

U-Small and U-Locally Small Categories¹

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Summary. This paper deals with the notions of U-small set, U-small category, and U-locally small category (U is non-empty Grothendieck universe). We reuse the first Mizar formalization of categories contained in **CAT_*** series of Mizar articles in order to show the expressive power of the Tarski-Grothendieck set theory (which is the base for the Mizar Mathematical Library) in this area. We encode parts of SGA 4 by Nicolas Bourbaki.

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INTRODUCTION

Category theory is developed from the beginning of the Mizar Mathematical Library [2]. Gradually it was expanded, with several different definitions for a category: **Category** [4] and **category** (in [9] or with another definition in [14]) based on [11] and [7]. In the following, we will only use, among these three definitions, the first, as well as the notion $\mathcal{U}$ for Grothendieck's non-empty Universe (see the Mizar formalisation of facts from [15] in [8] described in [3]). This choice is by no means accidental as the base set theory within the repository of Mizar texts is Tarski-Grothendieck set theory [12], [13].

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The first part of this work is devoted to the definitions of $\mathcal{U}$ -set and proper classes $\mathcal{U}$ -class.

The second part is largely influenced by the number 0 *Univers* of the first presentation of SGA 4 [1]: we define the notion of a $\mathcal{U}$ -small set (and of $\mathcal{U}$ -small group as well as of $\mathcal{U}$ -small Category).

1.1.1. Soient C et D deux catégories et $\text{Fonct}(C, D)$ la catégorie des foncteurs de C dans D . a) Si C et D sont éléments d'un univers $\underline{U}$ [...], la catégorie $\text{Fonct}(C, D)$ est un élément de $\underline{U}$ [...].

is reflected in theorems (70) and (91).

Our approach allows us to access the formalization of the definition of $\mathcal{U}$ -Category (see a wider general insight into the topic in less formal sense offered by [6]).

Finally, the notions of $\mathcal{U}$ -small Category and $\mathcal{U}$ -locally small Category are introduced. We conclude with some classic examples (adapted from “Example 1.1.4” by Emily Riehl in “Category theory in Context” [10]).

1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider a non empty set X . Then $\{\langle A, B \rangle, \text{ where } A, B \text{ are elements of } X : \text{ if } B = \emptyset, \text{ then } A = \emptyset\} = (X \times X) \setminus (X \setminus \{\emptyset\} \times \{\emptyset\})$.
- (2) Let us consider a non empty set X . Suppose $\{\emptyset\}$ is an element of X . Then $\{\emptyset\} \notin \text{Funcs } X$.
- (3) $\mathbb{N}_{\text{even}}$ is denumerable.
- (4) $\mathbb{N}_{\text{odd}}$ is denumerable.
- (5) Let us consider non empty sets X, Y , and an element y of Y . Then $X \times \{y\} \subseteq \bigcup Y^X$.
- (6) Let us consider a non empty set X , and a non zero natural number n . If X^n is finite, then X is finite.

Let us consider a non empty set X . Now we state the propositions:

- (7) $\bigcup \text{SmallestPartition}(X) = X$.
- (8) $X \approx \text{SmallestPartition}(X)$.
- (9) Let us consider a strict object-category $\mathcal{C}$. Then $(\mathcal{C}^{\text{op}})^{\text{op}} = \mathcal{C}$.

Let x_1, x_2, x_3, x_4, x_5 be objects. The functor $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ yielding an object is defined by the term

(Def. 1) $\langle \langle x_1, x_2, x_3, x_4 \rangle, x_5 \rangle$.

Let x be an object. We say that x is quintuple if and only if

(Def. 2) there exist objects x_1, x_2, x_3, x_4, x_5 such that $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Let us note that $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is quintuple.

Now we state the proposition:

(10) Let us consider objects $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$. Suppose $\langle x_1, x_2, x_3, x_4, x_5 \rangle = \langle y_1, y_2, y_3, y_4, y_5 \rangle$. Then

- (i) $x_1 = y_1$, and
- (ii) $x_2 = y_2$, and
- (iii) $x_3 = y_3$, and
- (iv) $x_4 = y_4$, and
- (v) $x_5 = y_5$.

One can verify that there exists an object which is quintuple and there exists a set which is quintuple.

Let x be an object. Assume x is quintuple. The functor $(x)_1$ yielding an object is defined by

(Def. 3) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_1$.

Assume x is quintuple. The functor $(x)_2$ yielding an object is defined by

(Def. 4) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_2$.

Assume x is quintuple. The functor $(x)_3$ yielding an object is defined by

(Def. 5) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_3$.

Assume x is quintuple. The functor $(x)_4$ yielding an object is defined by

(Def. 6) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_4$.

Assume x is quintuple. The functor $(x)_5$ yielding an object is defined by

(Def. 7) for every objects y_1, y_2, y_3, y_4, y_5 such that $x = \langle y_1, y_2, y_3, y_4, y_5 \rangle$ holds $it = y_5$.

Let x_1, x_2, x_3, x_4, x_5 be objects. Observe that $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_1$ reduces to x_1 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_2$ reduces to x_2 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_3$ reduces to x_3 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_4$ reduces to x_4 and $(\langle x_1, x_2, x_3, x_4, x_5 \rangle)_5$ reduces to x_5 .

Let x be a quintuple object. Observe that $\langle (x)_1, (x)_2, (x)_3, (x)_4, (x)_5 \rangle$ reduces to x .

