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Pascal's Triangle and Lucas's Theorem

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Summary. In this article we construct formally the Pascal's triangle using Mizar proof assistant. Using the same techniques, we show some similar constructions based on integer sequences. We also prove Lucas's theorem providing useful registrations of clusters to enable more automation in calculations.

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Introduction

The use of Pascal's triangle as an object of mathematical studies has occurred much earlier than its name may suggest, but still gathers a lot of interest due to its relevance to scientific studies and teaching [11], [15]. This work illustrates the creation of the Triangle in the Mizar system [2], [3], using finite sequences, and shows similar constructions developed more recently [1]. Additionally it provides some simple lemmas on the divisibility of factorials (Sect. 2) and binomial coefficients [7] (Sect. 3), and the Lucas's theorem [9], in the form sometimes referred to as Anton's Lemma [4] (Sect. 4), relatively recently formalized in Isabelle/HOL [6], [5]. Some of the properties are expressed in the form of registrations of clusters to simplify calculations within the Mizar Mathematical Library. This could enhance the encoding of elementary number theory in Mizar [10], [8] as described in [12] or even more complex topics there [13].

1. Preliminaries

Now we state the propositions:

(1) Let us consider non zero natural numbers k, n. If $k \bmod n = 0$, then $k - 1 \bmod n = n - 1$.

(2) Let us consider a natural number k, and a non zero natural number n. If $k + 1 \bmod n = 0$, then $k + 1 \operatorname{div} n = (k \operatorname{div} n) + 1$. The theorem is a consequence of (1).

Let a be a non zero natural number. Let us observe that $a - 1 \bmod a$ reduces to $a - 1$. Let us consider non zero natural numbers n, k. Now we state the propositions:

(3) If $n \bmod k > 0$, then $n - 1 \bmod k = (n \bmod k) - 1$.

(4) If $n \bmod k > 0$, then $n \operatorname{div} k = n - 1 \operatorname{div} k$. The theorem is a consequence of (3).

2. Properties of Factorial

Let a be a trivial natural number. Let us observe that $a!$ is trivial and non zero.

One can check that $1!$ reduces to 1 and $2!$ reduces to 2. Let a, b be natural numbers. Let us note that $(a + b)! \bmod b$ is zero.

Let us consider natural numbers n, k. Now we state the propositions:

(5) $n! \mid (n + k)!$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv n! \mid (n + \$_1)!$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m + 1]$. For every natural number c, $\mathcal{P}[c]$. □

(6) $(\min(n, k))! \mid n!$. The theorem is a consequence of (5).

Let n be a natural number. One can check that $\binom{n}{1}$ reduces to n. Let k be a natural number. Note that $\frac{(n+k)!}{n!}$ is natural and $\frac{(n+k)!}{n! \cdot (k!)}$ is natural and $\frac{n!}{(\min(n,k))!}$ is natural and $\frac{(n+k)!}{((\min(n,k))!)^2}$ is natural.

Let us consider natural numbers n, k. Now we state the propositions:

(7) $(n!)^k \mid n \cdot k!$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv (n!)^{\$_1} \mid n \cdot \$_1!$. $\mathcal{P}[0]$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m + 1]$ by [14, (8)]. For every natural number c, $\mathcal{P}[c]$. □

(8) If $k \geqslant 2 \cdot n$, then $2^n \mid k!$. The theorem is a consequence of (6) and (7).

3. Properties of Binomial Coefficient

Let n be a non zero natural number. Observe that $\binom{0}{n}$ reduces to 0. Let m be a natural number. Observe that $\binom{m \bmod n}{n}$ is zero. Now we state the propositions:

(9) Let us consider natural numbers k, n. Then $\binom{k+n}{n} = \binom{k+n}{k}$.

(10) Let us consider an odd natural number n. Then $\binom{n}{\frac{n+1}{2}} = \binom{n}{\frac{n-1}{2}}$. The theorem is a consequence of (9).

Let us consider a natural number n. Now we state the propositions:

(11) $\binom{2\cdot(n+1)}{n+1} = 2 \cdot \binom{2\cdot n+1}{n}$. The theorem is a consequence of (9).

(12) $\binom{n+1}{2} - \binom{n}{2} = n$.

Let n be a natural number. Observe that $\binom{n+1}{0}$ reduces to 1 and $\binom{n+1}{2} - \binom{n}{2}$ reduces to n. One can check that $\binom{2\cdot(n+1)}{n+1}$ is even. Now we state the proposition:

(13) Let us consider a natural number n, and a non zero natural number m. Then $\binom{n \bmod m}{m-1} < 2$.

