FOUNDATIONS OF COMPUTING AND DECISION SCIENCES
Vol. 50 (2025) No. 1
ISSN 0867-6356
e-ISSN 2300-3405
sciendo
DOI: 10.2478/fcds-2025-0003

A New Approach to Near Approximation in Fuzzy Ideals of an MV-Algebras

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Abstract. The rough set concept was originally proposed by Pawlak as a formal tool for modeling incompleteness and imprecision in information systems. Rough set theory is an extension of set theory in which a subset of a universe is described by a pair of sets called lower and upper approximations. Near set theory was introduced by Peters as a generalization of rough set theory. In this theory, Peters depends on the features of objects to define the nearness of objects and consequently, the classification of our universal set with respect to the available information of the objects. Many sets are naturally endowed with binary operations. One concept which does this is an MV-algebra. An MV-algebra has the structure $(M, \oplus, *, 0)$, where $\oplus$ is a binary operation, $*$ is a single operation, and 0 is a constant satisfying. Indeed, an MV-algebra is an algebraic structure which models Lukasiewicz multivalued logic, and the fragment of that calculus which deals with the basic logical connectives "and", "or", and "not", but in a multivalued context. This paper concerns a relationship between near sets, fuzzy sets and MV-algebra. We define lower and upper near approximations based on fuzzy ideals in an MV -algebra. Some characterizations of the above near approximations are made and some examples are presented.

Keywords: fuzzy set, rough set, near set, approximation space, MV-algebra, ideal, fuzzy ideal.

1. Introduction and preliminaries

In the first view, the concept of MV-algebra introduced by C.C. Chang in [7, 8]. An MV-algebra is represented by $(A, \oplus, *, 0)$ with specific axioms regulating governing the operations and constants. The structure must satisfy properties like forming an abelian monoid and obeying certain equations involving the operations. The structure

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must fulfill properties such as forming an abelian monoid and connecting to specific equations related to the operations. Additional operations $\odot$, $\wedge$, and $\neg$ are defined based on the original operations. The structure $(A; \odot; 1)$ forms an abelian monoid, and $(A; \neg; \wedge; 0; 1)$ constitutes a bounded distributive lattice. The relationship $a \leq b$ is characterized by $a^* \oplus b = 1$. In addition, an ideal in an MV-algebra is a subset that meets specific following conditions:

- If $a \in I$, $y \in A$, and $b \leq a$, then $b \in I$,
- If $a, b \in I$, then $a \oplus b \in I$.

The set of all ideals is denoted by $Id(A)$. A prime ideal in A has a special property related to the operation $\wedge$, and the set of all prime ideals is denoted by $Spec(A)$.

For example, an MV-algebra with at least two ideals is the MV-algebra of truth values $[0, 1]$ equipped with the standard operations of addition and multiplication defined as follows:
Addition: $a \oplus b = \min(1, a + b)$,
Multiplication: $a * b = a \wedge b$.

Another example, an MV-algebra considers which identified two non-trivial ideals as following bullet points:

- The ideal $\{0, 1/2, 1\}$, which contains the extreme truth values 0 and 1 as well as their midpoint 1/2.
- The ideal $[0, 1]$, which is the largest ideal containing all elements in the unit interval $[0, 1]$.

These two ideals exemplify the variety of ideals that can be found within an MV-algebra, demonstrating their differences in size and structure.

Lemma 1. ([38], Theorem 2.3) *In every MV-algebra, the following properties are valid for all elements $\nu_1, \nu_2, \nu_3 \in A$:*

1. $\nu_1 \leq \nu_2$ *if and only if* $\nu_2^* \leq \nu_1^*$,
2. *If* $\nu_1 \leq \nu_2$, *then* $\nu_1 \oplus \nu_3 \leq \nu_2 \oplus \nu_3$, $\nu_1 \odot \nu_3 \leq \nu_2 \odot \nu_3$,
3. $\nu_1 \leq \nu_2$ *if and only if* $\nu_1^* \oplus \nu_2 = 1$ *if and only if* $\nu_1 \odot \nu_2^* = 0$,
4. $\nu_1, \nu_2 \leq \nu_1 \oplus \nu_2$, $\nu_1 \odot \nu_2 \leq \nu_1, \nu_2$, $\nu_1 \leq n\nu_1 = \nu_1 \oplus \nu_1 \oplus \ldots \oplus \nu_1; \nu_1^n = \nu_1 \odot \nu_1 \odot \ldots \odot \nu_1 \leq \nu_1$, $\nu_1 \oplus \nu_1^* = 1$ *and* $\nu_1 \odot \nu_1^* = 0$,
5. $\nu_1 \odot \nu_2^* \wedge \nu_2 \odot \nu_1^* = 0$,
6. $(\nu_1 \oplus \nu_2) \wedge (\nu_1 \oplus \nu_3) = \nu_1 \oplus (\nu_2 \wedge \nu_3)$.

Definition 1. An MV-algebra, denoted as $(A, \oplus, *, 0)$, consists of a set A along with a binary operation $\oplus$, a unary operation $*$, and a distinguished constant 0. This structure must satisfy a set of specific equations:

1. $\nu_1 \oplus (\nu_2 \oplus \nu_3) = (\nu_1 \oplus \nu_2) \oplus \nu_3$,

2. $\nu_1 \oplus \nu_2 = \nu_2 \oplus \nu_1$,
3. $\nu_1 \oplus 0 = \nu_1$,
4. $(\nu_1^*)^* = \nu_1$,
5. $\nu_1 \oplus 0^* = 0^*$,
6. $(\nu_1^* \oplus \nu_2)^* \oplus \nu_2 = (\nu_2^* \oplus \nu_1)^* \oplus \nu_1$.

An MV-algebra A has a subalgebra S if S is a subset of A containing the zero element of A, closed under the operations of A, and with the operations of A restricted to S.

