , then $L \cdot u$ is differentiable $\$1$ times on Z and $\text{diff}_Z(L \cdot u, \$1) = (\text{LTRN}(\$1, L, E)) \cdot \text{diff}_Z(u, \$1)$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (25) Suppose u is differentiable i times on Z and $\text{diff}_Z(u, i)$ is continuous on Z . Then

- (i) $L \cdot u$ is differentiable i times on Z , and
- (ii) $\text{diff}_Z(L \cdot u, i)$ is continuous on Z .

The theorem is a consequence of (24) and (15).

- (26) Let us consider real normed spaces S, T, U , a point x of S , a partial function u from S to T , a partial function v from S to U , and a partial function w from S to $T \times U$. Suppose u is differentiable in x and v is differentiable in x and $w = \langle u, v \rangle$. Then

- (i) w is differentiable in x , and
- (ii) $w'(x) = \langle u'(x), v'(x) \rangle$, and
- (iii) for every point d_2 of S , $w'(x)(d_2) = \langle u'(x)(d_2), v'(x)(d_2) \rangle$.

PROOF: For every point t of S such that $t \in \text{dom } w$ holds $w_t = \langle f_{1/t}, f_{2/t} \rangle$. Consider N_1 being a neighbourhood of x_0 such that $N_1 \subseteq \text{dom } f_1$ and there exists a point L of the real norm space of bounded linear operators from S into T and there exists a rest R of S, T such that for every point x of S such that $x \in N_1$ holds $f_{1/x} - f_{1/x_0} = L(x - x_0) + R_{/x-x_0}$. Consider L_1 being a point of the real norm space of bounded linear operators from S into T , R_1 being a rest of S, T such that for every point x of S such that $x \in N_1$ holds $f_{1/x} - f_{1/x_0} = L_1(x - x_0) + R_{1/x-x_0}$. Consider N_2 being a neighbourhood of x_0 such that $N_2 \subseteq \text{dom } f_2$ and there exists a point L of the real norm space of bounded linear operators from S into U and there exists a rest R of S, U such that for every point x of S such that $x \in N_2$ holds $f_{2/x} - f_{2/x_0} = L(x - x_0) + R_{/x-x_0}$. Consider L_2 being a point of the real norm space of bounded linear operators from S into U , R_2 being a rest of S, U such that for every point x of S such that $x \in N_2$ holds $f_{2/x} - f_{2/x_0} = L_2(x - x_0) + R_{2/x-x_0}$. Define $\mathcal{O}(\text{object}) = \langle R_{1/\$1}, R_{2/\$1} \rangle$. Consider R being a function from S into $T \times U$ such that for every object d_2 such that $d_2 \in \text{the carrier of } S$ holds $R(d_2) = \mathcal{O}(d_2)$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R_{/z}\| < r$. Define $\mathcal{O}(\text{object}) = \langle L_1(\$1), L_2(\$1) \rangle$. For every object x such that $x \in \text{the carrier of } S$ holds $\mathcal{O}(x) \in \text{the carrier of } T \times U$. Consider L being a function from S into $T \times U$ such that for every object d_2 such that $d_2 \in \text{the carrier of } S$ holds $L(d_2) = \mathcal{O}(d_2)$. For every elements x, y of S , $L(x + y) = L(x) + L(y)$. For every vector x of S and for every real number a , $L(a \cdot x) = a \cdot L(x)$. Set $K = \|L_1\| + \|L_2\|$. For every vector w of S , $\|L(w)\| \leq K \cdot \|w\|$. Consider N being a neighbourhood of x_0 such that $N \subseteq N_1$ and $N \subseteq N_2$. $\square$

Let us consider real normed spaces S, T, U , a subset Z of S , a partial function u from S to T , a partial function v from S to U , and a partial function w from S to $T \times U$. Now we state the propositions:

- (27) Suppose u is differentiable on Z and v is differentiable on Z and $w = \langle u, v \rangle$. Then
- (i) w is differentiable on Z , and
 - (ii) for every point x of S such that $x \in Z$ holds $w'_{|Z/x} = \langle u'_{|Z/x}, v'_{|Z/x} \rangle$, and

- (iii) for every point x of S such that $x \in Z$ for every point d_2 of S ,
 $(w'_{\upharpoonright Z/x})(d_2) = \langle (u'_{\upharpoonright Z/x})(d_2), (v'_{\upharpoonright Z/x})(d_2) \rangle$.

PROOF: For every point x of S such that $x \in Z$ holds w is differentiable in x . For every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$. For every point x of S such that $x \in Z$ for every point d_2 of S , $(w'_{\upharpoonright Z/x})(d_2) = \langle (u'_{\upharpoonright Z/x})(d_2), (v'_{\upharpoonright Z/x})(d_2) \rangle$. $\square$

- (28) Suppose u is differentiable on Z and $u'_{\upharpoonright Z}$ is continuous on Z and v is differentiable on Z and $v'_{\upharpoonright Z}$ is continuous on Z and $w = \langle u, v \rangle$. Then

- (i) w is differentiable on Z , and
 (ii) $w'_{\upharpoonright Z}$ is continuous on Z .

PROOF: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$ and for every point x of S such that $x \in Z$ for every point d_2 of S , $(w'_{\upharpoonright Z/x})(d_2) = \langle (u'_{\upharpoonright Z/x})(d_2), (v'_{\upharpoonright Z/x})(d_2) \rangle$. Set $f = w'_{\upharpoonright Z}$. For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|f_{/x_1} - f_{/x_0}\| < r$. $\square$

- (29) Let us consider real normed spaces E , F , and a natural number i . Then $\text{diff}_{\text{SP}}(E^{(i+1)}, F) = \text{diff}_{\text{SP}}(E^i, (\text{the real norm space of bounded linear operators from } E \text{ into } F))$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{diff}_{\text{SP}}(E^{(\mathbb{S}_1+1)}, F) = \text{diff}_{\text{SP}}(E^{\mathbb{S}_1}, (\text{the real norm space of bounded linear operators from } E \text{ into } F))$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [4, (10)]. For every natural number i , $\mathcal{P}[i]$. $\square$

- (30) Let us consider real normed spaces E , F , a subset Z of E , a partial function g from E to F , and a partial function f from E to the real norm space of bounded linear operators from E into F . Suppose $(g \upharpoonright Z)'_{\upharpoonright Z} = f \upharpoonright Z$. Let us consider a natural number i . Then $\text{diff}_Z(g, i+1) = \text{diff}_Z(f, i)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{diff}_Z(g, \mathbb{S}_1+1) = \text{diff}_Z(f, \mathbb{S}_1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

Let us consider real normed spaces E , F , a natural number n , a subset Z of E , and a partial function g from E to F . Now we state the propositions:

- (31) If $g'_{\upharpoonright Z}$ is differentiable n times on Z and g is differentiable on Z , then g is differentiable $n+1$ times on Z .