Let m, n be natural numbers. One can check that $\binom{n \bmod (m+1)}{m}$ is trivial.

Now we state the propositions:

(14) Let us consider a natural number n, and a non zero natural number m. Then $\binom{n \bmod m}{m-1} = 1$ if and only if $n \bmod m = m - 1$.

PROOF: If $n \bmod m \neq m - 1$, then $\binom{n \bmod m}{m-1} = 0$. □

(15) Let us consider an odd prime number p. Then $p \mid \binom{p+1}{\frac{p+1}{2}}$. The theorem is a consequence of (10) and (11).

(16) Let us consider an odd prime number p, and a non zero natural number k. If $k + 1 < p$, then $p \mid \binom{p+1}{k+1}$.

Let us consider a prime number p and a non zero natural number k. Now we state the propositions:

(17) If $k \neq p$, then $\binom{p}{k} \bmod p = 0$.

(18) If $p \nmid k$, then $\binom{p}{k \bmod p} \bmod p = 0$. The theorem is a consequence of (17).

(19) Let us consider a prime number p, and an odd natural number n. Suppose $n < p$. Then

(i) $\binom{p-1}{n} \bmod p = p - 1$, and

(ii) $\binom{p-1}{n-1} \bmod p = 1$.

(20) Let us consider a prime number p, and an even natural number n. If $n < p$, then $\binom{p-1}{n} \bmod p = 1$. The theorem is a consequence of (19).

(21) Let us consider a prime number p, a natural number n, and a non zero natural number k. If $n + k < p$, then $\binom{p+n}{k+n} \bmod p = 0$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non zero natural number k such that $k + \$_1 < p$ holds $\binom{p+\$_1}{k+\$_1} \bmod p = 0$. $\mathcal{P}[0]$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number c, $\mathcal{P}[c]$. □

Let us consider a prime number p and a natural number n. Now we state the propositions:

(22) If $n + 2 < p$, then $\binom{p+1}{n+2} \bmod p = 0$. The theorem is a consequence of (21).

(23) If $n < p$, then $\binom{p+n}{n} \bmod p = 1$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 < p$, then $\binom{p+\$_1}{\$_1} \bmod p = 1$. $\mathcal{P}[0]$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number k, $\mathcal{P}[k]$. □

(24) Let us consider a prime number p, a natural number n, and a natural number k. Suppose $k \leqslant n < p$. Then $\binom{p+n}{k} \bmod p = \binom{n}{k} \bmod p$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number k such that $k \leqslant \$_1 < p$ holds $\binom{p+\$_1}{k} \bmod p = \binom{\$_1}{k} \bmod p$. $\mathcal{P}[0]$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number c, $\mathcal{P}[c]$. □

(25) Let us consider a prime number p, and a non zero natural number n. If $n < p$, then $\binom{2 \cdot p}{n} \bmod p = 0$. The theorem is a consequence of (24) and (17).

(26) Let us consider a prime number p, and natural numbers k, n. Suppose $k < n \leqslant p$. Then $\binom{p+n}{k} \bmod p = \binom{n}{k} \bmod p$. The theorem is a consequence of (25), (17), and (24).

(27) Let us consider a prime number p, and a natural number n. If $p \nmid n$, then $\binom{p}{n} \bmod p = 0$.

(28) Let us consider non zero natural numbers a, b. Then $\binom{a \cdot b}{1} \bmod b = 0$.

(29) Let us consider natural numbers a, b. Then $\binom{a \cdot b + 1}{1} \bmod b = 1 \bmod b$. The theorem is a consequence of (28).

(30) Let us consider natural numbers a, b, c. Then $\binom{a+1}{b+1} \bmod c = (\binom{a}{b} \bmod c) + (\binom{a}{b+1} \bmod c) \bmod c$.

(31) Let us consider a prime number p, and natural numbers n, k. Suppose $k \neq p$. Then $\binom{(n \bmod p)+1}{k} \bmod p = \binom{n+1 \bmod p}{k} \bmod p$. The theorem is a consequence of (17).