Definition 2. [9] Suppose that S and K are MV-algebras. A function $\phi : S \to K$ is called a homomorphism of MV-algebras if and only if

1. $\phi(0) = 0$,
2. $\phi(s \oplus k) = \phi(s) \oplus \phi(k)$, for every $s, k \in S$,
3. $\phi(s^*) = (\phi(s))^*$, for every $s \in S$.

Definition 3. [10] A mapping $\mu : A \to [0,1]$ represents a fuzzy set in A, and we assume that μ is a fuzzy set in A. A level subset of μ is defined as the set $\mu_t = \{\alpha \in A : \mu(\alpha) \geq t\}$, where $t \in [0,1]$.

The binary relation $\subseteq$ between fuzzy sets μ and ν in A is characterized by $\mu \subseteq \nu$ if and only if $\mu(\alpha) \leq \nu(\alpha)$ holds for all $\alpha \in A$.

Definition 4. [10] Suppose that S and K are fuzzy subsets of M.

- $S \subseteq K$ if and only if $\mu_S(a) \leqslant \mu_K(a)$, for all $a \in M$.
- $S = K$ if and only if $\mu_S(a) = \mu_K(a)$, for all $a \in M$.
- $H = S \cup K$ if and only if $\mu_H(a) = \max\{\mu_S(a), \mu_K(a)\}$, for all $a \in M$.
- $L = S \cap K$ if and only if $\mu_L(a) = \min\{\mu_S(a), \mu_K(a)\}$, for all $a \in M$.

The notation S^c is used to represent the complement of set S, defined as

$$\mu_{S^c}(\alpha) = 1 - \mu_S(a), \text{for all } a \in M.$$

Definition 5. [10] Suppose that S and K are two sets, where μ is a fuzzy subset of S, μ' is a fuzzy subset of K and $\varphi : S \to K$ be a function. The image of μ under φ, represented as $\varphi(\mu)$, defines a fuzzy set in K, as follows:

$$\varphi(\mu)(k) = \sup_{s \in \varphi^{-1}(k)} \mu(s), \text{ if } \varphi^{-1}(k) \neq \emptyset,$$
$$\varphi(\mu)(k) = 0, \text{ if } \varphi^{-1}(k) = \emptyset.$$

The fuzzy set $\varphi^{-1}(\mu')$, representing the preimage of μ' under the function φ, is defined on the set S as follows: for every element $s \in S$, the membership value of $\varphi^{-1}(\mu')$ at s is equal to the membership value of μ' at $\varphi(s)$. On the other word $\varphi^{-1}(\mu')(s) = \mu'(\varphi(s))$, for every element $s \in S$.

Definition 6. Suppose that A is an MV-algebra. A fuzzy set μ in A is a fuzzy ideal of A, if the following conditions hold:

- $\mu(0) \geq \mu(s)$, for all $s \in A$,
- $\mu(r) \geq \mu(s) \wedge \mu(r \odot s^*)$, for every $s, r \in A$.

Theorem 1. *Suppose that M is an MV-algebra and μ is a fuzzy ideal of M if and only if for every $t \in [0,1]$, μ_t $(\neq \emptyset)$ is an ideal of M.*

Proof. It is straightforward. □

Theorem 2. ([16] Theorem 3.8) *Suppose that A is an MV-algebra that is both a fuzzy Boolean and a fuzzy prime ideal. So, the following items satisfy for all $r, s, t \in A$:*

1. $\mu(r \vee s) = \mu(r) \wedge \mu(s)$,
2. $\mu(r \wedge s) = \mu(r) \vee \mu(s)$,
3. $\mu(r) \wedge \mu(r^*) = \mu(1)$,
4. $\mu(r) \vee \mu(r^*) = \mu(0)$,
5. $\mu(r \odot s^*) \geq \mu(r \odot t^*) \wedge \mu(t \odot s^*)$,
6. *If* $\mu(r \odot s^*) = \mu(0)$, *then* $\mu(r) \geq \mu(s)$.

Example 1. Let $A = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\}$ and we have the ideal sets $I = \{0, \alpha_1\}$, $J = \{0, \alpha_2\}$ and $K = I \oplus J = \{0, \alpha_1, \alpha_2, \alpha_3\}$. In addition, $\odot$, $\oplus$, and $*$ are defined in Tables 1, 2 and 3.

$*$	0	α_1	α_2	α_3	α_4	α_5	α_6	1
	1	α_6	α_5	α_4	α_3	α_2	α_1	0

Table 1. The unary operation * in the MV-algebra.

$\oplus$	0	α_1	α_2	α_3	α_4	α_5	α_6	1
0	0	α_1	α_2	α_3	α_4	α_5	α_6	1
α_1	α_1	α_1	α_2	α_3	α_4	α_5	1	1
α_2	α_2	α_3	α_2	α_3	α_6	1	α_6	1
α_3	α_3	α_3	α_3	α_3	1	1	1	1
α_4	α_4	α_5	α_6	1	α_4	α_5	α_6	1
α_5	α_5	α_5	1	1	α_5	α_5	1	1
α_6	α_6	1	α_6	1	α_6	1	α_6	1
1	1	1	1	1	1	1	1	1

Table 2. The binary operation $\oplus$ in the MV-algebra.

$\odot$	0	α_1	α_2	α_3	α_4	α_5	α_6	1
0	0	0	0	0	0	0	0	0
α_1	0	α_1	0	α_1	0	α_1	0	α_1
α_2	0	0	α_2	α_2	0	0	α_2	α_2
α_3	0	α_1	α_2	α_3	0	α_1	α_2	α_3
α_4	0	0	0	0	α_4	α_4	α_4	α_4
α_5	0	α_1	0	α_1	α_4	α_5	α_4	α_5
α_6	0	0	α_2	α_3	α_4	α_5	α_6	α_6
1	0	α_1	α_2	α_3	α_4	α_5	α_6	1

Table 3. The binary operation $\odot$ in the MV-algebra.