2. SOME ELEMENTARY PROPERTIES

From now on $\mathcal{U}$, $\mathcal{V}$ denote universal classes and x denotes an element of $\mathcal{U}$. Now we state the propositions:

- (11) Let us consider objects x_1, x_2, x_3 . Suppose $x = \langle x_1, x_2, x_3 \rangle$. Then
- (i) x_1 is an element of $\mathcal{U}$, and
 - (ii) x_2 is an element of $\mathcal{U}$, and
 - (iii) x_3 is an element of $\mathcal{U}$.
- (12) Let us consider objects x_1, x_2, x_3, x_4 . Suppose $x = \langle x_1, x_2, x_3, x_4 \rangle$. Then
- (i) x_1 is an element of $\mathcal{U}$, and
 - (ii) x_2 is an element of $\mathcal{U}$, and
 - (iii) x_3 is an element of $\mathcal{U}$, and
 - (iv) x_4 is an element of $\mathcal{U}$.

The theorem is a consequence of (11).

- (13) Let us consider elements x_1, x_2, x_3, x_4, x_5 of $\mathcal{U}$. Then $\langle x_1, x_2, x_3, x_4, x_5 \rangle$ is an element of $\mathcal{U}$.
- (14) Let us consider objects x_1, x_2, x_3, x_4, x_5 . Suppose $x = \langle x_1, x_2, x_3, x_4, x_5 \rangle$. Then
- (i) x_1 is an element of $\mathcal{U}$, and
 - (ii) x_2 is an element of $\mathcal{U}$, and
 - (iii) x_3 is an element of $\mathcal{U}$, and
 - (iv) x_4 is an element of $\mathcal{U}$, and
 - (v) x_5 is an element of $\mathcal{U}$.

The theorem is a consequence of (12).

Let $\mathcal{U}$ be a universal class and u_1, u_2, u_3 be elements of $\mathcal{U}$. Let us note that the functor $\langle u_1, u_2, u_3 \rangle$ yields an element of $\mathcal{U}$. Let u_4 be an element of $\mathcal{U}$. Observe that the functor $\langle u_1, u_2, u_3, u_4 \rangle$ yields an element of $\mathcal{U}$. Let u_5 be an element of $\mathcal{U}$. Let us note that the functor $\langle u_1, u_2, u_3, u_4, u_5 \rangle$ yields an element of $\mathcal{U}$. Now we state the propositions:

- (15) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is an element of $\mathbf{U}_0$.
- (16) Let us consider a finite set X . If $X \subseteq \mathbf{U}_0$, then $X \in \mathbf{U}_0$.

PROOF: Consider p being a function such that $\text{rng } p = X$ and $\text{dom } p \in \omega$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$_2 = \{p(\$_1)\}$. Consider g being a function such that $\text{dom } g = \text{dom } p$ and for every object x such that $x \in \text{dom } p$ holds $\mathcal{P}[x, g(x)]$. $\text{rng } g \subseteq \mathbf{U}_0$. $\bigcup \text{rng } g = X$. $\square$

(17) (i) $\bigcup\{\mathbb{N}\} \subseteq \mathbf{U}_0$, and

(ii) $\bigcup\{\mathbb{N}\} \notin \mathbf{U}_0$, and

(iii) $\{\mathbb{N}\} \not\subseteq \mathbf{U}_0$, and

(iv) $\{\mathbb{N}\} \notin \mathbf{U}_0$.

(18) Let us consider an object x . Then $x \in \mathcal{U}$ if and only if $\{x\} \in \mathcal{U}$.

Let us consider a set X and a non zero natural number n . Now we state the propositions:

(19) If $\{X\}^{\text{Seg } n}$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (18).

(20) If $\{X\}^n$ is an element of $\mathcal{U}$, then X is an element of $\mathcal{U}$. The theorem is a consequence of (19).

(21) Let us consider a set X . If $\bigcup X \in \mathcal{U}$, then $X \in \mathcal{U}$.

3. SET AND CLASS

Let X be a non empty set and x be an object. We say that x is X -Set if and only if

(Def. 8) $x \in X$.

A Set of X is a set defined by

(Def. 9) it is X -Set.

Now we state the propositions:

(22) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Let us consider an object x . If x is $\mathcal{U}$ -Set, then x is $\mathcal{V}$ -Set.

(23) Let us consider universal classes $\mathcal{U}$, $\mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Then every Set of $\mathcal{U}$ is a Set of $\mathcal{V}$.

(24) Every Set of $\mathbf{U}_0$ is finite.

(25) Let us consider a subset x of $\mathbf{U}_0$. If x is finite, then x is a Set of $\mathbf{U}_0$. The theorem is a consequence of (15).

(26) Let us consider an object x . Then x is a Set of $\mathbf{U}_0$ if and only if x is a set of a finite rank.

Let $\mathcal{U}$ be a universal class and x be an object. We say that x is $\mathcal{U}$ -Class if and only if

(Def. 10) $x \in 2^{\mathcal{U}}$ and $x \notin \mathcal{U}$.

Now we state the proposition:

(27) Let us consider a set x . If x is $\mathcal{U}$ -Class, then x is not empty.

Let $\mathcal{U}$ be a universal class.