4. ANTON'S LEMMA

Now we state the proposition:

(32) Let us consider a prime number p, a natural number n, and a natural number k. Then $\binom{n}{k} \bmod p = \binom{n \bmod p}{k \bmod p} \cdot \binom{n \operatorname{div} p}{k \operatorname{div} p} \bmod p$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number k, $\binom{\$_1}{k} \bmod p = \binom{\$_1 \bmod p}{k \bmod p} \cdot \binom{\$_1 \operatorname{div} p}{k \operatorname{div} p} \bmod p$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number c, $\mathcal{P}[c]$. $\square$

Let us consider a prime number p and a natural number n. Now we state the propositions:

(33) $\binom{n}{p} \bmod p = (n \operatorname{div} p) \bmod p$. The theorem is a consequence of (32).

(34) $\binom{p+n}{p} \bmod p = (n \operatorname{div} p) + 1 \bmod p$. The theorem is a consequence of (32).

(35) $\binom{p \cdot n}{p} \bmod p = n \bmod p$. The theorem is a consequence of (32).

(36) Let us consider a prime number p, and natural numbers n, k. Suppose $k < p$. Then $\binom{n}{k} \bmod p = \binom{n \bmod p}{k} \bmod p$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number k such that $k < p$ holds $\binom{\$_1}{k} \bmod p = \binom{\$_1 \bmod p}{k} \bmod p$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$. For every natural number c, $\mathcal{P}[c]$. $\square$

Let us consider natural numbers n, k and a prime number p. Now we state the propositions:

(37) If $k < p$, then $\binom{n \cdot p + k}{k} \bmod p = 1$. The theorem is a consequence of (36).

(38) If $k < p$, then $\binom{n \cdot p + k}{n \cdot p} \bmod p = 1$. The theorem is a consequence of (37) and (9).

(39) Let us consider a prime number p. Then $\binom{2 \cdot p}{p} \bmod p = 2 \bmod p$. The theorem is a consequence of (37) and (38).

(40) Let us consider a prime number p, and a natural number n. Then $\binom{n}{p-1} \bmod p = \binom{n \bmod p}{p-1} \bmod p$. The theorem is a consequence of (36).

(41) Let us consider a non zero natural number k, a natural number i, and a prime number p. Then $\binom{i \cdot p + (p - 'k)}{p - 'k} \bmod p = 1$. The theorem is a consequence of (36).

Let us consider a prime number p and natural numbers n, k. Now we state the propositions:

(42) If $k < p$, then $\binom{n \cdot p + k}{p} \bmod p = \binom{n \cdot p}{p} \bmod p$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 < p$, then $\binom{n \cdot p + \$_1}{p} \bmod p = \binom{n \cdot p}{p} \bmod p$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(43) $\binom{(n+k)\cdot p}{p} \bmod p = n + k \bmod p$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv \binom{(n+\$_1)\cdot p}{p} \bmod p = n + \$_1 \bmod p$. $\mathcal{P}[0]$. For every natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every natural number k, $\mathcal{P}[k]$. □

(44) Let us consider a prime number p, natural numbers k, n, and a non zero natural number m. If $k + m < p$, then $\binom{n\cdot p+k}{m+k} \bmod p = 0$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every natural number k for every non zero natural number m such that $k+m < p$ holds $\binom{\$_1\cdot p+k}{m+k} \bmod p = 0$. $\mathcal{P}[0]$. For every natural number q such that $\mathcal{P}[q]$ holds $\mathcal{P}[q+1]$. For every natural number c, $\mathcal{P}[c]$. □

(45) Let us consider a natural number n. Then $\mathrm{Parity}((n+1)!) = (\mathrm{Parity}(n+1)) \cdot (\mathrm{Parity}(n!))$.

(46) Let us consider an even natural number n. Then $\mathrm{Parity}(n!) = \mathrm{Parity}((n+1)!)$. The theorem is a consequence of (45).

(47) Let us consider a natural number n.
Then $\mathrm{Parity}((n+2)!) = 2 \cdot (\mathrm{Parity}(\mathrm{Triangle}(n+1))) \cdot (\mathrm{Parity}(n!))$.

5. Pascal's Triangle Step by Step

Let f be a 1-element finite sequence. Let us note that $\langle f(1)\rangle$ reduces to f. Let f be a 2-element finite sequence. One can verify that $\langle f(1), f(2)\rangle$ reduces to f. Let f be a 3-element finite sequence. One can verify that $\langle f(1), f(2), f(3)\rangle$ reduces to f. Let f be a 4-element finite sequence. Let us note that $\langle f(1), f(2), f(3), f(4)\rangle$ reduces to f.