Now, we introduce Fuzzy ideals of MV-algebra A such that $\mu_1, \mu_2, \mu_3 : A \to [0,1]$ and for every $a \in A$ as follow:

$$\mu_1(a) = \begin{cases} 1/5 & \text{if } a = 0, \alpha_1 \\ 1/6 & \text{if } a = \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1 \end{cases}$$

$$\mu_2(a) = \begin{cases} 2/3 & \text{if } a = 0, \alpha_2 \\ 1/6 & \text{if } a = \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1 \end{cases}$$

$$\mu_3(a) = \begin{cases} 2/3 & \text{if } a = 0, \alpha_1, \alpha_2, \alpha_3 \\ 1/2 & \text{if } a = \alpha_4, \alpha_5, \alpha_6, 1 \end{cases}$$

According to Theorem 1 μ_1, μ_2, μ_3 are fuzzy ideals of A for every $t \in [0,1]$ and then $(\mu_1)_t = \{0, \alpha_1\}$, $(\mu_2)_t = \{0, \alpha_2\}$ and $(\mu_3)_t = \{0, \alpha_1, \alpha_2, \alpha_3\}$ are ideals of A for every $t \in [0,1]$.

2. Approximation spaces

The rough set concept was originally proposed by Pawlak [30, 31]. The initial step typically involves considering a universal set U along with an equivalence relation h defined on U. The equivalence class of an element $\alpha \in U$, denoted by $[\alpha]_h$, includes all elements in U that are related to α. The set of all equivalence classes generated by the relation h on U is represented as U/h. Additionally, the set that complements ρ within U is denoted by ρ^c, which corresponds to $U \setminus \rho$, for any subset ρ of U. In the context of rough set theory, an approximation space is defined by a pair $(U; h)$, where U is a non-empty set and h is an equivalence relation on U. This framework implies that our understanding of the elements in U is limited to their association with the equivalence class h, and our comprehension of a subset ρ of U is restricted to the equivalence class h and its combinations. For more details we refer to [12, 20, 44, 45, 48].

Definition 7. A rough approximation is a function $\langle U, h, \varphi \rangle : \mathcal{P}(U) \to \mathcal{P}(U) \times \mathcal{P}(U)$, with considering an approximation space (U, h), that assigns both a lower rough approximation and an upper rough approximation to each subset ρ of U. The lower rough approximation of ρ within (U, h) is represented as $\langle \underline{U}, h, \rho\rangle$ and includes elements in U that are linked to α in ρ. The upper rough approximation of ρ in (U, h) is denoted by $\langle \overline{U}, h, \rho\rangle$ and comprises elements in U that are related to α but not part of ρ.

Moreover, for $\rho \in \mathcal{P}(U)$, we define $\langle U, h, \rho\rangle = \big(\langle \overline{U}, h, \rho\rangle, \langle \underline{U}, h, \rho\rangle\big)$, where

$$\langle U, h, \rho\rangle = \{\alpha \in U \mid [\alpha]_h \subseteq \rho\}, \quad \langle U, h, \rho\rangle = \{\alpha \in U \mid [\alpha]_h \cap \rho \neq \varnothing\}.$$

In [2], the authors studied novel rough-approximation operators inspired by an abstract structure called "supra-topology". This approach is more relaxed than topological ones and extends the scope of applications because an intersection condition of topology is dispensed. In [3], the authors focused on rough approximation operators generated from infra-topology spaces and examine their features. They investigated how infra-topology spaces are constructed from N_j-neighborhood systems under an arbitrary relation. Then, they exploited these infra-topology spaces to form new rough set models and scrutinize their master characterizations. In [1], Al-Shami familiarized a new class of orthopair fuzzy sets called "(2,1)-Fuzzy sets" which are good enough to control some real-life situations. He compared (2,1)-fuzzy sets with IFSs and some of their celebrated extensions. In [4], a generalized frame for orthopair fuzzy sets called "(m, n)-fuzzy sets" is introduced, which is an efficient tool to deal with issues that require different importance for the degrees of membership and non-membership and cannot be addressed by the fuzzification tools existing in the published literature. Also, also see [15, 17, 23, 26, 43].

3. Fuzzy ideals and congruence relations

In the domain of fuzzy theory as introduced by Zadeh [47], a fuzzy subset μ_1 of an MV-algebra M is commonly interpreted as a function from M to the unit interval $[0, 1]$. When μ_1 and μ_2 are two fuzzy subsets of M, their inclusion relationship, denoted as $\mu_2 \subseteq \mu_1$, is represented by $\mu_2(s) \leq \mu_1(s)$ for all $s \in M$. Moreover, the intersection of μ_1 and μ_2, denoted as $\mu_1 \cap \mu_2$, is defined as

$$(\mu_1 \cap \mu_2)(s) = \min\{\mu_1(s), \mu_2(s)\} \text{ for all } s \in M.$$

In addition to Zadeh's original definition, the field of fuzzy literature has delved into numerous alternative approaches to fuzzy set intersection. Those seeking foundational insights into fuzzy sets can turn to Zadeh's ground breaking paper [47]. Rosenfeld's work [40] served as a catalyst for the introduction of fuzzification in algebraic structures, while Liu [24] pioneered the concept and exploration of fuzzy ideals within ring theory. References [24], [25] and [27] offer further exploration on fuzzy ideals. Nevertheless, it is deemed crucial to revisit the following two definitions due to their frequent application throughout this paper.

Suppose that μ_1 and μ_2 represent two fuzzy subsets of an MV-algebra M. The sum $\mu_1 \oplus \mu_2$ is defined as

$$(\mu_1 \oplus \mu_2)(s) = \sup_{s=\vartheta_1 \oplus \vartheta_2} \{\min\{\mu_1(\vartheta_1), \mu_2(\vartheta_2)\}\} \text{ for all } s \in M.$$

This definition is based on Zadeh's extension principle [46].

A commonly recognized and easily observable assertion is the following: Suppose that μ_1 and μ_2 are fuzzy ideals of an MV-algebra M. Then, $\mu_1 \cap \mu_2$ also meets the criteria to be considered a fuzzy ideal of M.