A Class of $\mathcal{U}$ is a non empty set defined by

(Def. 11) *it is $\mathcal{U}$ -Class.*

Now we state the propositions:

- (28) Let us consider a finite subset X of $\mathcal{U}$. Then $X \in \mathcal{U}$. The theorem is a consequence of (18) and (7).
- (29) Every Class of $\mathcal{U}$ is not finite. The theorem is a consequence of (28).
- (30) Let us consider a Set X of $\mathcal{U}$. Then $\mathcal{U} \setminus X$ is a Class of $\mathcal{U}$.
- (31) Every non finite subset of $\mathbf{U}_0$ is a Class of $\mathbf{U}_0$.
- (32) $\mathbf{N}$ is a Class of $\mathbf{U}_0$.
- (33) $\mathbf{N}_{\text{even}}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (3) and (31).
- (34) $\mathbf{N}_{\text{odd}}$ is a Class of $\mathbf{U}_0$. The theorem is a consequence of (4) and (31).
- (35) Let us consider an object x . Then
 - (i) x is not $\mathcal{U}$ -Class, or
 - (ii) x is not $\mathcal{U}$ -Set.
- (36) Let us consider universal classes $\mathcal{U}, \mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Let us consider an object x . If x is $\mathcal{U}$ -Class, then x is $\mathcal{V}$ -Set.
- (37) (i) $\bigcup\{\mathbf{N}\}$ is $\mathbf{U}_0$ -Class, and
 - (ii) $\{\mathbf{N}\}$ is not $\mathbf{U}_0$ -Class, and
 - (iii) $\{\mathbf{N}\}$ is not $\mathbf{U}_0$ -Set.

4. SOME OPERATIONS ON SETS AND CLASSES

Now we state the propositions:

- (38) Let us consider an object x . Then there exists $\mathcal{U}$ such that x is $\mathcal{U}$ -Set.
- (39) Every set is $(\text{GrothendieckUniverse}(x))$ -Set.

Let $\mathcal{U}, \mathcal{V}$ be universal classes. The functor $\text{sup}(\mathcal{U}, \mathcal{V})$ yielding a universal class is defined by the term

$$(\text{Def. 12}) \quad \begin{cases} \mathcal{U}, & \text{if } \mathcal{V} \in \mathcal{U}, \\ \mathcal{V}, & \text{otherwise.} \end{cases}$$

Now we state the propositions:

- (40) Let us consider universal classes $\mathcal{U}, \mathcal{V}$, a Set x of $\mathcal{U}$, and a Set y of $\mathcal{V}$. Then there exists a Set z of $\text{sup}(\mathcal{U}, \mathcal{V})$ such that for every object a , $a \in z$ iff $a = x$ or $a = y$.
- (41) Let us consider a Set X of $\mathcal{U}$. Then $\bigcup X$ is a Set of $\mathcal{U}$.

Let us consider a set X . Now we state the propositions:

- (42) If $\bigcup X$ is a Set of $\mathcal{U}$, then X is a Set of $\mathcal{U}$. The theorem is a consequence of (21).
- (43) If $\bigcup X$ is empty, then X is $\mathcal{U}$ -Set.
- (44) Let us consider a Class X of $\mathcal{U}$. Then $\bigcup X$ is a Class of $\mathcal{U}$. The theorem is a consequence of (43) and (21).
- (45) There exists a set X such that
 - (i) $\bigcup X$ is a Class of $\mathbf{U}_0$, and
 - (ii) X is not a Class of $\mathbf{U}_0$, and
 - (iii) X is not a Set of $\mathbf{U}_0$, and
 - (iv) X is a Set of $\mathbf{U}_1$.

The theorem is a consequence of (17).

- (46) Let us consider a Set X of $\mathcal{U}$, and a set Y . If $Y \in X$, then Y is a Set of $\mathcal{U}$.
- (47) Let us consider a Class X of $\mathcal{U}$, and a set Y . If $Y \in X$, then Y is a Set of $\mathcal{U}$.

5. U-PETIT

Let $\mathcal{U}$ be a universal class and x be a set. We say that x is $\mathcal{U}$ -petit if and only if

- (Def. 13) there exists an element u of $\mathcal{U}$ such that $u \approx x$.

Now we state the proposition:

- (48) Every element of $\mathcal{U}$ is $\mathcal{U}$ -petit.

Let us consider a set x . Now we state the propositions:

- (49) x is $\mathcal{U}$ -petit if and only if $\overline{\overline{x}} \in \overline{\overline{\mathcal{U}}}$.
- (50) $\{x\}$ is $\mathcal{U}$ -petit.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$ -element if and only if

- (Def. 14) the carrier of G is an element of $\mathcal{U}$.

Now we state the proposition:

- (51) Let us consider a group G . Suppose G is $\mathcal{U}$ -element. Then the multiplication of G is an element of $\mathcal{U}$.

Let $\mathcal{U}$ be a universal class and G be a group. We say that G is $\mathcal{U}$ -petit if and only if

(Def. 15) there exists a group H such that H is $\mathcal{U}$ -element and G and H are isomorphic.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$ -element if and only if

(Def. 16) the carrier of $\mathcal{C}$ is an element of $\mathcal{U}$ and the carrier' of $\mathcal{C}$ is an element of $\mathcal{U}$.