Let f be a 5-element finite sequence. Let us note that $\langle f(1), f(2), f(3), f(4), f(5)\rangle$ reduces to f. Let f be a 6-element finite sequence. One can verify that $\langle f(1), f(2), f(3), f(4), f(5), f(6)\rangle$ reduces to f. Let f be a 7-element finite sequence. One can verify that $\langle f(1), f(2), f(3), f(4), f(5), f(6), f(7)\rangle$ reduces to f. Let f be an 8-element finite sequence. Let us note that $\langle f(1), f(2), f(3), f(4), f(5), f(6), f(7), f(8)\rangle$ reduces to f.

Let n be a natural number. Let us observe that $\langle 0\rangle(n)$ reduces to 0. Let a_1, a_2, a_3, a_4, a_5 be complex numbers. One can verify that $\langle a_1, a_2, a_3, a_4, a_5\rangle$ is complex-valued. Let a_6 be a complex number. One can check that $\langle a_1, a_2, a_3, a_4, a_5, a_6\rangle$ is complex-valued. Let a_7 be a complex number. One can verify that $\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7\rangle$ is complex-valued. Let a_8 be a complex number. Let us note that $\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\rangle$ is complex-valued. Now we state the propositions:

(48) Let us consider a non zero natural number n, and finite sequences f, g. Then $(f \frown g)(\mathrm{len}\, f + n) = g(n)$.

(49) Let us consider a non zero natural number n, a complex number c, and a finite sequence f. Then $(\langle c\rangle \frown f)(n+1) = f(n)$. The theorem is a consequence of (48).

(50) Let us consider a finite sequence f, and a natural number n. Then $(f \frown \langle 0\rangle)(n) = f(n)$. The theorem is a consequence of (48).

(51) Let us consider a natural number n. Then $\langle \binom{n+1}{0}, \ldots, \binom{n+1}{n+1}\rangle = \langle 0\rangle \frown \langle \binom{n}{0}, \ldots, \binom{n}{n}\rangle + \langle \binom{n}{0}, \ldots, \binom{n}{n}\rangle \frown \langle 0\rangle$.
PROOF: For every natural number i such that $1 \leqslant i \leqslant \operatorname{len}\langle \binom{n+1}{0}, \ldots, \binom{n+1}{n+1}\rangle$ holds $\langle \binom{n+1}{0}, \ldots, \binom{n+1}{n+1}\rangle(i) = (\langle 0\rangle \frown \langle \binom{n}{0}, \ldots, \binom{n}{n}\rangle + \langle \binom{n}{0}, \ldots, \binom{n}{n}\rangle \frown \langle 0\rangle)(i)$. □

(52) Let us consider complex numbers a_1, a_2, b_1, b_2, a natural number n, and n-element, complex-valued finite sequences f, g. Then $f \frown \langle a_1, a_2\rangle + g \frown \langle b_1, b_2\rangle = (f+g) \frown \langle a_1+b_1, a_2+b_2\rangle$.

(53) Let us consider complex numbers a_1, a_2, a_3, b_1, b_2, b_3, a natural number n, and n-element, complex-valued finite sequences f, g. Then $f \frown \langle a_1, a_2, a_3\rangle + g \frown \langle b_1, b_2, b_3\rangle = (f+g) \frown \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$.

(54) Let us consider complex numbers a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5. Then $\langle a_1, a_2, a_3, a_4, a_5\rangle + \langle b_1, b_2, b_3, b_4, b_5\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5\rangle$.

(55) Let us consider complex numbers a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6. Then $\langle a_1, a_2, a_3, a_4, a_5, a_6\rangle + \langle b_1, b_2, b_3, b_4, b_5, b_6\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6\rangle$. The theorem is a consequence of (54).

(56) Let us consider complex numbers a_1, a_2, a_3, a_4, a_5, a_6, a_7, b_1, b_2, b_3, b_4, b_5, b_6, b_7. Then $\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7\rangle + \langle b_1, b_2, b_3, b_4, b_5, b_6, b_7\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7\rangle$. The theorem is a consequence of (54) and (52).

(57) Let us consider complex numbers a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8. Then $\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\rangle + \langle b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7, a_8+b_8\rangle$. The theorem is a consequence of (54) and (53).

(58) $\langle \binom{0}{0}, \ldots, \binom{0}{0}\rangle = \langle 1\rangle$.

(59) $\langle \binom{1}{0}, \ldots, \binom{1}{1}\rangle = \langle 1, 1\rangle$.

(60) $\langle \binom{2}{0}, \ldots, \binom{2}{2}\rangle = \langle 1, 2, 1\rangle$.