PROOF: Set $f = g'_{\upharpoonright Z}$. $f \upharpoonright Z = (g \upharpoonright Z)'_{\upharpoonright Z}$. For every natural number i such that $i \leq n+1-1$ holds $\text{diff}_Z(g, i)$ is differentiable on Z . $\square$

(32) Suppose $g'_{\upharpoonright Z}$ is differentiable n times on Z and g is differentiable on Z and $\text{diff}_Z(g'_{\upharpoonright Z}, n)$ is continuous on Z . Then

- (i) g is differentiable $n + 1$ times on Z , and
- (ii) $\text{diff}_Z(g, n + 1)$ is continuous on Z .

The theorem is a consequence of (31), (3), (30), and (29).

(33) Let us consider real normed spaces S, E, F, G , a Lipschitzian bilinear operator B from $E \times F$ into G , a partial function W from S to G , a partial function w from S to $E \times F$, a partial function u from S to E , a partial function v from S to F , and a point x of S . Suppose u is differentiable in x and v is differentiable in x and $x \in \text{dom } w$ and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

- (i) W is differentiable in x , and
- (ii) w is differentiable in x , and
- (iii) $W'(x) = B'(\langle u_{/x}, v_{/x} \rangle) \cdot w'(x)$, and
- (iv) $w'(x) = \langle u'(x), v'(x) \rangle$, and
- (v) for every point d_1 of S , $W'(x)(d_1) = B(u'(x)(d_1), v_{/x}) + B(u_{/x}, v'(x)(d_1))$.

The theorem is a consequence of (26).

(34) Let us consider real normed spaces S, E, F, G , a subset Z of S , a Lipschitzian bilinear operator B from $E \times F$ into G , a partial function W from S to G , a partial function w from S to $E \times F$, a partial function u from S to E , and a partial function v from S to F . Suppose u is differentiable on Z and v is differentiable on Z and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

- (i) W is differentiable on Z , and
- (ii) for every point x of S such that $x \in Z$ for every point d_1 of S , $(W'_{\upharpoonright Z/x})(d_1) = B((u'_{\upharpoonright Z/x})(d_1), v_{/x}) + B(u_{/x}, (v'_{\upharpoonright Z/x})(d_1))$.

PROOF: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$ and for every point x of S such that $x \in Z$ for every point d_2 of S , $(w'_{\upharpoonright Z/x})(d_2) = \langle (u'_{\upharpoonright Z/x})(d_2), (v'_{\upharpoonright Z/x})(d_2) \rangle$. For every point x of S such that $x \in Z$ for every point d_1 of S , $(W'_{\upharpoonright Z/x})(d_1) = B((u'_{\upharpoonright Z/x})(d_1), v_{/x}) + B(u_{/x}, (v'_{\upharpoonright Z/x})(d_1))$. $\square$

Let us consider real normed spaces S, E , a subset Z of S , a partial function u from S to E , and a natural number i . Now we state the propositions:

(35) If u is differentiable $i + 1$ times on Z , then $u'_{\upharpoonright Z}$ is differentiable i times on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if u is differentiable $\$1 + 1$ times on Z , then $u'_{|Z}$ is differentiable $\$1$ times on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (36) Suppose u is differentiable $i+1$ times on Z and $\text{diff}_Z(u, i+1)$ is continuous on Z . Then

- (i) $u'_{|Z}$ is differentiable i times on Z , and
- (ii) $\text{diff}_Z(u'_{|Z}, i)$ is continuous on Z .

The theorem is a consequence of (35), (29), (3), and (30).

- (37) Let us consider real normed spaces E, F, G . Then there exists a Lipschitzian bilinear operator B from the real norm space of bounded linear operators from E into $F \times$ the real norm space of bounded linear operators from F into G into the real norm space of bounded linear operators from E into G such that for every point u of the real norm space of bounded linear operators from E into F for every point v of the real norm space of bounded linear operators from F into G , $B(u, v) = v \cdot u$.

PROOF: Set $E_3 =$ the carrier of the real norm space of bounded linear operators from E into F . Set $F_2 =$ the carrier of the real norm space of bounded linear operators from F into G . Set $E_4 =$ the carrier of the real norm space of bounded linear operators from E into G . Define $\mathcal{P}[\text{element of } E_3, \text{element of } F_2, \text{object}] \equiv \$3 = \$2 \cdot \1 . Consider L being a function from $E_3 \times F_2$ into E_4 such that for every element x of E_3 and for every element y of F_2 , $\mathcal{P}[x, y, L(x, y)]$. Set $L_4 =$ the real norm space of bounded linear operators from E into F . Set $L_5 =$ the real norm space of bounded linear operators from F into G . For every points x_1, x_2 of L_4 and for every point y of L_5 , $L(x_1 + x_2, y) = L(x_1, y) + L(x_2, y)$. For every point x of L_4 and for every point y of L_5 and for every real number a , $L(a \cdot x, y) = a \cdot L(x, y)$. For every point x of L_4 and for every points y_1, y_2 of L_5 , $L(x, y_1 + y_2) = L(x, y_1) + L(x, y_2)$. For every point x of L_4 and for every point y of L_5 and for every real number a , $L(x, a \cdot y) = a \cdot L(x, y)$. Set $K = 1$. For every vector x of L_4 and for every vector y of L_5 , $\|L(x, y)\| \leq K \cdot \|x\| \cdot \|y\|$. $\square$

- (38) Let us consider a natural number i , real normed spaces S, E, F, G , a subset Z of S , a Lipschitzian bilinear operator B from $E \times F$ into G , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, and a partial function W from S to G . Suppose $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable i times on Z and v is differentiable i times on Z . Then W is differentiable i times on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces $S, E,$

F, G for every subset Z of S for every Lipschitzian bilinear operator B from $E \times F$ into G for every partial function u from S to E for every partial function v from S to F for every partial function w from S to $E \times F$ for every partial function W from S to G such that $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable $\$1$ times on Z and v is differentiable $\$1$ times on Z holds W is differentiable $\$1$ times on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (39) Let us consider real normed spaces S, E, F, G , a subset Z of S , a Lipschitzian bilinear operator B from $E \times F$ into G , a partial function W from S to G , a partial function w from S to $E \times F$, a partial function u from S to E , and a partial function v from S to F . Suppose u is differentiable on Z and $u'|_Z$ is continuous on Z and v is differentiable on Z and $v'|_Z$ is continuous on Z and $W = B \cdot w$ and $w = \langle u, v \rangle$. Then

- (i) W is differentiable on Z , and
- (ii) $W'|_Z$ is continuous on Z .