Now we state the propositions:

(52) Let us consider an object-category $\mathcal{C}$. Suppose $\mathcal{C}$ is $\mathcal{U}$ -element. Then

- (i) the source of $\mathcal{C}$ is an element of $\mathcal{U}$, and
- (ii) the target of $\mathcal{C}$ is an element of $\mathcal{U}$, and
- (iii) the composition of $\mathcal{C}$ is an element of $\mathcal{U}$.

(53) Let us consider elements o, m of $\mathcal{U}$. Then $\dot{\mathcal{C}}(o, m)$ is $\mathcal{U}$ -element.

Let $\mathcal{U}$ be a universal class. Note that there exists an object-category which is $\mathcal{U}$ -element.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$ -petit if and only if

(Def. 17) there exists a strict object-category $\mathcal{D}$ such that $\mathcal{D}$ is $\mathcal{U}$ -element and $\mathcal{C} \cong \mathcal{D}$.

Now we state the propositions:

(54) Let us consider an object-category $\mathcal{C}$, and an object a of $\mathcal{C}$. Then $\langle \langle \text{id}_a, \text{id}_a \rangle, \text{id}_a \rangle \in$ the composition of $\mathcal{C}$.

(55) Let us consider objects o, m . Then the composition of $\dot{\mathcal{C}}(o, m) = \{ \langle \langle m, m \rangle, m \rangle \}$.

PROOF: Set $\mathcal{C} = \dot{\mathcal{C}}(o, m)$. The composition of $\mathcal{C} \subseteq \{ \langle \langle m, m \rangle, m \rangle \}$. $\square$

(56) Let us consider objects o, m , and an object c of $\dot{\mathcal{C}}(o, m)$. Then $c = o$.

(57) Let us consider objects o, m , and an element c of $\dot{\mathcal{C}}(o, m)$. Then

- (i) c is an object of $\dot{\mathcal{C}}(o, m)$, and
- (ii) $c = o$, and
- (iii) $\text{id}_c = m$.

(58) Let us consider objects o_1, o_2, m_1, m_2 . Then $\dot{\mathcal{C}}(o_1, m_1) \cong \dot{\mathcal{C}}(o_2, m_2)$. The theorem is a consequence of (57).

(59) Let us consider objects o, m . Then $\dot{\mathcal{C}}(o, m)$ is $\mathcal{U}$ -petit. The theorem is a consequence of (53) and (58).

Let $\mathcal{U}$ be a universal class. Let us note that there exists an object-category which is $\mathcal{U}$ -petit. Now we state the propositions:

(60) There exists a $\mathcal{U}$ -petit object-category $\mathcal{C}$ such that

- (i) the carrier of $\mathcal{C}$ is not an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(61) There exists a $\mathcal{U}$ -petit object-category $\mathcal{C}$ such that

(i) the carrier of $\mathcal{C}$ is not an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(62) There exists a $\mathcal{U}$ -petit object-category $\mathcal{C}$ such that

(i) the carrier of $\mathcal{C}$ is an element of $\mathcal{U}$, and

(ii) the carrier' of $\mathcal{C}$ is not an element of $\mathcal{U}$.

The theorem is a consequence of (59) and (18).

(63) There exists a $\mathcal{U}$ -petit object-category $\mathcal{C}$ such that $\mathcal{C}$ is not $\mathcal{U}$ -element.

The theorem is a consequence of (62).

(64) Let us consider a strict object-category $\mathcal{C}$. Suppose $\mathcal{C}$ is $\mathcal{U}$ -element. Then $\mathcal{C}$ is $\mathcal{U}$ -petit.

Let $\mathcal{U}$ be a universal class and $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$ -category if and only if

(Def. 18) for every objects x, y of $\mathcal{C}$, $\text{hom}(x, y)$ is $\mathcal{U}$ -petit.

Let us note that there exists an object-category which is $\mathcal{U}$ -category. Now we state the proposition:

(65) Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$, and a functor F from $\mathcal{C}$ to $\mathcal{D}$.

Then $F \subseteq (\text{the carrier' of } \mathcal{C}) \times (\text{the carrier' of } \mathcal{D})$.

Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$. Now we state the propositions:

(66) $\text{Func}(\mathcal{C}, \mathcal{D}) \subseteq 2^{\alpha \times \beta}$, where α is the carrier' of $\mathcal{C}$ and β is the carrier' of $\mathcal{D}$.

(67) $\text{NatTrans}(\mathcal{C}, \mathcal{D}) \subseteq (2^{\alpha \times \beta} \times 2^{\alpha \times \beta}) \times 2^{\gamma \times \beta}$, where α is the carrier' of $\mathcal{C}$, β is the carrier' of $\mathcal{D}$, and γ is the carrier of $\mathcal{C}$.

(68) Let us consider sets X, Y, Z . Suppose $X, Y, Z \in \mathcal{U}$.

Then $2^{(2^{X \times Y} \times 2^{X \times Y}) \times 2^{Z \times Y}} \in \mathcal{U}$.

(69) Let us consider non empty sets X, Y . Suppose Y^X is an element of $\mathcal{U}$. Then X is an element of $\mathcal{U}$. The theorem is a consequence of (5).

(70) PROP 1.1.1 A) SGA 4:

Let us consider object-categories $\mathcal{C}$, $\mathcal{D}$. Suppose $\mathcal{C}$ is $\mathcal{U}$ -element and $\mathcal{D}$ is $\mathcal{U}$ -element. Then $\text{Func}(\mathcal{D}, \mathcal{C})$ is $\mathcal{U}$ -element. The theorem is a consequence of (66) and (67).

