FORMALIZED MATHEMATICS
Vol. 32, No. 1, pp. 195–201, 2024
DOI: 10.2478/forma-2024-0016
e-ISSN: 1898–9934

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sciendo.com/journal/forma

On the Properties of Curves and Parametrization-Independent Isoperimetric Inequality[1]

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Summary. In this article we formalize in Mizar several properties of curves. We introduce the definition of the `ArcLenP` function and define arc length parametrization with its fundamental properties. Finally we prove an isoperimetric inequality that holds regardless of the curve's parametrization.

MSC: 26B15 49Q20 68V20

Keywords: isoperimetric theorem; differential geometry; parametric curve

MML identifier: `CURVE`, version: `8.1.14 5.86.1479`

Introduction

In this article we formalize in Mizar [1], [2] several properties of curves and establish a parametrization-independent isoperimetric inequality [3]. The paper is structured into three main sections: Section 1 introduces fundamental definitions, notational conventions and initial theorems, including the definition of the `ArcLenP` function (defined as the Mizar functor). In the second section arc length parametrization is constructed and some of its properties, including differentiability and characteristics of its inverse function, are explored. Section 3 proves an isoperimetric inequality [16] that holds regardless of the curve's parametrization [11].

[1]This work was supported by JSPS KAKENHI Grant Number 24K14897.

We use our earlier formalization of Peter D. Lax paper [12] contained in [13]. It provides a rigorous foundation for further work in differential geometry [15] and analysis (compare recent results encoding this area within the Mizar Mathematical Library in [4], [5]). It allows also for further formalization of the results strictly connected with the isoperimetric theorem [14]. This work, as a continuation of [13], represents the solution of the problem #43 of Freek Wiedijk's "Formalizing 100 Theorems" project [18] (compare the formal development in HOL Light [9]: [10] and [17]).

1. Preliminaries and Basic Theorems

From now on a, b, r denote real numbers, A denotes a non empty set, X, x denote sets, f, g, F, G denote partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n denotes an element of $\mathbb{N}$.

Let a, b be real numbers and x, y be partial functions from $\mathbb{R}$ to $\mathbb{R}$. The functor $\mathrm{ArcLenP}(x, y, a, b)$ yielding a partial function from $\mathbb{R}$ to $\mathbb{R}$ is defined by

(Def. 1) $\mathrm{dom}\, it = [a, b]$ and for every real number t such that $t \in [a, b]$ holds
$$it(t) = \int\limits_{a}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx.$$

Now we state the propositions:

(1) Let us consider real numbers a, b, d, and a partial function f from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq \mathrm{dom}\, f$ and $f \restriction [a, b]$ is continuous and $f(a) < d < f(b)$. Then there exists a real number c such that

(i) $a < c < b$, and

(ii) $d = f(c)$.

Proof: Reconsider $g = f \restriction [a, b]$ as a function from $[a,\, b]_{\mathrm{T}}$ into $\mathbb{R}^{\mathbf{1}}$. Set $T = [a,\, b]_{\mathrm{T}}$. For every point p of T and for every positive real number r, there exists an open subset W of T such that $p \in W$ and $g^{\circ}W \subseteq \,]g(p) - r, g(p) + r[$ by [8, (39)]. Consider c being a real number such that $g(c) = d$ and $a < c < b$. $\square$

(2) Let us consider real numbers a, b, and an open subset Z of $\mathbb{R}$. Suppose $a < b$ and $[a, b] \subseteq Z$. Then there exist real numbers a_1, b_1 such that

(i) $a_1 < a$, and

(ii) $b < b_1$, and

(iii) $a_1 < b_1$, and

(iv) $[a_1, b_1] \subseteq Z$, and

(v) $[a, b] \subseteq \,]a_1, b_1[$.

2. Arc Length Parametrization

Let us consider real numbers a, b and partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$. Now we state the propositions:

(3) Suppose $a < b$ and x is differentiable and y is differentiable and $[a, b] \subseteq \operatorname{dom} x$ and $[a, b] \subseteq \operatorname{dom} y$ and $x'_{\restriction \operatorname{dom} x}$ is continuous and $y'_{\restriction \operatorname{dom} y}$ is continuous and for every real number t such that $t \in \operatorname{dom} x \cap \operatorname{dom} y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then there exist real numbers a_1, b_1 and there exists a partial function l from $\mathbb{R}$ to $\mathbb{R}$ and there exists an open subset Z of $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and $Z = \operatorname{dom} x \cap \operatorname{dom} y$ and $[a, b] \subseteq]a_1, b_1[$ and $[a_1, b_1] \subseteq Z$ and $\operatorname{dom} l = Z$ and for every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$ and l is differentiable on $]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[} = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y}) \restriction]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[}$ is continuous and for every real number t such that $t \in]a_1, b_1[$ holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and for every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$.