The theorem is a consequence of (28) and (22).

- (40) Let us consider real normed spaces S, E, F, G , a subset Z of S , a partial function u from S to E , a partial function v from S to F , and a partial function w from S to $E \times F$. Suppose $w = \langle u, v \rangle$ and u is continuous on Z and v is continuous on Z . Then w is continuous on Z .

PROOF: For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|w_{/x_1} - w_{/x_0}\| < r$ by [9, (18)], (16). $\square$

- (41) Let us consider a natural number i , real normed spaces S, E, F, G , a subset Z of S , a Lipschitzian bilinear operator B from $E \times F$ into G , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, and a partial function W from S to G . Suppose $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable i times on Z and $\text{diff}_Z(u, i)$ is continuous on Z and v is differentiable i times on Z and $\text{diff}_Z(v, i)$ is continuous on Z . Then

- (i) W is differentiable i times on Z , and
- (ii) $\text{diff}_Z(W, i)$ is continuous on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces S, E, F, G for every subset Z of S for every Lipschitzian bilinear operator B from $E \times F$ into G for every partial function u from S to E for every partial function v from S to F for every partial function w from S to

$E \times F$ for every partial function W from S to G such that $W = B \cdot w$ and $w = \langle u, v \rangle$ and u is differentiable $\$1$ times on Z and $\text{diff}_Z(u, \$1)$ is continuous on Z and v is differentiable $\$1$ times on Z and $\text{diff}_Z(v, \$1)$ is continuous on Z holds W is differentiable $\$1$ times on Z and $\text{diff}_Z(W, \$1)$ is continuous on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

Let us consider a natural number i , real normed spaces E, F, G , a subset Z of E , a subset T of F , a partial function u from E to F , and a partial function v from F to G . Now we state the propositions:

- (42) If $u^\circ Z \subseteq T$ and u is differentiable i times on Z and v is differentiable i times on T , then $v \cdot u$ is differentiable i times on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces E, F, G for every subset Z of E for every subset T of F for every partial function u from E to F for every partial function v from F to G such that $u^\circ Z \subseteq T$ and u is differentiable $\$1$ times on Z and v is differentiable $\$1$ times on T holds $v \cdot u$ is differentiable $\$1$ times on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (43) Suppose $u^\circ Z \subseteq T$ and u is differentiable i times on Z and $\text{diff}_Z(u, i)$ is continuous on Z and v is differentiable i times on T and $\text{diff}_T(v, i)$ is continuous on T . Then

- (i) $v \cdot u$ is differentiable i times on Z , and
- (ii) $\text{diff}_Z(v \cdot u, i)$ is continuous on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces E, F, G for every subset Z of E for every subset T of F for every partial function u from E to F for every partial function v from F to G such that $u^\circ Z \subseteq T$ and u is differentiable $\$1$ times on Z and $\text{diff}_Z(u, \$1)$ is continuous on Z and v is differentiable $\$1$ times on T and $\text{diff}_T(v, \$1)$ is continuous on T holds $v \cdot u$ is differentiable $\$1$ times on Z and $\text{diff}_Z(v \cdot u, \$1)$ is continuous on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

3. PARTIAL DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

Now we state the proposition:

- (44) Let us consider real normed spaces E, F, G , a subset Z of $E \times F$, and a partial function f from $E \times F$ to G . Suppose f is differentiable on Z . Then

- (i) f is partially differentiable on Z w.r.t. 1, and

- (ii) f is partially differentiable on Z w.r.t. 2, and
- (iii) for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E , $((f \upharpoonright^1 Z)_{/z})(d_2) = f'_{\upharpoonright Z/z}(d_2, 0_F)$ and for every point d_4 of F , $((f \upharpoonright^2 Z)_{/z})(d_4) = f'_{\upharpoonright Z/z}(0_E, d_4)$.

PROOF: For every point z of $E \times F$ such that $z \in Z$ holds f is partially differentiable in z w.r.t. 1. For every point z of $E \times F$ such that $z \in Z$ holds f is partially differentiable in z w.r.t. 2. For every point d_2 of E , $((f \upharpoonright^1 Z)_{/z})(d_2) = f'_{\upharpoonright Z/z}(d_2, 0_F)$. For every point d_4 of F , $((f \upharpoonright^2 Z)_{/z})(d_4) = f'_{\upharpoonright Z/z}(0_E, d_4)$. $\square$

Let us consider real normed spaces E, F . Now we state the propositions:

- (45) There exists a Lipschitzian linear operator L_{10} from E into $E \times F$ such that for every point d_2 of E , $L_{10}(d_2) = \langle d_2, 0_F \rangle$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point d_2 of E such that $d_2 = \$_1$ and $\$_2 = \langle d_2, 0_F \rangle$. For every object x such that $x \in$ the carrier of E there exists an object y such that $y \in$ the carrier of $E \times F$ and $\mathcal{P}[x, y]$. Consider L_1 being a function from the carrier of E into the carrier of $E \times F$ such that for every object x such that $x \in$ the carrier of E holds $\mathcal{P}[x, L_1(x)]$. For every point d_2 of E , $L_1(d_2) = \langle d_2, 0_F \rangle$. For every elements x, y of E , $L_1(x + y) = L_1(x) + L_1(y)$. For every vector x of E and for every real number a , $L_1(a \cdot x) = a \cdot L_1(x)$. Set $K = 1$. For every vector x of E , $\|L_1(x)\| \leq K \cdot \|x\|$. $\square$

- (46) There exists a Lipschitzian linear operator L_{20} from F into $E \times F$ such that for every point d_4 of F , $L_{20}(d_4) = \langle 0_E, d_4 \rangle$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point d_2 of F such that $d_2 = \$_1$ and $\$_2 = \langle 0_E, d_2 \rangle$. For every object x such that $x \in$ the carrier of F there exists an object y such that $y \in$ the carrier of $E \times F$ and $\mathcal{P}[x, y]$. Consider L_1 being a function from the carrier of F into the carrier of $E \times F$ such that for every object x such that $x \in$ the carrier of F holds $\mathcal{P}[x, L_1(x)]$. For every point d_2 of F , $L_1(d_2) = \langle 0_E, d_2 \rangle$. For every elements x, y of F , $L_1(x + y) = L_1(x) + L_1(y)$. For every vector x of F and for every real number a , $L_1(a \cdot x) = a \cdot L_1(x)$. Set $K = 1$. For every vector x of F , $\|L_1(x)\| \leq K \cdot \|x\|$. $\square$

- (47) Let us consider real normed spaces E, F , a non empty subset Z of E , a partial function L_1 from E to F , and a point L_0 of F . Suppose Z is open and $L_1 = Z \mapsto L_0$. Then

- (i) L_1 is differentiable on Z , and
- (ii) $L_1'_{\upharpoonright Z}$ is continuous on Z , and
- (iii) $L_1'_{\upharpoonright Z} = Z \mapsto 0_\alpha$,

where α is the real norm space of bounded linear operators from E into F .