For any fuzzy subset μ_1 of M, the set $(\mu_1)_t = \{s \in M \mid \mu_1(s) \geqslant t\}$, for every $t \in [0,1]$, is known as a t-level subset of μ_1. The notion of t-level subsets plays a crucial role in linking fuzzy sets with crisp sets. It is a firmly established fact that each fuzzy set can be distinctly represented by the ensemble of all its t-level subsets. Furthermore, a fuzzy subset μ_1 of an MV-algebra M is categorized as a fuzzy ideal of M if and only if the t-level subsets $(\mu_1)_t$, $t \in \mathrm{Im}\ell$, are conventional ideals of M.

Definition 8. For every $t \in [0,1]$, we have defined the set $U(\ell,t)$ as follows, assuming that ℓ is a fuzzy ideal of an MV-algebra M:

$$U(\ell,t) = \{(s,r) \in M \times M \mid \ell(s \oplus r) \leq t\}$$

This set is termed a t-level relation of ℓ.

h as an equivalence relation on an MV-algebra M is recognized as a congruence relation if $(s,r) \in h$ infers $(s \oplus \alpha, r \oplus \alpha \oplus s, \alpha \oplus r) \in h$ for all $\alpha \in M$. This statement is derived from Lemma 6.1 of [22]. To ensure thoroughness, we will present a proof.

Lemma 2. *Suppose that ℓ be a fuzzy ideal of an MV-algebra M, and $t \in [0,1]$. Then $U(\ell,t)$ is a congruence relation on M.*

Proof. Consider an element s in M. It follows that $\ell(s \oplus s) = \ell(0) \leq t$, implying $(s,s) \in U(\ell,t)$. For any $(s,r) \in U(\ell,t)$, we have $\ell(s \oplus r) \leq t$. Given that ℓ is a fuzzy ideal of M, it holds that $\ell(r \oplus s) \geqslant \ell(r) \wedge \ell(s)$, where $\ell(r) \wedge \ell(s) = \ell(s) \wedge \ell(r)$. Moreover, as $\ell(s \oplus r) \leq t$, we deduce $(r,s) \in U(\ell,t)$. If $(s,r) \in U(\ell,t)$ and $(r,k) \in U(\ell,t)$, then by the nature of ℓ as a fuzzy ideal of M, we find that $\ell(s \oplus k) \leq t$, thus establishing $(s,k) \in U(\ell,t)$. Consequently, $U(\ell,t)$ forms an equivalence relation on M.

Now, assuming $(s,r) \in U(\ell,t)$ and α is an arbitrary element of M, with $\ell(s \oplus r) \leq t$, $t \in [0,1]$, we observe:

$$\begin{aligned}\ell((s \oplus \alpha) \oplus (r \oplus \alpha)) &= \ell((s \oplus \alpha) \oplus (\alpha \oplus r)) \\ &= \ell(s \oplus (\alpha \oplus \alpha) \oplus r) = \ell(s \oplus 0 \oplus r) = \ell(s \oplus r) \leq t;\end{aligned}$$

thus, $(s \oplus \alpha, r \oplus \alpha \oplus s, \alpha \oplus r) \in U(\ell,t)$. Hence, $U(\ell,t)$ qualifies as a congruence relation on M, thereby completing the proof. □

We define a to be congruent to b modulo the fuzzy ideal ℓ, denoted as $a \equiv b \pmod{\ell}$, when $\ell(a \oplus b) \leq t$ for $t \in [0,1]$.

Lemma 3. *Assume that ℓ and k are fuzzy ideals of an MV-algebra M, and considering $t \in [0,1]$, it follows from these conditions that $U(\ell \cap k, t) = U(\ell,t) \cap U(k,t)$.*

Proof. The proof is easy to see. □

In an MV-algebra M, we denote the equivalence class of $U(\ell, t)$ containing α as $[\alpha]_{(\ell,t)}$.

Lemma 4. *Suppose that ℓ is a fuzzy ideal of an MV-algebra M. If $s, r \in M$ and $t \in [0,1]$, then*

1. $[s]_{(\ell,t)} \oplus [r]_{(\ell,t)} = [s \oplus r]_{(\ell,t)}$,
2. $[s^*]_{(\ell,t)} = ([s]_{(\ell,t)})^*$.

Proof. 1. If we suppose that $\nu_1 \in [s]_{(\ell,t)} \oplus [r]_{(\ell,t)}$ so we can say that there are $\nu_2 \in [s]_{(\ell,t)}$ and $\nu_3 \in [r]_{(\ell,t)}$ such that $\nu_1 = \nu_2 \oplus \nu_3$. As $(s, \nu_2) \in U(\ell, t)$ and $(r, \nu_3) \in U(\ell, t)$, by Lemma 2, we have $(s \oplus r, \nu_2 \oplus \nu_3) \in U(\ell, t)$ or $(s \oplus r, \nu_1) \in U(\ell, t)$, and then $\nu_1 \in [s \oplus r]_{(\ell,t)}$.

Conversely, suppose that there is $\nu_1 \in [s \oplus r]_{(\ell,t)}$ then $(\nu_1, s \oplus r) \in U(\ell, t)$. Therefore $(\nu_1 \oplus r, s) \in U(\ell, t)$ and so $\nu_1 \oplus r \in [s]_{(\ell,t)}$ or $\nu_1 \in [s]_{(\ell,t)} \oplus r$, which induces $\nu_1 \in [s]_{(\ell,t)} \oplus [r]_{(\ell,t)}$.