(61) $\langle \binom{3}{0}, \ldots, \binom{3}{3}\rangle = \langle 1, 3, 3, 1\rangle$.

(62) $\langle \binom{4}{0}, \ldots, \binom{4}{4}\rangle = \langle 1, 4, 6, 4, 1\rangle$. The theorem is a consequence of (51), (61), and (54).

(63) $\langle \binom{5}{0}, \ldots, \binom{5}{5}\rangle = \langle 1, 5, 10, 10, 5, 1\rangle$. The theorem is a consequence of (51), (62), and (55).

(64) $\langle \binom{6}{0}, \ldots, \binom{6}{6} \rangle = \langle\!\langle 1, 6, 15, 20, 15, 6, 1 \rangle\!\rangle$. The theorem is a consequence of (51), (63), and (56).

(65) $\langle \binom{7}{0}, \ldots, \binom{7}{7} \rangle = \langle\!\langle 1, 7, 21, 35, 35, 21, 7, 1 \rangle\!\rangle$. The theorem is a consequence of (51), (64), and (57).

(66) $\langle \binom{8}{0}, \ldots, \binom{8}{8} \rangle = \langle\!\langle 1, 8, 28, 56, 70, 56, 28, 8, 1 \rangle\!\rangle$. The theorem is a consequence of (51), (65), and (57).

Let us consider a natural number n. Now we state the propositions:

(67) $\binom{n}{0} + \binom{n+2}{1} + \binom{n+4}{2} = \binom{n+5}{2} - \binom{n+5}{0}$.

(68) $\binom{n}{0} + \binom{n+2}{1} + \binom{n+4}{2} + \binom{n+6}{3} = \binom{n+7}{3} - \binom{n+6}{1}$.

(69) Let us consider natural numbers n, k. Suppose $k \in \mathrm{Seg}(n+1)$. Then there exist natural numbers l, m such that

(i) $l = k - 1$, and

(ii) $m = n - l$.

(70) Let us consider complex numbers a, b, and natural numbers n, k. Suppose $k \in \mathrm{Seg}(n+1)$. Then there exists an object c and there exist natural numbers l, m such that $m = k - 1$ and $l = n - m$ and $c = a^l \cdot b^m$. The theorem is a consequence of (69).

6. Harmonic Triangle

Let n be a non zero natural number. The functor $\mathrm{HTriangle}_{\mathrm{R}}(n)$ yielding a finite sequence is defined by the term

(Def. 1) $n \cdot \langle \binom{n-1}{0}, \ldots, \binom{n-1}{n-1} \rangle$.

One can check that $\mathrm{HTriangle}_{\mathrm{R}}(n)$ is n-element and $\mathrm{HTriangle}_{\mathrm{R}}(n)$ is $\mathbb{N}$-valued. Now we state the propositions:

(71) Let us consider non zero natural numbers n, k. Then $(\mathrm{HTriangle}_{\mathrm{R}}(n))(k) = n \cdot \binom{n-1}{k-1}$.

(72) $\mathrm{HTriangle}_{\mathrm{R}}(1) = \langle 1 \rangle$.

(73) $\mathrm{HTriangle}_{\mathrm{R}}(2) = \langle 2, 2 \rangle$.

(74) $\mathrm{HTriangle}_{\mathrm{R}}(3) = \langle 3, 6, 3 \rangle$.

(75) $\mathrm{HTriangle}_{\mathrm{R}}(4) = \langle 4, 12, 12, 4 \rangle$.

(76) $\mathrm{HTriangle}_{\mathrm{R}}(5) = \langle 5, 20, 30, 20, 5 \rangle$. The theorem is a consequence of (62).

(77) $\mathrm{HTriangle}_{\mathrm{R}}(6) = \langle\!\langle 6, 30, 60, 60, 30, 6 \rangle\!\rangle$. The theorem is a consequence of (63).

(78) $\mathrm{HTriangle}_{\mathrm{R}}(7) = \langle\!\langle 7, 42, 105, 140, 105, 42, 7 \rangle\!\rangle$. The theorem is a consequence of (64).

(79) $\mathrm{HTriangle}_{\mathrm{R}}(8) = \langle 8, 56, 168, 280, 280, 168, 56, 8\rangle$. The theorem is a consequence of (65).

Let n be a non zero natural number. The functor $\mathrm{HTriangle}(n)$ yielding a finite sequence is defined by the term

(Def. 2) $(\mathrm{HTriangle}_{\mathrm{R}}(n))^{-1}$.