PROOF: For every object z such that $z \in \text{dom } L_1'_{|Z}$ holds $L_1'_{|Z}(z) = 0_\alpha$, where α is the real norm space of bounded linear operators from E into F . $\square$

- (48) Let us consider real normed spaces E, F , a non empty subset Z of E , a partial function L_1 from E to F , and a point L_0 of F . Suppose Z is open and $L_1 = Z \mapsto L_0$. Let us consider a natural number i . Then

- (i) there exists a point P of $\text{diff}_{\text{SP}}(E^i, F)$ such that $\text{diff}_Z(L_1, i) = Z \mapsto P$, and
- (ii) $\text{diff}_Z(L_1, i)$ is differentiable on Z , and
- (iii) $\text{diff}_Z(L_1, i)'_{|Z}$ is continuous on Z .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists a point P of $\text{diff}_{\text{SP}}(E^{\$1}, F)$ such that $\text{diff}_Z(L_1, \$1) = Z \mapsto P$ and $\text{diff}_Z(L_1, \$1)$ is differentiable on Z and $\text{diff}_Z(L_1, \$1)'_{|Z}$ is continuous on Z . $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (49) Let us consider a natural number i , real normed spaces E, F , a non empty subset Z of E , a partial function L_1 from E to F , and a point L_0 of F . Suppose Z is open and $L_1 = Z \mapsto L_0$. Then

- (i) L_1 is differentiable i times on Z , and
- (ii) $\text{diff}_Z(L_1, i)'_{|Z}$ is continuous on Z .

The theorem is a consequence of (48).

- (50) Let us consider a natural number n , a real normed space S , a subset Z of S , and a partial function f from S to S . Suppose Z is open and $f = \text{id}_{\Omega_S}$. Then

- (i) f is differentiable n times on Z , and
- (ii) $\text{diff}_Z(f, n)$ is continuous on Z .

The theorem is a consequence of (9).

Let us consider a natural number i , real normed spaces E, F, G , a non empty subset Z of $E \times F$, and a partial function f from $E \times F$ to G . Now we state the propositions:

- (51) If f is differentiable $i+1$ times on Z , then $f \upharpoonright^1 Z$ is differentiable i times on Z and $f \upharpoonright^2 Z$ is differentiable i times on Z .

PROOF: f is differentiable on Z . f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E , $((f \upharpoonright^1 Z)_{/z})(d_2) = f'_{|Z/z}(d_2, 0_F)$ and for every point d_4 of F , $((f \upharpoonright^2 Z)_{/z})(d_4) = f'_{|Z/z}(0_E, d_4)$.

Set $P_1 = f \upharpoonright^1 Z$. Set $P_2 = f \upharpoonright^2 Z$. Consider L_{10} being a Lipschitzian linear operator from E into $E \times F$ such that for every point d_2 of E , $L_{10}(d_2) = \langle d_2, 0_F \rangle$. Consider L_{20} being a Lipschitzian linear operator from F into $E \times F$ such that for every point d_4 of F , $L_{20}(d_4) = \langle 0_E, d_4 \rangle$. Set $B_3 =$ the real norm space of bounded linear operators from E into $E \times F$. Set $B_2 =$ the real norm space of bounded linear operators from F into $E \times F$. Reconsider $L_1 = Z \mapsto L_{10}$ as a partial function from $E \times F$ to B_3 . L_1 is differentiable i times on Z . For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^1 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{1/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z . Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from E into $E \times F$ into the real norm space of bounded linear operators from $E \times F$ into G such that for every point u of the real norm space of bounded linear operators from E into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G , $B(u, v) = v \cdot u$. Set $w_2 = \langle L_1, f'_{\upharpoonright Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from E into G . W is differentiable i times on Z . For every object x_0 such that $x_0 \in \text{dom } P_1$ holds $P_1(x_0) = W(x_0)$. Reconsider $L_2 = Z \mapsto L_{20}$ as a partial function from $E \times F$ to B_2 . L_2 is differentiable i times on Z . For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^2 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{2/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z . Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from F into $E \times F$ into the real norm space of bounded linear operators from $E \times F$ into G such that for every point u of the real norm space of bounded linear operators from F into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G , $B(u, v) = v \cdot u$. Set $w_2 = \langle L_2, f'_{\upharpoonright Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from F into G . For every object x_0 such that $x_0 \in \text{dom } P_2$ holds $P_2(x_0) = W(x_0)$. $\square$

(52) Suppose f is differentiable $i+1$ times on Z and $\text{diff}_Z(f, i+1)$ is continuous on Z . Then

- (i) $f \upharpoonright^1 Z$ is differentiable i times on Z , and
- (ii) $\text{diff}_Z(f \upharpoonright^1 Z, i)$ is continuous on Z , and
- (iii) $f \upharpoonright^2 Z$ is differentiable i times on Z , and
- (iv) $\text{diff}_Z(f \upharpoonright^2 Z, i)$ is continuous on Z .

PROOF: f is differentiable on Z . f is partially differentiable on Z w.r.t.