2. We have

$$\begin{aligned} \alpha \in [s^*]_{(\ell,t)} &\iff (\nu_1, s^*) \in U(\ell,t) \iff (0, s^* \oplus \nu_1) \in U(\ell,t) \\ &\iff (s, \nu_1^*) \in U(\ell,t) \iff \nu_1^* \in [s]_{(\ell,t)} \iff \nu_1 \in ([s]_{(\ell,t)})^*. \end{aligned}$$

□

Definition 9. Suppose that ℓ_1 is a fuzzy subset of R and ℓ_2 is a fuzzy subset of S. In this case, the product $\ell_1 \times \ell_2$ is a fuzzy subset of $R \times S$, such that the membership function is defined as following:

$$(\ell_1 \times \ell_2)(\alpha, \beta) = \min\{\ell_1(\alpha), \ell_2(\beta)\} \text{ for every } \alpha \in R \text{ and } \beta \in S.$$

Theorem 3. *Suppose that μ and λ are fuzzy ideals of M_1 and M_2, respectively. Then, $\mu \times \lambda$ is a fuzzy ideal of $M_1 \times M_2$.*

Proof. It is straightforward. □

4. A survey in approximations based on fuzzy ideals

Assume that ℓ is a fuzzy ideal of an MV-algebra M and $t \in [0,1]$. The relation $U(\ell, t)$ is proven to be an equivalence relation, as demonstrated in Lemma 2. Consequently, if $U = M$ and h denotes this equivalence relation, we designate the approximation space as (M, ℓ, t) instead of (U, h).

Also, assume that ℓ a fuzzy ideal of an MV-algebra M, and let $U(\ell, t)$ be a congruence relation at the t-level induced by ℓ on M, where $t \in [0,1]$. Consider a non-empty subset N of M. The lower and upper approximations of the set N concerning $U(\ell, t)$ are defined as follows.

$$\underline{U}(\ell, t, N) = \{n \in M \mid [n]_{(\ell,t)} \subseteq N\};$$
$$\overline{U}(\ell, t, N) = \{n \in M \mid [n]_{(\ell,t)} \cap N \neq \emptyset\};$$

The following proposition marks the beginning of our investigation in this paper.

Proposition 1. *For any given approximation space (M, ℓ, t) and subsets S, K of M with $t \in [0, 1]$, the following condition is satisfied:*

1. $\underline{U}(\ell, t, S) \subseteq S \subseteq \overline{U}(\ell, t, S)$;
2. $\underline{U}(\ell, t, \emptyset) = \emptyset = \overline{U}(\ell, t, \emptyset)$;
3. $\underline{U}(\ell, t, M) = M = \overline{U}(\ell, t, M)$;
4. *If* $S \subseteq K$, *then* $\underline{U}(\ell, t, S) \subseteq \underline{U}(\ell, t, K)$ *and* $\overline{U}(\ell, t, S) \subseteq \overline{U}(\ell, t, K)$;
5. $\underline{U}(\ell, t, \underline{U}(\ell, t, S)) = \underline{U}(\ell, t, S)$;
6. $\overline{U}(\ell, t, \overline{U}(\ell, t, S)) = \overline{U}(\ell, t, K)$;
7. $\overline{U}(\ell, t, \underline{U}(\ell, t, S)) = \underline{U}(\ell, t, K)$;
8. $\underline{U}(\ell, t, \overline{U}(\ell, t, S)) = \overline{U}(\ell, t, S)$;
9. $\underline{U}(\ell, t, S) = (\overline{U}(\ell, t, S^c))^c$;
10. $\overline{U}(\ell, t, S) = (\underline{U}(\ell, t, S^c))^c$;
11. $\underline{U}(\ell, t, S \cap B) = \underline{U}(\ell, t, S) \cap \underline{U}(\ell, t, K)$;
12. $\overline{U}(\ell, t, S \cap K) = \overline{U}(\ell, t, S) \cap \overline{U}(\ell, t, K)$;
13. $\underline{U}(\ell, t, S \cup K) \supseteq \underline{U}(\ell, t, S) \cup \underline{U}(\ell, t, K)$;
14. $\overline{U}(\ell, t, S \cup K) = \overline{U}(\ell, t, S) \cup \overline{U}(\ell, t, K)$;
15. $\underline{U}(\ell, t, [x]_{(\ell,t)}) = \overline{U}(\ell, t, [x]_{(\ell,t)})$ *for all* $x \in M$.

Proof. The proof follows from a related structure to the proof of Theorem 2.1 in reference [21] and Proposition 4.1 in reference [13].

□

The following properties hold for a fuzzy ideal ℓ of an MV-algebra M:

1. $\ell(m) \leq \ell(0)$ and $\ell(m) = \ell(m^*)$ for all $m \in M$, where 0 represents the additive identity of M.
2. If $\ell(m \oplus n) = \ell(0)$, then $\ell(m) = \ell(n)$ for $m, n \in M$.

Lemma 5. *Assume ℓ is a fuzzy ideal of an MV-algebra M and $t \in [0,1]$. For any $m \in M$, it holds that $m \oplus [0]_{(\ell,t)} = [m]_{(\ell,t)}$.*

Proof. Assume that $m \in M$, then we have

$$\begin{aligned} n \in m \oplus [0]_{(\ell,t)} &\iff n \oplus m \in [0]_{(\ell,t)} \iff (n \oplus m, 0) \in U(\ell,t) \\ &\iff (n,m) \in U(\ell,t) \iff n \in [m]_{(\ell,t)}. \end{aligned}$$

□

Lemma 6. *Suppose that ℓ and k are fuzzy ideals of an MV-algebra M with $k \subseteq \ell$ and $t \in [0,1]$, then we have $[r]_{(k,t)} \subseteq [r]_{(\ell,t)}$ for all $r \in M$.*

Proof. We have $s \in [r]_{(k,t)} \Rightarrow (r,s) \in U(k,t) \Rightarrow k(r \oplus s) \geq t \Rightarrow \ell(r \oplus s) \geq t \Rightarrow (r,s) \in U(\ell,t) \Rightarrow y \in [r]_{(\ell,t)}$. □

5. The study on near sets and near approximation in a fuzzy MV-algebra

Peters studied near sets in [36], where objects with affinities were considered perceptually near to each other, i.e., objects with similar descriptions to some degrees. Although any rough set has a non-empty boundary region, the near set may also be a null set. In fact, this is the main difference between a rough set and a near set. Peters and Wasilewski [37] introduced an approach to the foundations of information science considered in the context of near sets. Perceptual information systems (or, more concisely, perceptual systems) provide stepping stones leading to nearness relations, near sets and a framework for classifying perceptual objects. More recent work considers a generalized approach theory in the study of the nearness of nonempty sets that resemble each other [34, 35] and a topological framework for the study of nearness and apartness of sets (see, e.g., [33, 32]). Several authors applied the notion of near sets to algebraic structures, for example see [5, 6, 14, 18, 19, 28, 29, 41].