(71) Let us consider a set c . Suppose $c \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^c}} \in \mathcal{U}$.

- (72) Let us consider cardinal numbers c_1, c_2 . Suppose $c_1, c_2 \in \overline{\overline{\mathcal{U}}}$. Then $\overline{\overline{2^{c_1 \times c_2}}} \in \mathcal{U}$.

6. CATEGORIES OF GROUPS AND UNIVERSES

Let x be an object. The functor $\text{op}_0(x)$ yielding an element of $\{x\}$ is defined by the term

- (Def. 19) x .

The functor $\text{op}_1(x)$ yielding a unary operation on $\{x\}$ is defined by the term

- (Def. 20) $x \mapsto x$.

The functor $\text{op}_2(x)$ yielding a binary operation on $\{x\}$ is defined by the term

- (Def. 21) $[\langle x, x \rangle \mapsto x]$.

Now we state the proposition:

- (73) (i) $\text{op}_0(0) = \text{op}_0$, and
 (ii) $\text{op}_1(0) = \text{op}_1$, and
 (iii) $\text{op}_2(0) = \text{op}_2$.

Let x be an object. The functor $\text{Trivial-addLoopStr}(x)$ yielding a non empty additive loop structure is defined by the term

- (Def. 22) $\langle \{x\}, \text{op}_2(x), \text{op}_0(x) \rangle$.

Now we state the propositions:

- (74) $\text{Trivial-addLoopStr} = \text{Trivial-addLoopStr}(0)$.

- (75) Let us consider an object x . Then $\text{Trivial-addLoopStr}(x)$ is a strict group.

- (76) (i) $\text{op}_0(x)$ is an element of $\mathcal{U}$, and
 (ii) $\text{op}_1(x)$ is an element of $\mathcal{U}$, and
 (iii) $\text{op}_2(x)$ is an element of $\mathcal{U}$.

- (77) $\text{comp Trivial-addLoopStr}(x)$ is an element of $\mathcal{U}$.

- (78) There exists an element y of $\mathcal{U}$ such that $\text{P}_{\text{ob}} y, \text{Trivial-addLoopStr}(x)$. The theorem is a consequence of (76) and (77).

- (79) $\bigcup$ the set of all the carrier of $\text{Trivial-addLoopStr}(x)$ where x is an element of $\mathcal{U} = \mathcal{U}$.

- (80) $\text{Trivial-addLoopStr}(x) \in \text{GroupObj}(\mathcal{U})$. The theorem is a consequence of (78).

- (81) $\text{GroupObj}(\mathcal{U}) \approx \mathcal{U}$.

PROOF: Set $G_1 = \text{GroupObj}(\mathcal{U})$. Reconsider $G_2 = G_1$ as a non empty set. Define $\mathcal{P}[\text{object}, \text{object}] \equiv \$2 \in \mathcal{U}$ and $\text{P}_{\text{ob}} \$2, \1 . For every element x of G_2 , there exists an element y of $\mathcal{U}$ such that $\mathcal{P}[x, y]$. Consider f being

a function from G_2 into $\mathcal{U}$ such that for every element x of G_2 , $\mathcal{P}[x, f(x)]$. Define $\mathcal{Q}(\text{object}) = \text{Trivial-addLoopStr}(\$1)$. For every object x such that $x \in \mathcal{U}$ holds $\mathcal{Q}(x) \in \text{GroupObj}(\mathcal{U})$. Consider g being a function from $\mathcal{U}$ into $\text{GroupObj}(\mathcal{U})$ such that for every object x such that $x \in \mathcal{U}$ holds $g(x) = \mathcal{Q}(x)$. $\square$

(82) $\text{GroupObj}(\mathcal{U})$ is not $\mathcal{U}$ -petit. The theorem is a consequence of (81).

7. OBJECT-CATEGORY REPRESENTED BY A SET

Let $\mathcal{C}$ be an object-category. The functor $\text{CatToSet}(\mathcal{C})$ yielding a set is defined by the term

(Def. 23) $\langle \text{the carrier of } \mathcal{C}, \text{the carrier' of } \mathcal{C}, \text{the source of } \mathcal{C}, \text{the target of } \mathcal{C}, \text{the composition of } \mathcal{C} \rangle$.

Let S be a quintuple set. We say that S is StrCategory -like if and only if

(Def. 24) there exist sets y_1, y_2, y_3, y_4, y_5 such that $y_1 = (S)_1$ and $y_2 = (S)_2$ and $y_3 = (S)_3$ and $y_4 = (S)_4$ and $y_5 = (S)_5$ and y_3 is a function from y_2 into y_1 and y_4 is a function from y_2 into y_1 and y_5 is a partial function from $y_2 \times y_2$ to y_2 .

Note that there exists a quintuple set which is StrCategory -like.

Let S be a StrCategory -like, quintuple set. The functor $\text{SetToCat}(S)$ yielding a strict category structure is defined by

(Def. 25) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_1$ and $y_2 = (S)_2$ and $y_3 = (S)_3$ and $y_4 = (S)_4$ and $y_5 = (S)_5$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

We say that S is category-like if and only if

(Def. 26) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_1$ and $y_2 = (S)_2$ and $y_3 = (S)_3$ and $y_4 = (S)_4$ and $y_5 = (S)_5$ and $\langle y_1, y_2, y_3, y_4, y_5 \rangle$ is an object-category.