1 and f is partially differentiable on Z w.r.t. 2 and for every point z of $E \times F$ such that $z \in Z$ holds for every point d_2 of E , $((f \upharpoonright^1 Z)_{/z})(d_2) = f'_{\upharpoonright Z/z}(d_2, 0_F)$ and for every point d_4 of F , $((f \upharpoonright^2 Z)_{/z})(d_4) = f'_{\upharpoonright Z/z}(0_E, d_4)$. Set $P_1 = f \upharpoonright^1 Z$. Set $P_2 = f \upharpoonright^2 Z$. Consider L_{10} being a Lipschitzian linear operator from E into $E \times F$ such that for every point d_2 of E , $L_{10}(d_2) = \langle d_2, 0_F \rangle$. Consider L_{20} being a Lipschitzian linear operator from F into $E \times F$ such that for every point d_4 of F , $L_{20}(d_4) = \langle 0_E, d_4 \rangle$. Set $B_3 =$ the real norm space of bounded linear operators from E into $E \times F$. Set $B_2 =$ the real norm space of bounded linear operators from F into $E \times F$. Reconsider $L_1 = Z \longmapsto L_{10}$ as a partial function from $E \times F$ to B_3 . L_1 is differentiable i times on Z . L_1 is differentiable $i + 1$ times on Z . For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^1 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{1/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z and $\text{diff}_Z(f'_{\upharpoonright Z}, i)$ is continuous on Z . Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from E into $E \times F \times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from E into G such that for every point u of the real norm space of bounded linear operators from E into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G , $B(u, v) = v \cdot u$. Set $w_2 = \langle L_1, f'_{\upharpoonright Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from E into G . W is differentiable i times on Z and $\text{diff}_Z(W, i)$ is continuous on Z . For every object x_0 such that $x_0 \in \text{dom } P_1$ holds $P_1(x_0) = W(x_0)$. Reconsider $L_2 = Z \longmapsto L_{20}$ as a partial function from $E \times F$ to B_2 . L_2 is differentiable i times on Z . L_2 is differentiable $i + 1$ times on Z . For every point z of $E \times F$ such that $z \in Z$ holds $(f \upharpoonright^2 Z)_{/z} = f'_{\upharpoonright Z/z} \cdot (L_{2/z})$. $f'_{\upharpoonright Z}$ is differentiable i times on Z and $\text{diff}_Z(f'_{\upharpoonright Z}, i)$ is continuous on Z . Consider B being a Lipschitzian bilinear operator from the real norm space of bounded linear operators from F into $E \times F \times$ the real norm space of bounded linear operators from $E \times F$ into G into the real norm space of bounded linear operators from F into G such that for every point u of the real norm space of bounded linear operators from F into $E \times F$ and for every point v of the real norm space of bounded linear operators from $E \times F$ into G , $B(u, v) = v \cdot u$. Set $w_2 = \langle L_2, f'_{\upharpoonright Z} \rangle$. Reconsider $W = B \cdot w_2$ as a partial function from $E \times F$ to the real norm space of bounded linear operators from F into G . W is differentiable i times on Z and $\text{diff}_Z(W, i)$ is continuous on Z . For every object x_0 such that $x_0 \in \text{dom } P_2$ holds $P_2(x_0) = W(x_0)$. $\square$

- (53) Let us consider real normed spaces S , E , F , a partial function u from S to E , a partial function v from S to F , a partial function w from S to

$E \times F$, and a point x of S . Suppose $w = \langle u, v \rangle$ and u is differentiable in x and v is differentiable in x . Then

- (i) w is differentiable in x , and
- (ii) $w'(x) = \langle u'(x), v'(x) \rangle$.

PROOF: Consider N_3 being a neighbourhood of x_0 such that $N_3 \subseteq \text{dom } u$ and there exists a rest R_3 of S, E such that for every point x of S such that $x \in N_3$ holds $u/x - u/x_0 = u'(x_0)(x - x_0) + R_{3/x-x_0}$. Consider R_3 being a rest of S, E such that for every point x of S such that $x \in N_3$ holds $u/x - u/x_0 = u'(x_0)(x - x_0) + R_{3/x-x_0}$. Consider N_4 being a neighbourhood of x_0 such that $N_4 \subseteq \text{dom } v$ and there exists a rest R_4 of S, F such that for every point x of S such that $x \in N_4$ holds $v/x - v/x_0 = v'(x_0)(x - x_0) + R_{4/x-x_0}$. Consider R_4 being a rest of S, F such that for every point x of S such that $x \in N_4$ holds $v/x - v/x_0 = v'(x_0)(x - x_0) + R_{4/x-x_0}$. Consider N being a neighbourhood of x_0 such that $N \subseteq N_3$ and $N \subseteq N_4$. Set $L = \langle u'(x_0), v'(x_0) \rangle$. For every elements x, y of S , $L(x + y) = L(x) + L(y)$. For every vector x of S and for every real number a , $L(a \cdot x) = a \cdot L(x)$. Set $K = \|u'(x_0)\| + \|v'(x_0)\|$. For every vector x of S , $\|L(x)\| \leq K \cdot \|x\|$. Set $R = \langle R_3, R_4 \rangle$. For every point d_2 of S , $R/d_2 = \langle R_{3/d_2}, R_{4/d_2} \rangle$. For every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every point z of S such that $z \neq 0_S$ and $\|z\| < d$ holds $\|z\|^{-1} \cdot \|R/z\| < r$. For every point x of S such that $x \in N$ holds $w/x - w/x_0 = L(x - x_0) + R_{/x-x_0}$. $\square$

- (54) Let us consider real normed spaces S, E, F , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, and a subset Z of S . Suppose $w = \langle u, v \rangle$ and u is differentiable on Z and v is differentiable on Z . Then

- (i) w is differentiable on Z , and
- (ii) for every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$.

PROOF: For every point x of S such that $x \in Z$ holds w is differentiable in x and $w'(x) = \langle u'(x), v'(x) \rangle$. For every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$. $\square$

Let S, E, F be real normed spaces. The functor $\text{CTP}(S, E, F)$ yielding a Lipschitzian linear operator from (the real norm space of bounded linear operators from S into E) $\times$ (the real norm space of bounded linear operators from S into F) into the real norm space of bounded linear operators from S into $E \times F$ is defined by

- (Def. 3) for every Lipschitzian linear operator f from S into E and for every Lipschitzian linear operator g from S into F , $it(f, g) = \langle f, g \rangle$.

Now we state the proposition:

- (55) Let us consider real normed spaces S , E , F , and a natural number i . Then $\text{CTP}(S, \text{diff}_{\text{SP}}(S^i, E), \text{diff}_{\text{SP}}(S^i, F))$ is a Lipschitzian linear operator from $\text{diff}_{\text{SP}}(S^{(i+1)}, E) \times \text{diff}_{\text{SP}}(S^{(i+1)}, F)$ into the real norm space of bounded linear operators from S into $\text{diff}_{\text{SP}}(S^i, E) \times \text{diff}_{\text{SP}}(S^i, F)$.

Let us consider real normed spaces S , E , F , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, and a subset Z of S . Now we state the propositions:

- (56) Suppose $w = \langle u, v \rangle$ and u is differentiable on Z and v is differentiable on Z . Then

- (i) w is differentiable on Z , and
- (ii) $w'_{\upharpoonright Z} = (\text{CTP}(S, \text{diff}_{\text{SP}}(S^0, E), \text{diff}_{\text{SP}}(S^0, F))) \cdot \langle u'_{\upharpoonright Z}, v'_{\upharpoonright Z} \rangle$.