Now, as described in [37], near sets possess the same definitions and properties. For ease of work, we introduce and define the following symbols as follows and use them in the rest of this article.

$\mathbb{R}$: This is a collection of real numbers,

$\mathcal{O}$: This represents a set of objects for perception ,

$\alpha : \alpha \subseteq \mathcal{O}$, This serves as a representative object,

$\nu : \nu \in \mathcal{O}$, ,

$\mathbb{F}$: This constitutes a group of functions that depict the features of an object,

$(\mathcal{B}_r, t) : (\mathcal{B}_r, t) \subseteq \mathbb{F}$,

$t : t \in [0, 1]$,

Ψ : The mapping from object set $\Psi : \mathcal{O} \to \mathbb{R}^n$ describes the object explanation,

n : The length of the explanation for the object is denoted by n,

$\Psi(\nu) : \Psi(\nu) = (\psi_1(\nu), \ldots, \psi_n(\nu))$,

$\psi_i : \psi_i \in \mathcal{B},\ where\ \psi_i : \mathcal{O} \to \mathbb{R}$.

Definition 10. [37] Assuming $\langle \mathcal{O}, \mathbb{F} \rangle$ is a perceptual system where $\mathcal{O}$ is a set of objects and $\mathbb{F}$ is a set of probe functions that can be enumerated, a definitive total information system holds significant value.

Each object description includes various values of the function $\psi(\nu)$ linked to the object $\nu \in A$. An object of interest is characterized through functions $\psi_i \in (\mathcal{B}_r, t)$.

We can consider the set of functions $(\mathcal{B}_r, t) \subseteq \mathbb{F}$, where t ranges from 0 to 1, as features representing sample objects $\alpha \subseteq \mathcal{O}$.

Definition 11. Suppose that we have $\langle \mathcal{O}, \mathbb{F} \rangle$ as a perceptual system. For everyone $(\mathcal{B}_r, t) \subseteq \mathbb{F}$, $t \in [0, 1]$ the being unrecognizable relation $\sim_{(\mathcal{B}_r,t)}$ define, as follows:

$$\sim_{(\mathcal{B}_r,t)} = \{(\rho_1, \rho_2) \in \mathcal{O} \times \mathcal{O} : \forall \psi_i \in (\mathcal{B}_r, t), \psi_i(\rho_1) = \psi_i(\rho_2)\}, t \in [0, 1].$$

If $(\mathcal{B}_r, t) = \{\psi\}$, $t \in [0, 1]$, then we have $\sim_{\{\psi\}}$ according to the above definition for all $\psi \in \mathbb{F}$ but we write $\sim_\psi$ instead of it.

This relation leads to the partitioning of $\mathcal{O}$ into distinct, non-empty subsets, known as equivalence classes. This was demonstrated through $[\rho_1]_{(\mathcal{B}_r,t)}$ as $[\rho_1]_{(\mathcal{B}_r,t)} = \{\rho_2 \in \mathcal{O} : \rho_1 \sim_\psi \rho_2\}$. Within the quotient set $\mathcal{O}/\sim_\psi$, these classes exist as

$$\mathcal{O}/\sim_\psi = \{[\rho_1]_{(\mathcal{B}_r,t)} : \rho_1 \in \mathcal{O}\}, t \in [0, 1].$$

Definition 12. The set of neighborhoods associated with a single object feature is described as follows:

$$\mathcal{N}_1(\mathcal{B}_r, t) = \cup\{[\nu]_{\psi_i} : \nu \in \mathcal{O}, \psi_i \in (\mathcal{B}_r, t)\}, t \in [0, 1].$$

The set of neighborhoods due to two object features is defined as:

$$\mathcal{N}_2(\mathcal{B}_r, t) = \cup\{[\nu]_{\psi_i,\psi_j} : \nu \in \mathcal{O}, \psi_i, \psi_2 \in (\mathcal{B}_r, t), i \neq j\}, t \in [0, 1].$$

Lastly, the set of neighborhoods due to the all object features is given by:

$$\mathcal{N}_{|(\mathcal{B}_r,t)|}(\mathcal{B}_r, t) = \cup\{[\nu]_{\psi_i,\ldots,\psi_{|(\mathcal{B}_r,t)|}} : \nu \in \mathcal{O}, \psi_i \in (\mathcal{B}_r, t)\}, t \in [0, 1].$$

Definition 13. [36] Consider a pair $NAS = (\mathcal{O}, \mathbb{F}, \sim_{(\mathcal{B}_r,t)}, (\mathcal{B}_r, t), \mathcal{N}_M)$, where $t \in [0,1]$, representing a nearness approximation space. It is important to mention that we will use NAS as an abbreviation for nearness approximation space throughout this article. The NAS approximation space comprises a collection of objects $\mathcal{O}$, a set of feature functions $\mathbb{F}$ reflecting object attributes, indistinguishability relation $\sim_{(\mathcal{B}_r,t)}$ determined by $(\mathcal{B}_r, t) \subseteq (\mathcal{B}, t) \subseteq \mathbb{F}$, and a set of divisions (neighborhood sets) $\mathcal{N}_M((\mathcal{B}_r, t))$ with $t \in [0,1]$.