PROOF: Reconsider $Z_1 = \operatorname{dom} x$, $Z_2 = \operatorname{dom} y$ as an open subset of $\mathbb{R}$. Reconsider $Z = Z_1 \cap Z_2$ as an open subset of $\mathbb{R}$. Consider d_1 being a real number such that $0 < d_1$ and $]a - d_1, a + d_1[\subseteq Z$. Consider d_2 being a real number such that $0 < d_2$ and $]b - d_2, b + d_2[\subseteq Z$. Reconsider $d = \min(d_1, d_2)$ as a real number. Set $a_1 = a - \frac{d}{2}$. Set $b_1 = b + \frac{d}{2}$. $[a_1, b_1] \subseteq Z$. Define $\mathcal{F}(\text{real number}) = (\int\limits_{a_1}^{\$_1} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx) (\in \mathbb{R})$. Consider l_0 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element t of $\mathbb{R}$, $l_0(t) = \mathcal{F}(t)$. For every real number t, $l_0(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$. Set $l = l_0 \restriction Z$. Set $X_2 = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})$. For every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} X_2(x)dx$. For every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$ by [6, (10), (11)], [7, (17)]. For every real number t such that $t \in]a_1, b_1[$ holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ by [7, (28)]. $\square$

(4) Suppose $a < b$ and x is differentiable and y is differentiable and $[a, b] \subseteq$

$\operatorname{dom} x$ and $[a,b] \subseteq \operatorname{dom} y$ and $x'_{\restriction \operatorname{dom} x}$ is continuous and $y'_{\restriction \operatorname{dom} y}$ is continuous and for every real number t such that $t \in \operatorname{dom} x \cap \operatorname{dom} y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then there exist real numbers a_1, b_1 and there exists a one-to-one partial function L from $\mathbb{R}$ to $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and $[a_1, b_1] \subseteq \operatorname{dom} x \cap \operatorname{dom} y$ and $\operatorname{dom} L =]a_1, b_1[$ and for every real number t such that $t \in]a_1, b_1[$ holds $L(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$ and for every real number t such that $t \in [a,b]$ holds (ArcLenP$(x,y,a,b))(t) = L(t) - L(a)$ and L is increasing and $L \restriction [a,b]$ is continuous and $L^{\circ}[a,b] = [L(a), L(b)]$ and for every real number t such that $t \in]a_1, b_1[$ holds L is differentiable in t and L is differentiable on $]a_1, b_1[$ and for every real number t such that $t \in]a_1, b_1[$ holds $L'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and L^{-1} is differentiable on $\operatorname{dom}(L^{-1})$ and for every real number t such that $t \in \operatorname{dom}(L^{-1})$ holds $(L^{-1})'(t) = \frac{1}{L'((L^{-1})(t))}$ and L^{-1} is continuous and for every real number s such that $s \in \operatorname{rng} L$ holds $x \cdot (L^{-1})$ is differentiable in s and $y \cdot (L^{-1})$ is differentiable in s and $(x \cdot (L^{-1}))'(s) = x'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(y \cdot (L^{-1}))'(s) = y'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ and $(x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))} = x'_{\restriction \operatorname{dom} x} \cdot (L^{-1}) \cdot (L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ and $(y \cdot (L^{-1}))'_{\restriction \operatorname{dom}(y \cdot (L^{-1}))} = y'_{\restriction \operatorname{dom} y} \cdot (L^{-1}) \cdot (L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ and $(L^{-1})'_{\restriction \operatorname{dom}(L^{-1})} = \frac{1}{L'_{\restriction \operatorname{dom} L} \cdot (L^{-1})}$ and $(L^{-1})'_{\restriction \operatorname{dom}(L^{-1})}$ is continuous and $[L(a), L(b)] \subseteq \operatorname{dom}(L^{-1})$ and $[L(a), L(b)] \subseteq \operatorname{dom}(x \cdot (L^{-1}))$ and $[L(a), L(b)] \subseteq \operatorname{dom}(y \cdot (L^{-1}))$ and $[L(a), L(b)] \subseteq \operatorname{rng} L$ and $\operatorname{dom}(x \cdot (L^{-1})) = \operatorname{dom}(L^{-1})$ and $\operatorname{dom}(y \cdot (L^{-1})) = \operatorname{dom}(L^{-1})$ and $x \cdot (L^{-1})$ is differentiable and $y \cdot (L^{-1})$ is differentiable and $(x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))}$ is continuous and $(y \cdot (L^{-1}))'_{\restriction \operatorname{dom}(y \cdot (L^{-1}))}$ is continuous and for every real number s such that $s \in \operatorname{dom}(x \cdot (L^{-1})) \cap \operatorname{dom}(y \cdot (L^{-1}))$ holds $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$ and $\int\limits_{a}^{b} (y \cdot x'_{\restriction \operatorname{dom} x})(x)dx = \int\limits_{L(a)}^{L(b)} (y \cdot (L^{-1}) \cdot (x \cdot (L^{-1}))'_{\restriction \operatorname{dom}(x \cdot (L^{-1}))})(x)dx$.