Let us note that there exists a StrCategory -like, quintuple set which is category-like and there exists a StrCategory -like, quintuple set which is non empty and category-like.

Let S be a category-like, StrCategory -like, quintuple set. The functor $\text{Obj } S$ yielding a set is defined by the term

(Def. 27) $(S)_1$.

The functor $\text{Mor } S$ yielding a set is defined by the term

(Def. 28) $(S)_2$.

We say that S is non-empty if and only if

(Def. 29) $\text{Obj } S$ is not empty.

Note that there exists a category-like, StrCategory -like, quintuple set which is non-empty.

A CategorySet is a non-empty, category-like, StrCategory -like, quintuple set. Now we state the proposition:

(83) Every CategorySet is not empty.

Note that every CategorySet is non empty.

Let S be a CategorySet . The functor $\text{SetToCat}(S)$ yielding a strict object-category is defined by

(Def. 30) there exist sets y_1, y_2 and there exist functions y_3, y_4 from y_2 into y_1 and there exists a partial function y_5 from $y_2 \times y_2$ to y_2 such that $y_1 = (S)_1$ and $y_2 = (S)_2$ and $y_3 = (S)_3$ and $y_4 = (S)_4$ and $y_5 = (S)_5$ and $it = \langle y_1, y_2, y_3, y_4, y_5 \rangle$.

Let $\mathcal{C}$ be a strict object-category. Let us observe that the functor $\text{CatToSet}(\mathcal{C})$ yields a CategorySet . Now we state the propositions:

(84) Let us consider a $\text{CategorySet } S$. Then $\text{CatToSet}(\text{SetToCat}(S)) = S$.

(85) Let us consider a strict object-category $\mathcal{C}$.

Then $\text{SetToCat}(\text{CatToSet}(\mathcal{C})) = \mathcal{C}$.

(86) Let us consider an object-category $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{U}$ -element if and only if $\text{CatToSet}(\mathcal{C})$ is $\mathcal{U}$ -Set. The theorem is a consequence of (52), (13), and (14).

(87) Let us consider a $\text{CategorySet } S$. Then S is $\mathcal{U}$ -Set if and only if $\text{SetToCat}(S)$ is $\mathcal{U}$ -element. The theorem is a consequence of (14) and (84).

Let S, T be CategorySets . We say that $S \cong T$ if and only if

(Def. 31) $\text{SetToCat}(S) \cong \text{SetToCat}(T)$.

Now we state the proposition:

(88) Let us consider strict object-categories $\mathcal{C}, \mathcal{D}$. Suppose $\mathcal{C} \cong \mathcal{D}$. Then $\text{CatToSet}(\mathcal{C}) \cong \text{CatToSet}(\mathcal{D})$. The theorem is a consequence of (85).

Let $\mathcal{U}$ be a universal class and S be a CategorySet . We say that S is $\mathcal{U}$ -petit if and only if

(Def. 32) there exists a $\text{CategorySet } T$ such that T is $\mathcal{U}$ -Set and $S \cong T$.

Now we state the proposition:

(89) Let us consider a strict object-category $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{U}$ -petit if and only if $\text{CatToSet}(\mathcal{C})$ is $\mathcal{U}$ -petit. The theorem is a consequence of (86), (88), (85), and (87).

Let S, T be CategorySets . The functor $\text{Func}(S, T)$ yielding a set is defined by the term

(Def. 33) $\text{Func}(\text{SetToCat}(S), \text{SetToCat}(T))$.

The functor $\text{Func}(T, S)$ yielding a CategorySet is defined by the term

(Def. 34) $\text{CatToSet}(\text{Func}(\text{SetToCat}(T), \text{SetToCat}(S)))$.

Now we state the proposition:

(90) Let us consider CategorySets S, T . Then

- (i) $\text{Obj Func}(T, S) = \text{Func}(\text{SetToCat}(S), \text{SetToCat}(T))$, and
- (ii) $\text{Mor Func}(T, S) = \text{NatTrans}(\text{SetToCat}(S), \text{SetToCat}(T))$.

(91) PROP 1.1.1 A) SGA 4:

Let us consider CategorySets S, T . Suppose S is $\mathcal{U}\text{-Set}$ and T is $\mathcal{U}\text{-Set}$. Then $\text{Func}(T, S)$ is $\mathcal{U}\text{-Set}$. The theorem is a consequence of (87), (70), and (86).

8. SMALL AND LOCALLY-SMALL CATEGORIES

Let $\mathcal{U}$ be a universal class and $\mathcal{C}$ be an object-category. We introduce the notation $\mathcal{C}$ is $\mathcal{U}$ -small as a synonym of $\mathcal{C}$ is $\mathcal{U}$ -element.

Note that there exists an object-category which is $\mathcal{U}$ -small. Now we state the propositions:

(92) Let us consider sets o, m . Suppose m is not $\mathcal{U}\text{-Set}$ or o is not $\mathcal{U}\text{-Set}$.

Then $\dot{\mathcal{C}}(o, m)$ is not $\mathcal{U}$ -small. The theorem is a consequence of (18).

(93) Let us consider objects o, m . Suppose $\dot{\mathcal{C}}(o, m)$ is $\mathcal{U}$ -small. Then

- (i) m is $\mathcal{U}\text{-Set}$, and
- (ii) o is $\mathcal{U}\text{-Set}$.