Let us observe that $\mathrm{HTriangle}(n)$ is n-element and $\mathrm{HTriangle}(n)$ is $\mathbb{R}$-valued.

Now we state the propositions:

(80) Let us consider non zero natural numbers n, k. Then $(\mathrm{HTriangle}(n))(k) = \frac{1}{n\cdot\binom{n-1}{k-1}}$. The theorem is a consequence of (71).

(81) Let us consider a non zero natural number n. Then $\sum \mathrm{HTriangle}_{\mathrm{R}}(n) = n \cdot 2^{n-1}$.

(82) $\mathrm{HTriangle}(1) = \langle 1\rangle$.

(83) $\mathrm{HTriangle}(2) = \langle \frac{1}{2}, \frac{1}{2}\rangle$. The theorem is a consequence of (73).

(84) $\mathrm{HTriangle}(3) = \langle \frac{1}{3}, \frac{1}{6}, \frac{1}{3}\rangle$. The theorem is a consequence of (74).

(85) $\mathrm{HTriangle}(4) = \langle \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{4}\rangle$. The theorem is a consequence of (75).

(86) $\mathrm{HTriangle}(5) = \langle \frac{1}{5}, \frac{1}{20}, \frac{1}{30}, \frac{1}{20}, \frac{1}{5}\rangle$. The theorem is a consequence of (76).

(87) $\mathrm{HTriangle}(6) = \langle \frac{1}{6}, \frac{1}{30}, \frac{1}{60}, \frac{1}{60}, \frac{1}{30}, \frac{1}{6}\rangle$. The theorem is a consequence of (77).

(88) $\mathrm{HTriangle}(7) = \langle \frac{1}{7}, \frac{1}{42}, \frac{1}{105}, \frac{1}{140}, \frac{1}{105}, \frac{1}{42}, \frac{1}{7}\rangle$. The theorem is a consequence of (78).

(89) $\mathrm{HTriangle}(8) = \langle \frac{1}{8}, \frac{1}{56}, \frac{1}{168}, \frac{1}{280}, \frac{1}{280}, \frac{1}{168}, \frac{1}{56}, \frac{1}{8}\rangle$. The theorem is a consequence of (79).

7. RASCAL TRIANGLE

Let n be a natural number. The functor $\mathrm{Rascal}(n)$ yielding a finite sequence is defined by

(Def. 3) $\mathrm{dom}\, it = \mathrm{Seg}(n+1)$ and for every natural number i such that $i \in \mathrm{dom}\, it$ holds $it(i) = (i-1)\cdot(n+1-i)+1$.

Let n be a natural number. Let us observe that $\mathrm{Rascal}(n)$ is $(n+1)$-element.

Now we state the propositions:

(90) $\mathrm{Rascal}(0) = \langle 1\rangle$.

(91) $\mathrm{Rascal}(1) = \langle 1, 1\rangle$.

(92) $\mathrm{Rascal}(2) = \langle 1, 2, 1\rangle$.

(93) $\mathrm{Rascal}(3) = \langle 1, 3, 3, 1\rangle$.

(94) $\mathrm{Rascal}(4) = \langle 1, 4, 5, 4, 1\rangle$.

(95) $\mathrm{Rascal}(5) = \langle 1, 5, 7, 7, 5, 1 \rangle$.

(96) $\mathrm{Rascal}(6) = \langle 1, 6, 9, 10, 9, 6, 1 \rangle$.

(97) $\mathrm{Rascal}(7) = \langle 1, 7, 11, 13, 13, 11, 7, 1 \rangle$.

(98) $(\mathrm{Rascal}(7))(4) = 13$.

(99) Let us consider a natural number n. Then $(\mathrm{Rascal}(n))(1) = 1$.

(100) Let us consider a non zero natural number n. Then $(\mathrm{Rascal}(n))(2) = n$.

Let n, m be natural numbers. Let us note that $(\mathrm{Rascal}(n))(m)$ is natural.

Let us consider natural numbers k, n. Now we state the propositions:

(101) $(\mathrm{Rascal}(k+n))(n+1) + (\mathrm{Rascal}(k+n+2))(n+2) = (\mathrm{Rascal}(k+n+1))(n+1) + (\mathrm{Rascal}(k+n+1))(n+2) + 1$.

(102) $(\mathrm{Rascal}(k+n))(n+1) \cdot (\mathrm{Rascal}(k+n+2))(n+2) = (\mathrm{Rascal}(k+n+1))(n+1) \cdot (\mathrm{Rascal}(k+n+1))(n+2) + 1$.

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