PROOF: Consider a_1, b_1 being real numbers, l being a partial function from $\mathbb{R}$ to $\mathbb{R}$, Z being an open subset of $\mathbb{R}$ such that $a_1 < a$ and $b < b_1$ and $Z = \operatorname{dom} x \cap \operatorname{dom} y$ and $[a,b] \subseteq]a_1, b_1[$ and $[a_1, b_1] \subseteq Z$ and $\operatorname{dom} l = Z$ and for every real number t such that $t \in [a_1, b_1]$ holds $l(t) = \int\limits_{a_1}^{t} (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y})(x)dx$ and l is differentiable on $]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[} = (\square^{\frac{1}{2}}) \cdot (x'_{\restriction \operatorname{dom} x} \cdot x'_{\restriction \operatorname{dom} x} + y'_{\restriction \operatorname{dom} y} \cdot y'_{\restriction \operatorname{dom} y}) \restriction]a_1, b_1[$ and $l'_{\restriction]a_1, b_1[}$ is continuous and for every real number t such that $t \in]a_1, b_1[$

holds l is differentiable in t and $l'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$ and for every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = l(t) - l(a)$. Set $L = l\restriction]a_1, b_1[$. For every real number t such that $t \in]a_1, b_1[$ holds $L(t) = \int\limits_{a_1}^{t} (\Box^{\frac{1}{2}}) \cdot (x'_{\restriction \mathrm{dom}\, x} \cdot x'_{\restriction \mathrm{dom}\, x} + y'_{\restriction \mathrm{dom}\, y} \cdot y'_{\restriction \mathrm{dom}\, y})(x) dx$. For every real number t such that $t \in [a, b]$ holds $(\mathrm{ArcLenP}(x, y, a, b))(t) = L(t) - L(a)$. For every real number t such that $t \in]a_1, b_1[$ holds $0 < l'(t)$. For every real number t such that $t \in]a_1, b_1[$ holds $L'(t) = (x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}})^{\frac{1}{2}}$. For every real number t such that $t \in]a_1, b_1[$ holds $0 < L'(t)$. For every real number s such that $s \in \mathrm{rng}\, L$ holds $x \cdot (L^{-1})$ is differentiable in s and $y \cdot (L^{-1})$ is differentiable in s and $(x \cdot (L^{-1}))'(s) = x'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(y \cdot (L^{-1}))'(s) = y'((L^{-1})(s)) \cdot (L^{-1})'(s)$ and $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$. Set $L_1 = (L^{-1})'_{\restriction \mathrm{dom}(L^{-1})}$. $L'_{\restriction \mathrm{dom}\, L} \cdot (L^{-1})^{-1}(\{0\}) = \emptyset$. For every real number t such that $t \in \mathrm{dom}\, L_1$ holds L_1 is continuous in t. For every object s, $s \in L^{\circ}[a, b]$ iff $s \in [L(a), L(b)]$. Set $e_1 = \frac{a - a_1}{2}$. Set $e_2 = \frac{b_1 - b}{2}$. Set $a_2 = a_1 + e_1$. Set $b_2 = b_1 - e_2$. $a_2 < a$ and $a_2 < b < b_2$ and $a_2 < b_2$ and $[a, b] \subseteq]a_2, b_2[$ and $[a_2, b_2] \subseteq]a_1, b_1[$. Define $\mathcal{FX}$(real number) $= (\int\limits_{a_2}^{\$_1} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x) dx) (\in \mathbb{R})$. Consider F_0 being a function from $\mathbb{R}$ into $\mathbb{R}$ such that for every element t of $\mathbb{R}$, $F_0(t) = \mathcal{FX}(t)$. For every real number t, $F_0(t) = \int\limits_{a_2}^{t} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x) dx$. Set $F = F_0\restriction [a_2, b_2]$. For every real number t such that $t \in]a_2, b_2[$ holds $F(t) = \int\limits_{a_2}^{t} (y \cdot x'_{\restriction \mathrm{dom}\, x})(x) dx$. $[a_2, b_2] \subseteq \mathrm{dom}(y \cdot x'_{\restriction \mathrm{dom}\, x})$. $[a_2, b] \subseteq \mathrm{dom}(y \cdot x'_{\restriction \mathrm{dom}\, x})$. For every real number t such that $t \in]a_2, b_2[$ holds F is differentiable in t and $F'(t) = (y \cdot x'_{\restriction \mathrm{dom}\, x})(t)$. $[L(a), L(b)] \subseteq]L(a_2), L(b_2)[$. Set $G = F \cdot (L^{-1}\restriction]L(a_2), L(b_2)[)$. For every object s, $s \in L^{\circ}[a_2, b_2]$ iff $s \in [L(a_2), L(b_2)]$. $]L(a_2), L(b_2)[\subseteq \mathrm{rng}\, L$. $\mathrm{rng}(L^{-1}\restriction]L(a_2), L(b_2)[) \subseteq \mathrm{dom}\, F$. For every real number t such that $t \in]L(a_2), L(b_2)[$ holds G is differentiable in t and $(L^{-1}\restriction]L(a_2), L(b_2)[)(t) \in]a_2, b_2[$ and $G'(t) = F'((L^{-1}\restriction]L(a_2), L(b_2)[)(t)) \cdot (L^{-1}\restriction]L(a_2), L(b_2)[)'(t)$. For every object s such that $s \in \mathrm{dom}\, G'_{\restriction]L(a_2), L(b_2)[}$ holds $G'_{\restriction]L(a_2), L(b_2)[}(s) = ((y \cdot (L^{-1}) \cdot (x \cdot (L^{-1}))'_{\restriction \mathrm{dom}(x \cdot (L^{-1}))})\restriction]L(a_2), L(b_2)[)(s)$. For every real number s such that $s \in \mathrm{dom}(x \cdot (L^{-1})) \cap \mathrm{dom}(y \cdot (L^{-1}))$ holds $(x \cdot (L^{-1}))'(s)^{\mathbf{2}} + (y \cdot (L^{-1}))'(s)^{\mathbf{2}} = 1$. $\Box$