Lemma 7. *[36] Consider a nearness approximation space $(M, \mathbb{F})$ where t ranges from 0 to 1. It follows that for any $\nu \in M$, we have $[\nu]_{(\mathcal{B}_r,t)} = [\nu]_{\cap\mu,t}$, where μ belongs to $\mathcal{B}_r$.*

Definition 14. Suppose that there is a NAS with form of $(M, \mathbb{F}, \sim_{\mathcal{B}_M}, (\mathcal{B}_r, t), \mathcal{N}_M)$ with considering $t \in [0,1]$ and also ω is an arbitrary subset of M, then we define:

$$\underline{\mathcal{N}_M(\mathcal{B}_r, t)}(\omega) = \{\rho_1 \in M : [\rho_1]_{(\mathcal{B}_r,t)} \subseteq \omega\},$$
$$\overline{\mathcal{N}_M(\mathcal{B}_r, t)}(\omega) = \{\rho_1 \in M : [\rho_1]_{(\mathcal{B}_r,t)} \cap \omega \neq \emptyset\}.$$

In the context of $(M, \mathbb{F}, \sim_{(\mathcal{B}_r,t)}, (\mathcal{B}_r, t), \mathcal{N}_M)$, the terms $\underline{\mathcal{N}_M(\mathcal{B}_r, t)}(\omega)$, lower approximation of ω due to the fuzzy ideals of M under $\mathcal{N}_M(\mathcal{B}_r, t)$, and $\overline{\mathcal{N}_M(\mathcal{B}_r, t)}(\omega)$, upper approximation of ω under $\mathcal{N}_M(\mathcal{B}_r, t)$, are defined.

Proposition 2. *Given a nearness approximation space $(M, \mathbb{F})$ where $\mathcal{B} \subseteq \mathbb{F}$ and t belongs to $[0,1]$, consider a subset ϑ of M.*

1. $\underline{\mathcal{N}_M(\mathcal{B}_r, t)}(\vartheta) = \bigcup_{(\mathcal{B}_r,t)\subseteq\mathcal{B}} \underline{Apr_{(\mathcal{B}_r,t)}}(\vartheta)$,
2. $\overline{\mathcal{N}_M(\mathcal{B}_r, t)}(\vartheta) = \bigcup_{(\mathcal{B}_r,t)\subseteq\mathcal{B}} \overline{Apr_{(\mathcal{B}_r,t)}}(\vartheta)$.

Proof.

$$\begin{aligned}
(1):\ \text{For every } \nu \in \underline{\mathcal{N}_M(\mathcal{B}_r, t)}(\vartheta) &\Longleftrightarrow [\nu]_{(\mathcal{B}_r,t)} \subseteq \vartheta \text{for a } \mathcal{B}_r \in \mathcal{B} \subseteq \mathbb{F},\\
&\Longleftrightarrow \nu \in \underline{Apr_{(\mathcal{B}_r,t)}}(\vartheta) \text{for a } \mathcal{B}_r \in \mathcal{B} \subseteq \mathbb{F},\\
&\Longleftrightarrow \nu \in \bigcup_{(\mathcal{B}_r,t)\subseteq\mathcal{B}} \underline{Apr_{(\mathcal{B}_r,t)}}(\vartheta).\\
(2):\ \text{For every } \nu \in \overline{\mathcal{N}_M(\mathcal{B}_r, t)}(\vartheta) &\Longleftrightarrow [\nu]_{(\mathcal{B}_r,t)} \cap \vartheta \neq \emptyset \text{for a} \mathcal{B}_r \in \mathcal{B} \subseteq \mathbb{F},\\
&\Longleftrightarrow \nu \in \overline{Apr_{(\mathcal{B}_r,t)}}(\vartheta) \text{for a} \mathcal{B}_r \in \mathcal{B} \subseteq \mathbb{F},\\
&\Longleftrightarrow \nu \in \bigcup_{(\mathcal{B}_r,t)\subseteq\mathcal{B}} \overline{Apr_{(\mathcal{B}_r,t)}}(\vartheta).
\end{aligned}$$

□

Proposition 3. *We give that for every approximation space $(M, \mathbb{F})$ and every subsets $M_1, M_2 \subseteq M$ with considering $t \in [0,1]$*

1. $\underline{\mathcal{N}_M(\mathcal{B}, t)}(M_1) \subseteq \underline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)$ *and* $\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1) \subseteq \overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)$, *whenever* $M_1 \subseteq \beta$,

2. $\underline{\mathcal{N}_M(\mathcal{B},t)}(\emptyset) = \emptyset = \overline{\mathcal{N}_M(\mathcal{B},t)}(\emptyset)$,

3. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M) = M = \overline{\mathcal{N}_M(\mathcal{B})}(M)$,

4. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \subseteq M_1 \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$,

5. $\underline{\mathcal{N}_M(\mathcal{B},t)}\Big(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\Big) = \underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$,

6. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}\Big(\underline{\mathcal{N}_M(\mathcal{B})}(M_1)\Big)$,

7. $\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}\Big(\overline{\mathcal{N}_M(\mathcal{B})}(M_1)\Big)$,

8. $\underline{\mathcal{N}_M(\mathcal{B},t)}\Big(\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\Big) = \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$,

9. $\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1^c)^c \subseteq \underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$,

10. $\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1 \cap M_2) \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \cap \overline{\mathcal{N}_M(\mathcal{B},t)}(M_2)$,

11. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1^c)^c \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$,

12. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1 \cap M_2) = \underline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \cap \underline{\mathcal{N}_M(\mathcal{B},t)}(M_2)$,

13. $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \cup \underline{\mathcal{N}_M(\mathcal{B},t)}(M_2) \subseteq \underline{\mathcal{N}_M(\mathcal{B},t)}(M_1 \cup M_2)$,

14. $\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1 \cup M_2) = \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1) \cup \overline{\mathcal{N}_M(\mathcal{B},t)}(M_2)$.

Proof. (1), (4), (11), (12), (13) and (14): The proof results from Proposition 3 in [6] due to the fact that every ideal is a member of a normal subgroup.

(2) and (3): The proof results from Definition 14.

(5): The proof results from Proposition 5.2. in [39].

(6) and (7): The proof results by part (1).

(8): The proof results by part (1) and Proposition 5.2. in [39].