The theorem is a consequence of (92).

Let $\mathcal{U}$ be a universal class. Observe that there exists an object-category which is non $\mathcal{U}$ -small.

Let $\mathcal{C}$ be an object-category. We say that $\mathcal{C}$ is $\mathcal{U}$ -locally small if and only if

(Def. 35) for every objects x, y of $\mathcal{C}$, $\text{hom}(x, y)$ is $\mathcal{U}\text{-Set}$.

One can verify that there exists an object-category which is $\mathcal{U}$ -locally small and there exists a non void, non empty object-category which is $\mathcal{U}$ -locally small.

Now we state the propositions:

(94) Every $\mathcal{U}$ -small object-category is $\mathcal{U}$ -locally small.

(95) Let us consider an object o . Then $\dot{\mathcal{C}}(o, \mathcal{U})$ is not $\mathcal{U}$ -locally small.

PROOF: Set $\mathcal{C} = \dot{\mathcal{C}}(o', \mathcal{U})$. $\mathcal{C}$ is not $\mathcal{U}$ -locally small. $\square$

Let $\mathcal{U}$ be a universal class. Let us note that there exists an object-category which is non $\mathcal{U}$ -locally small.

Let us consider a $\mathcal{U}$ -locally small object-category $\mathcal{C}$. Now we state the propositions:

- (96) Suppose the carrier of $\mathcal{C}$ is $\mathcal{U}$ -Set. Then $\bigcup$ the set of all $\text{hom}(a, b)$ where a, b are objects of $\mathcal{C}$ is an element of $\mathcal{U}$.

PROOF: Define $\mathcal{P}[\text{object of } \mathcal{C}, \text{element of } \mathcal{U}] \equiv \bigcup$ the set of all $\text{hom}(\$_1, b)$ where b is an object of $\mathcal{C} = \$_2$. Consider f being a function from the carrier of $\mathcal{C}$ into $\mathcal{U}$ such that for every element x of the carrier of $\mathcal{C}$, $\mathcal{P}[x, f(x)]$. For every object x such that $x \in \text{dom } f$ holds $f(x) \in \mathcal{U}$. $\square$

- (97) If the carrier of $\mathcal{C}$ is $\mathcal{U}$ -Set, then $\mathcal{C}$ is $\mathcal{U}$ -small. The theorem is a consequence of (96).

- (98) Let us consider $\mathcal{U}$ -small object-categories $\mathcal{C}, \mathcal{D}$. Then

- (i) $\text{Functors}(\mathcal{D}, \mathcal{C})$ is $\mathcal{U}$ -small, and
- (ii) $\text{NatTrans}(\mathcal{C}, \mathcal{D})$ is $\mathcal{U}$ -Set.

The theorem is a consequence of (70).

- (99) Let us consider a $\mathcal{U}$ -small object-category $\mathcal{C}$. Then $\mathcal{C}^{\text{op}}$ is a $\mathcal{U}$ -small object-category.

- (100) Let us consider a $\mathcal{U}$ -locally small object-category $\mathcal{C}$. Then $\mathcal{C}^{\text{op}}$ is a $\mathcal{U}$ -locally small object-category.

9. EXAMPLES

Let X be a set. Observe that the functor id_X yields an element of X^X . Now we state the propositions:

- (101) $\text{Funcs}\mathcal{U} \subset \mathcal{U}$. The theorem is a consequence of [5, (81)].

- (102) $\text{Funcs}\mathcal{U}$ is $\mathcal{U}$ -Class. The theorem is a consequence of (101).

- (103) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\})$ is not an element of $\mathcal{U}$.

- (104) (i) $\pi_1(\text{Maps}\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$, and

- (ii) $(\mathcal{U} \times \mathcal{U}) \setminus (\mathcal{U} \setminus \{\emptyset\} \times \{\emptyset\}) \subseteq \pi_1(\text{Maps}\mathcal{U})$, and

- (iii) $\pi_2(\text{Maps}\mathcal{U}) = \text{Funcs}\mathcal{U}$.

PROOF: $\pi_1(\text{Maps}\mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}$. $(\mathcal{U} \times \mathcal{U}) \cap \{\langle A, B \rangle, \text{ where } A, B \text{ are elements of } \mathcal{U} : \text{if } B = \emptyset, \text{ then } A = \emptyset\} \subseteq \pi_1(\text{Maps}\mathcal{U})$. $\pi_2(\text{Maps}\mathcal{U}) = \text{Funcs}\mathcal{U}$. $\square$

- (105) $\text{Maps}\mathcal{U} \subseteq \mathcal{U}$.

- (106) The carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$ -Class. The theorem is a consequence of (102), (104), and (105).

(107) (i) $\mathbf{Ens}_{\mathcal{U}}$ is a non $\mathcal{U}$ -small object-category, and

(ii) the carrier of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$ -Class, and

(iii) the carrier' of $\mathbf{Ens}_{\mathcal{U}}$ is $\mathcal{U}$ -Class.

The theorem is a consequence of (106).

(108) Let us consider universal classes $\mathcal{U}, \mathcal{V}$. Suppose $\mathcal{U} \in \mathcal{V}$. Then $\mathbf{Ens}_{\mathcal{U}}$ is a $\mathcal{V}$ -small object-category. The theorem is a consequence of (107).