3. Parametrization-Independent Isoperimetric Inequality

Now we state the proposition:

(5) Let us consider real numbers a, b, l, and partial functions x, y from $\mathbb{R}$ to $\mathbb{R}$. Suppose $a < b$ and $(\mathrm{ArcLenP}(x, y, a, b))(b) = l$ and $y(a) = 0$ and $y(b) = 0$ and x is differentiable and y is differentiable and $[a, b] \subseteq \mathrm{dom}\, x$ and $[a, b] \subseteq \mathrm{dom}\, y$ and $x'_{\restriction \mathrm{dom}\, x}$ is continuous and $y'_{\restriction \mathrm{dom}\, y}$ is continuous and for every real number t such that $t \in \mathrm{dom}\, x \cap \mathrm{dom}\, y$ holds $0 < x'(t)^{\mathbf{2}} + y'(t)^{\mathbf{2}}$. Then

(i) $\int\limits_a^b (y \cdot x'_{\restriction \mathrm{dom}\, x})(x) dx \leqslant \dfrac{\frac{1}{2} \cdot l^{\mathbf{2}}}{\pi}$, and

(ii) $\int\limits_a^b (y \cdot x'_{\restriction \mathrm{dom}\, x})(x) dx = \dfrac{\frac{1}{2} \cdot l^{\mathbf{2}}}{\pi}$ iff for every real number s such that $s \in [a, b]$ holds $y(s) = \frac{l}{\pi} \cdot (\text{the function } \sin)(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l})$ and $x(s) = \frac{l}{\pi} \cdot (-(\text{the function } \cos)(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l}) + (\text{the function } \cos)(0) + \frac{\pi}{l} \cdot x(a))$ or for every real number s such that $s \in [a, b]$ holds $y(s) = -\frac{l}{\pi} \cdot (\text{the function } \sin)(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l})$ and $x(s) = \frac{l}{\pi} \cdot ((\text{the function } \cos)(\frac{\pi \cdot (\mathrm{ArcLenP}(x,y,a,b))(s)}{l}) - (\text{the function } \cos)(0) + \frac{\pi}{l} \cdot x(a))$.

The theorem is a consequence of (4).

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Accepted December 14, 2024