PROOF: w is differentiable on Z and for every point x of S such that $x \in Z$ holds $w'_{\upharpoonright Z/x} = \langle u'_{\upharpoonright Z/x}, v'_{\upharpoonright Z/x} \rangle$. $\text{CTP}(S, \text{diff}_{\text{SP}}(S^0, E), \text{diff}_{\text{SP}}(S^0, F))$ is a Lipschitzian linear operator from $\text{diff}_{\text{SP}}(S^{(0+1)}, E) \times \text{diff}_{\text{SP}}(S^{(0+1)}, F)$ into the real norm space of bounded linear operators from S into $\text{diff}_{\text{SP}}(S^0, E) \times \text{diff}_{\text{SP}}(S^0, F)$. For every object x_0 such that $x_0 \in \text{dom } w'_{\upharpoonright Z}$ holds $w'_{\upharpoonright Z}(x_0) = ((\text{CTP}(S, \text{diff}_{\text{SP}}(S^0, E), \text{diff}_{\text{SP}}(S^0, F))) \cdot \langle u'_{\upharpoonright Z}, v'_{\upharpoonright Z} \rangle)(x_0)$. $\square$

- (57) Suppose $w = \langle u, v \rangle$ and u is differentiable on Z and v is differentiable on Z . Then

- (i) $\text{diff}_Z(w, 0)$ is differentiable on Z , and
- (ii) there exists a Lipschitzian linear operator T from $\text{diff}_{\text{SP}}(S^1, E) \times \text{diff}_{\text{SP}}(S^1, F)$ into $\text{diff}_{\text{SP}}(S^1, (E \times F))$ such that $T = \text{CTP}(S, \text{diff}_{\text{SP}}(S^0, E), \text{diff}_{\text{SP}}(S^0, F))$ and $\text{diff}_Z(w, 1) = T \cdot \langle \text{diff}_Z(u, 1), \text{diff}_Z(v, 1) \rangle$.

The theorem is a consequence of (56), (3), (55), and (2).

- (58) Suppose $w = \langle u, v \rangle$ and u is differentiable 2 times on Z and v is differentiable 2 times on Z . Then

- (i) w is differentiable 2 times on Z , and
- (ii) there exists a Lipschitzian linear operator L_1 from $\text{diff}_{\text{SP}}(S^1, E) \times \text{diff}_{\text{SP}}(S^1, F)$ into $\text{diff}_{\text{SP}}(S^1, (E \times F))$ and there exists a Lipschitzian linear operator L_2 from $\text{diff}_{\text{SP}}(S^2, E) \times \text{diff}_{\text{SP}}(S^2, F)$ into $\text{diff}_{\text{SP}}(S^1, (\text{diff}_{\text{SP}}(S^1, E) \times \text{diff}_{\text{SP}}(S^1, F)))$ and there exists a Lipschitzian linear operator T from $\text{diff}_{\text{SP}}(S^2, E) \times \text{diff}_{\text{SP}}(S^2, F)$ into $\text{diff}_{\text{SP}}(S^2, (E \times F))$ such that $L_1 = \text{CTP}(S, \text{diff}_{\text{SP}}(S^0, E), \text{diff}_{\text{SP}}(S^0, F))$ and $L_2 = \text{CTP}(S, \text{diff}_{\text{SP}}(S^1, E), \text{diff}_{\text{SP}}(S^1, F))$ and $T = (\text{LTRN}(1, L_1, S)) \cdot L_2$ and $\text{diff}_Z(w, 2) = T \cdot \langle \text{diff}_Z(u, 2), \text{diff}_Z(v, 2) \rangle$.

The theorem is a consequence of (2), (57), (3), (55), and (24).

- (59) Let us consider a natural number i , real normed spaces S, E, F , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, and a subset Z of S . Suppose $w = \langle u, v \rangle$ and u is differentiable $i + 1$ times on Z and v is differentiable $i + 1$ times on Z . Then

- (i) w is differentiable $i + 1$ times on Z , and
- (ii) there exists a Lipschitzian linear operator T from $\text{diff}_{\text{SP}}(S^{(i+1)}, E) \times \text{diff}_{\text{SP}}(S^{(i+1)}, F)$ into $\text{diff}_{\text{SP}}(S^{(i+1)}, (E \times F))$ such that $\text{diff}_Z(w, i + 1) = T \cdot \langle \text{diff}_Z(u, i + 1), \text{diff}_Z(v, i + 1) \rangle$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real normed spaces S, E, F for every partial function u from S to E for every partial function v from S to F for every partial function w from S to $E \times F$ for every subset Z of S such that $w = \langle u, v \rangle$ and u is differentiable $\$1 + 1$ times on Z and v is differentiable $\$1 + 1$ times on Z holds w is differentiable $\$1 + 1$ times on Z and there exists a Lipschitzian linear operator T from $\text{diff}_{\text{SP}}(S^{(\$1+1)}, E) \times \text{diff}_{\text{SP}}(S^{(\$1+1)}, F)$ into $\text{diff}_{\text{SP}}(S^{(\$1+1)}, (E \times F))$ such that $\text{diff}_Z(w, \$1 + 1) = T \cdot \langle \text{diff}_Z(u, \$1 + 1), \text{diff}_Z(v, \$1 + 1) \rangle$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (60) Let us consider real normed spaces S, E, F , a partial function u from S to E , a partial function v from S to F , a partial function w from S to $E \times F$, a subset Z of S , and a natural number i . Suppose $w = \langle u, v \rangle$ and u is differentiable $i + 1$ times on Z and $\text{diff}_Z(u, i + 1)$ is continuous on Z and v is differentiable $i + 1$ times on Z and $\text{diff}_Z(v, i + 1)$ is continuous on Z . Then

- (i) w is differentiable $i + 1$ times on Z , and
- (ii) $\text{diff}_Z(w, i + 1)$ is continuous on Z .