Let G be an Abelian group. The functor $\#G$ yielding a function from (the carrier of G) $\times$ (the carrier of G) into the carrier of G is defined by the term

(Def. 36) the addition of G .

Let F be a field, o be an object, and n be a natural number. The functor $\mathbf{Mat}_F(o, n)$ yielding a non empty, non void, strict category structure is defined by the term

(Def. 37) $\langle \{o\}, \text{the carrier of } F_G^{n \times n}, ((\text{the carrier of } F_G^{n \times n}) \mapsto o), ((\text{the carrier of } F_G^{n \times n}) \mapsto o), \# F_G^{n \times n} \rangle$.

Observe that $\mathbf{Mat}_F(o, n)$ is category-like and $\mathbf{Mat}_F(o, n)$ is transitive and $\mathbf{Mat}_F(o, n)$ is associative and $\mathbf{Mat}_F(o, n)$ is reflexive.

Let K be a field. Let us observe that $\mathbf{Mat}_K(o, n)$ has identities. Now we state the proposition:

(109) Let us consider a field K , an element o of $\mathcal{U}$, and a non zero natural number n . Suppose the carrier of K is an element of $\mathcal{U}$. Then

(i) the carrier of $\mathbf{Mat}_K(o, n)$ is trivial, and

(ii) $\mathbf{Mat}_K(o, n)$ is $\mathcal{U}$ -small object-category and $\mathcal{U}$ -locally small object-category.

The theorem is a consequence of (18) and (94).

Let us consider an element o of $\mathbf{U}_0$ and a non zero natural number n . Now we state the propositions:

(110) (i) the carrier of $\mathbf{Mat}_{\mathbb{R}_F}(o, n)$ is trivial and $\mathbf{U}_0$ -Set, and

(ii) $\mathbf{Mat}_{\mathbb{R}_F}(o, n)$ is not a $\mathbf{U}_0$ -small object-category, and

(iii) $\mathbf{Mat}_{\mathbb{R}_F}(o, n)$ is not a $\mathbf{U}_0$ -locally small object-category, and

(iv) $\mathbf{Mat}_{\mathbb{R}_F}(o, n)$ is $\mathbf{U}_1$ -small object-category and $\mathbf{U}_1$ -locally small object-category.

The theorem is a consequence of (18), (6), and (109).

(111) (i) the carrier of $\mathbf{Mat}_{\mathbb{C}_F}(o, n)$ is trivial and $\mathbf{U}_0$ -Set, and

(ii) $\mathbf{Mat}_{\mathbb{C}_F}(o, n)$ is not a $\mathbf{U}_0$ -small object-category, and

(iii) $\mathbf{Mat}_{\mathbb{C}_F}(o, n)$ is not a $\mathbf{U}_0$ -locally small object-category, and

- (iv) $\mathbf{Mat}_{\mathbf{C}_F}(o, n)$ is $\mathbf{U}_1$ -small object-category and $\mathbf{U}_1$ -locally small object-category.

The theorem is a consequence of (18), (6), and (109).

REFERENCES

- [1] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos (exposés i à iv). In *Séminaire de Géométrie Algébrique du Bois Marie, 1963/64, SGA 4*, volume 269 of *Lecture Notes in Mathematics*. Springer, 1972.
- [2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.
- [3] Chad E. Brown and Karol Pąk. A tale of two set theories. In Cezary Kaliszyk, Edwin Brady, Andrea Kohlhasse, and Claudio Sacerdoti Coen, editors, *Intelligent Computer Mathematics – 12th International Conference, CICM 2019, CIIRC, Prague, Czech Republic, July 8-12, 2019, Proceedings*, volume 11617 of *Lecture Notes in Computer Science*, pages 44–60. Springer, 2019. doi:10.1007/978-3-030-23250-4_4.
- [4] Czesław Byliński. Introduction to categories and functors. *Formalized Mathematics*, 1(2):409–420, 1990.
- [5] Roland Coghetto. Non-trivial universes and sequences of universes. *Formalized Mathematics*, 30(1):53–66, 2022. doi:10.2478/forma-2022-0005.
- [6] Masaki Kashiwara and Pierre Schapira. *Categories and Sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften*. Springer, 2006. doi:10.1007/3-540-27950-4.
- [7] Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, Heidelberg, Berlin, 1971.
- [8] Karol Pąk. Grothendieck universes. *Formalized Mathematics*, 28(2):211–215, 2020. doi:10.2478/forma-2020-0018.
- [9] Marco Riccardi. Object-free definition of categories. *Formalized Mathematics*, 21(3):193–205, 2013. doi:10.2478/forma-2013-0021.
- [10] Emily Riehl. *Category Theory in Context*. Courier Dover Publications, 2017.
- [11] Zbigniew Semadeni and Antoni Wiweger. *Wstęp do teorii kategorii i funktorów*, volume 45 of *Biblioteka Matematyczna*. PWN, Warszawa, 1978.
- [12] Alfred Tarski. Über unerreichbare Kardinalzahlen. *Fundamenta Mathematicae*, 30:68–89, 1938.
- [13] Alfred Tarski. On well-ordered subsets of any set. *Fundamenta Mathematicae*, 32:176–183, 1939.
- [14] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(2):259–267, 1996.
- [15] N. H. Williams. On Grothendieck universes. *Compositio Mathematica*, 21(1):1–3, 1969.

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