PROOF: Consider T being a Lipschitzian linear operator from $\text{diff}_{\text{SP}}(S^{(i+1)}, E) \times \text{diff}_{\text{SP}}(S^{(i+1)}, F)$ into $\text{diff}_{\text{SP}}(S^{(i+1)}, (E \times F))$ such that $\text{diff}_Z(w, i + 1) = T \cdot \langle \text{diff}_Z(u, i + 1), \text{diff}_Z(v, i + 1) \rangle$. Set $u_1 = \text{diff}_Z(u, i + 1)$. Set $v_1 = \text{diff}_Z(v, i + 1)$. Set $G = \langle u_1, v_1 \rangle$. For every point x_0 of S and for every real number r such that $x_0 \in Z$ and $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in Z$ and $\|x_1 - x_0\| < s$ holds $\|G_{/x_1} - G_{/x_0}\| < r$. T is continuous on $\Omega_{\text{diff}_{\text{SP}}(S^{(i+1)}, E) \times \text{diff}_{\text{SP}}(S^{(i+1)}, F)}$. $\square$

- (61) Let us consider real normed spaces X, Y , a subset V of $X \times Y$, a subset D of X , and a subset E of Y . Suppose D is open and E is open and $V = D \times E$. Then V is open.

PROOF: For every point x of X and for every point y of Y such that $\langle x, y \rangle \in V$ there exist real numbers r_1, r_2 such that $0 < r_1$ and $0 < r_2$ and $\text{Ball}(x, r_1) \times \text{Ball}(y, r_2) \subseteq V$. $\square$

4. HIGHER-ORDER DIFFERENTIABILITY OF INVERSE FUNCTION THEOREM

Now we state the propositions:

- (62) Let us consider real normed spaces E, F, G , a point x of the real norm space of bounded linear operators from E into F , and a point L of the real norm space of bounded linear operators from the real norm space of bounded linear operators from F into E into the real norm space of bounded linear operators from E into E . Suppose x is invertible and for every point y of the real norm space of bounded linear operators from F into E , $L(y) = y \cdot x$. Then L is invertible.

PROOF: Set $F_4 =$ the real norm space of bounded linear operators from F into E . Set $E_2 =$ the real norm space of bounded linear operators from E into E . Reconsider $L_1 = L$ as a Lipschitzian linear operator from F_4 into E_2 . Reconsider $d_2 = x^{-1}$ as a point of the real norm space of bounded linear operators from F into E . For every objects x_1, x_2 such that $x_1, x_2 \in \Omega_{F_4}$ and $L_1(x_1) = L_1(x_2)$ holds $x_1 = x_2$. For every object y such that $y \in \Omega_{E_2}$ there exists an object z such that $z \in \Omega_{F_4}$ and $y = L_1(z)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point y of E_2 such that $y = \$_1$ and $\$_2 = y \cdot d_2$. For every object y such that $y \in$ the carrier of E_2 there exists an object z such that $z \in$ the carrier of F_4 and $\mathcal{P}[y, z]$. Consider R being a function from the carrier of E_2 into the carrier of F_4 such that for every object y such that $y \in$ the carrier of E_2 holds $\mathcal{P}[y, R(y)]$. For every point y of E_2 , $R(y) = y \cdot d_2$. For every element y of F_4 , $(R \cdot L_1)(y) = y$. Set $K = \|d_2\|$. For every vector y of E_2 , $\|R(y)\| \leq K \cdot \|y\|$. $\square$

- (63) Let us consider a non trivial real Banach space F . Then the real norm space of bounded linear operators from F into F is a non trivial real Banach space.
- (64) Let us consider a real Banach space E , non trivial real Banach spaces F, G , a non empty subset Z of $E \times F$, a point c of G , a subset A of E , and a subset B of F . Suppose Z is open and A is open and B is open and $A \times B \subseteq Z$. Let us consider a natural number i , a partial function f from $E \times F$ to G , and a partial function g from E to F . Suppose $\text{dom } f = Z$ and f is differentiable $i + 1$ times on Z and $\text{diff}_Z(f, i + 1)$ is continuous on Z and $\text{dom } g = A$ and $\text{rng } g \subseteq B$ and g is continuous on A and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and for every point x

of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $\text{partdiff}(f, z)$ w.r.t. 2 is invertible. Then

- (i) g is differentiable $i + 1$ times on A , and
- (ii) $\text{diff}_A(g, i + 1)$ is continuous on A , and
- (iii) for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -(\text{Inv partdiff}(f, z) \text{ w.r.t. } 2) \cdot (\text{partdiff}(f, z) \text{ w.r.t. } 1)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every real Banach space E for every non trivial real Banach spaces F, G for every non empty subset Z of $E \times F$ for every point c of G for every subset A of E for every subset B of F such that Z is open and A is open and B is open and $A \times B \subseteq Z$ for every partial function f from $E \times F$ to G for every partial function g from E to F such that $\text{dom } f = Z$ and f is differentiable $\$1 + 1$ times on Z and $\text{diff}_Z(f, \$1 + 1)$ is continuous on Z and $\text{dom } g = A$ and $\text{rng } g \subseteq B$ and g is continuous on A and for every point x of E such that $x \in A$ holds $f(x, g(x)) = c$ and for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $\text{partdiff}(f, z)$ w.r.t. 2 is invertible holds g is differentiable $\$1 + 1$ times on A and $\text{diff}_A(g, \$1 + 1)$ is continuous on A and for every point x of E and for every point z of $E \times F$ such that $x \in A$ and $z = \langle x, g(x) \rangle$ holds $g'(x) = -(\text{Inv partdiff}(f, z) \text{ w.r.t. } 2) \cdot (\text{partdiff}(f, z) \text{ w.r.t. } 1)$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number i , $\mathcal{P}[i]$. $\square$

- (65) Let us consider non trivial real Banach spaces F, G . Then there exists a partial function I from the real norm space of bounded linear operators from F into G to the real norm space of bounded linear operators from G into F such that

- (i) $\text{dom } I = \text{InvertOps}(F, G)$, and
- (ii) $\text{rng } I = \text{InvertOps}(G, F)$, and
- (iii) I is one-to-one and continuous on $\text{InvertOps}(F, G)$, and
- (iv) there exists a partial function J from the real norm space of bounded linear operators from G into F to the real norm space of bounded linear operators from F into G such that $J = I^{-1}$ and J is one-to-one and $\text{dom } J = \text{InvertOps}(G, F)$ and $\text{rng } J = \text{InvertOps}(F, G)$ and J is continuous on $\text{InvertOps}(G, F)$, and
- (v) for every point u of the real norm space of bounded linear operators from F into G such that $u \in \text{InvertOps}(F, G)$ holds $I(u) = \text{Inv } u$, and

- (vi) for every natural number n , I is differentiable $n+1$ times on $\text{InvertOps}(F, G)$ and $\text{diff}_{\text{InvertOps}(F, G)}(I, n+1)$ is continuous on $\text{InvertOps}(F, G)$.

PROOF: Set $E_1 =$ the real norm space of bounded linear operators from F into G . Set $F_1 =$ the real norm space of bounded linear operators from G into F . Set $G_1 =$ the real norm space of bounded linear operators from F into F . G_1 is a non trivial real Banach space. Set $A_1 = \text{InvertOps}(F, G)$. Set $B_1 = \text{InvertOps}(G, F)$. Consider g_1 being a partial function from E_1 to F_1 such that $\text{dom } g_1 = A_1$ and $\text{rng } g_1 = B_1$ and g_1 is one-to-one and continuous on A_1 and there exists a partial function J from F_1 to E_1 such that $J = g_1^{-1}$ and J is one-to-one and $\text{dom } J = B_1$ and $\text{rng } J = A_1$ and J is continuous on B_1 and for every point u of E_1 such that $u \in A_1$ holds $g_1(u) = \text{Inv } u$. Set $Z_1 = \Omega_{E_1 \times F_1}$. Reconsider $a = \text{id}_{\Omega_F}$ as a Lipschitzian linear operator from F into F . Consider f_0 being a Lipschitzian bilinear operator from $E_1 \times F_1$ into G_1 such that for every point u of E_1 and for every point v of F_1 , $f_0(u, v) = v \cdot u$. Reconsider $f_1 = f_0|_{Z_1}$ as a partial function from $E_1 \times F_1$ to G_1 . For every point x of E_1 such that $x \in A_1$ holds $f_1(x, g_1(x)) = a$ by [6, (22)]. For every point x of E_1 and for every point z of $E_1 \times F_1$ such that $x \in A_1$ and $z = \langle x, g_1(x) \rangle$ for every point y of F_1 , $(\text{partdiff}(f_1, z) \text{ w.r.t. } 2)(y) = y \cdot x$ by [8, (4)]. For every point x of E_1 and for every point z of $E_1 \times F_1$ such that $x \in A_1$ and $z = \langle x, g_1(x) \rangle$ holds $\text{partdiff}(f_1, z) \text{ w.r.t. } 2$ is invertible. g_1 is differentiable $i+1$ times on A_1 and $\text{diff}_{A_1}(g_1, i+1)$ is continuous on A_1 . $\square$

- (66) Let us consider non trivial real Banach spaces E, F , a subset Z of E , a partial function f from E to F , a point a of E , a point b of F , and a natural number n . Suppose Z is open and $\text{dom } f = Z$ and f is differentiable $n+1$ times on Z and $\text{diff}_Z(f, n+1)$ is continuous on Z and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible. Then there exists a subset A of E and there exists a subset B of F and there exists a partial function g from F to E such that A is open and B is open and $A \subseteq \text{dom } f$ and $a \in A$ and $b \in B$ and $f^\circ A = B$ and $\text{dom } g = B$ and $\text{rng } g = A$ and $\text{dom}(f|_A) = A$ and $\text{rng}(f|_A) = B$ and $f|_A$ is one-to-one and g is one-to-one and $g = (f|_A)^{-1}$ and $f|_A = g^{-1}$ and $g(b) = a$ and for every point y of F such that $y \in B$ holds $f'(g/y)$ is invertible and for every point y of F such that $y \in B$ holds $g'(y) = \text{Inv } f'(g/y)$ and f is differentiable $n+1$ times on A and $\text{diff}_A(f, n+1)$ is continuous on A and g is differentiable $n+1$ times on B and $\text{diff}_B(g, n+1)$ is continuous on B .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if Z is open and $\text{dom } f = Z$ and f is differentiable $\$_1 + 1$ times on Z and $\text{diff}_Z(f, \$_1 + 1)$ is continuous on Z

and $a \in Z$ and $f(a) = b$ and $f'(a)$ is invertible, then there exists a subset A of E and there exists a subset B of F and there exists a partial function g from F to E such that A is open and B is open and $A \subseteq \text{dom } f$ and $a \in A$ and $b \in B$ and $f^\circ A = B$ and $\text{dom } g = B$ and $\text{rng } g = A$ and $\text{dom}(f \upharpoonright A) = A$ and $\text{rng}(f \upharpoonright A) = B$ and $f \upharpoonright A$ is one-to-one and g is one-to-one and $g = (f \upharpoonright A)^{-1}$ and $f \upharpoonright A = g^{-1}$ and $g(b) = a$ and for every point y of F such that $y \in B$ holds $f'(g/y)$ is invertible and for every point y of F such that $y \in B$ holds $g'(y) = \text{Inv } f'(g/y)$ and f is differentiable $\$1 + 1$ times on A and $\text{diff}_A(f, \$1 + 1)$ is continuous on A and g is differentiable $\$1 + 1$ times on B and $\text{diff}_B(g, \$1 + 1)$ is continuous on B . $\mathcal{P}[0]$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$. For every natural number n , $\mathcal{P}[n]$. $\square$

REFERENCES

- [1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.
- [2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, and Karol Pąk. The role of the Mizar Mathematical Library for interactive proof development in Mizar. *Journal of Automated Reasoning*, 61(1):9–32, 2018. doi:10.1007/s10817-017-9440-6.
- [3] Nelson Dunford and Jacob T. Schwartz. *Linear operators I*. Interscience Publ., 1958.
- [4] Noboru Endou and Yasunari Shidama. Differentiation in normed spaces. *Formalized Mathematics*, 21(2):95–102, 2013. doi:10.2478/forma-2013-0011.
- [5] Isao Miyadera. *Functional Analysis*. Riko-Gaku-Sya, 1972.
- [6] Kazuhisa Nakasho. Invertible operators on Banach spaces. *Formalized Mathematics*, 27(2):107–115, 2019. doi:10.2478/forma-2019-0012.
- [7] Kazuhisa Nakasho and Yasunari Shidama. On implicit and inverse function theorems on Euclidean spaces. *Formalized Mathematics*, 30(3):159–168, 2022. doi:10.2478/forma-2022-0012.
- [8] Kazuhisa Nakasho and Yasunari Shidama. Differentiability properties of Lipschitzian bilinear operators in real normed spaces. *Formalized Mathematics*, 32(1):165–172, 2024. doi:10.2478/forma-2024-0013.
- [9] Hiroyuki Okazaki, Noboru Endou, and Yasunari Shidama. Cartesian products of family of real linear spaces. *Formalized Mathematics*, 19(1):51–59, 2011. doi:10.2478/v10037-011-0009-2.
- [10] Laurent Schwartz. *Théorie des ensembles et topologie, tome 1. Analyse*. Hermann, 1997.
- [11] Laurent Schwartz. *Calcul différentiel, tome 2. Analyse*. Hermann, 1997.
- [12] Kōsaku Yosida. *Functional Analysis*. Springer, 1980.

Accepted December 24, 2024