(9) and (10): The proof results from Proposition 5.2. in [39]. □

Example 2. Regarding to Examples 1, we achieve following results of this section. Let $\mathcal{B} = \{\mu_1, \mu_2, \mu_3\}$ and $\vartheta_1 = \{0, \alpha_1, \alpha_3, \alpha_5, 1\}$ and $\vartheta_2 = \{0, \alpha_1, \alpha_4, \alpha_6\}$ are sub-MV-

algebras of A. Then

$$\begin{aligned}
&N_1(\mathcal{B},1/5)=\{[\alpha]_{\mu_1,1/5},\text{for each } \alpha\in A\}=\{[0]_{\mu_1}\},\\
&N_1(\mathcal{B},1/6)=\{[\alpha]_{\mu_1,1/6},[\alpha]_{\mu_2,1/6},\text{for each } \alpha\in A\}=\{[\alpha_2]_{\mu_1},[\alpha_1]_{\mu_2}\},\\
&N_1(\mathcal{B},2/3)=\{[\alpha]_{\mu_2,2/3},[\alpha]_{\mu_3,2/3},\text{for each } \alpha\in A\}=\{[0]_{\mu_2},[0]_{\mu_3}\},\\
&N_1(\mathcal{B},1/2)=\{[\alpha]_{\mu_3,1/2},\text{for each } \alpha\in A\}=\{[\alpha_4]_{\mu_3}\},\\
&\underline{N_1(\mathcal{B},1/5)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/5)}\subseteq\vartheta_1\}=\{0,\alpha_1\},\\
&\overline{N_1(\mathcal{B},1/5)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/5)}\cap\vartheta_1\neq\emptyset\}=\{0,\alpha_1\},\\
&\underline{N_1(\mathcal{B},1/5)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/5)}\subseteq\vartheta_2\}=\{0,\alpha_1\},\\
&\overline{N_1(\mathcal{B},1/5)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/5)}\cap\vartheta_2\neq\emptyset\}=\{0,\alpha_1\},\\
&\underline{N_1(\mathcal{B},1/6)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/6)}\subseteq\vartheta_1\}=\emptyset,\\
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/6)}\cap\vartheta_1\neq\emptyset\}=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\\
&\underline{N_1(\mathcal{B},1/6)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/6)}\subseteq\vartheta_2\}=\emptyset,\\
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/6)}\cap\vartheta_2\neq\emptyset\}=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\\
&\underline{N_1(\mathcal{B},2/3)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,2/3)}\subseteq\vartheta_1\}=\emptyset,\\
&\overline{N_1(\mathcal{B},2/3)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,2/3)}\cap\vartheta_1\neq\emptyset\}=\{0,\alpha_1,\alpha_2,\alpha_3\},\\
&\underline{N_1(\mathcal{B},2/3)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,2/3)}\subseteq\vartheta_2\}=\emptyset,\\
&\overline{N_1(\mathcal{B},2/3)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,2/3)}\cap\vartheta_2\neq\emptyset\}=\{0,\alpha_1\},\\
&\underline{N_1(\mathcal{B},1/2)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\subseteq\vartheta_1\}=\emptyset,\\
&\overline{N_1(\mathcal{B},1/2)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\cap\vartheta_1\neq\emptyset\}=\{\alpha_5,1\},\\
&\underline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\subseteq\vartheta_2\}=\emptyset,\\
&\overline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\cap\vartheta_2\neq\emptyset\}=\{\alpha_4,\alpha_5\},
\end{aligned}$$

Also, we obtain $\vartheta_1\cap\vartheta_2=\{0,\alpha_1\}$, $\vartheta_1\cup\vartheta_2=\{0,\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\}$,

$$\underline{N_1(\mathcal{B},1/5)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/5)} \subseteq \vartheta_1 \cap \vartheta_2\} = \{0,\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/5)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/5)} \cap \vartheta_1 \cap \vartheta_2 \neq \emptyset\} = \{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/5)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/5)} \subseteq \vartheta_1 \cup \vartheta_2\} = \{0,\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/5)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/5)} \cap \vartheta_1 \cup \vartheta_2 \neq \emptyset\} = \{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/6)} \subseteq \vartheta_1 \cap \vartheta_2\} = \emptyset,$$
$$\overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/6)} \cap \vartheta_1 \cap \vartheta_2 \neq \emptyset\} = \{\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/6)} \subseteq \vartheta_1 \cup \vartheta_2\} = \emptyset,$$
$$\overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/6)} \cap \vartheta_1 \cup \vartheta_2 \neq \emptyset\} = \{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},$$
$$\underline{N_1(\mathcal{B},2/3)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,2/3)} \subseteq \vartheta_1 \cap \vartheta_2\} = \emptyset,$$
$$\overline{N_1(\mathcal{B},2/3)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,2/3)} \cap \vartheta_1 \cap \vartheta_2 \neq \emptyset\} = \{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},2/3)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,2/3)} \subseteq \vartheta_1 \cup \vartheta_2\} = \emptyset,$$
$$\overline{N_1(\mathcal{B},2/3)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,2/3)} \cap \vartheta_1 \cup \vartheta_2 \neq \emptyset\} = \{0,\alpha_1,\alpha_3\},$$
$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \subseteq \vartheta_1 \cap \vartheta_2\} = \emptyset,$$
$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cap \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \cap \vartheta_1 \cap \vartheta_2 \neq \emptyset\} = \emptyset,$$
$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \subseteq \vartheta_1 \cup \vartheta_2\} = \{\alpha_4,\alpha_5,\alpha_6,1\},$$
$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cup \vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \cap \vartheta_1 \cup \vartheta_2 \neq \emptyset\} = \{\alpha_4,\alpha_5,\alpha_6,1\},$$
$$\underline{N_1(\mathcal{B},1/5)}(\vartheta_1) \cap \underline{N_1(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},\ \underline{N_1(\mathcal{B},1/5)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/5)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},\ \overline{N_1(\mathcal{B},1/5)}(\vartheta_1) \cup \overline{N_1(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cap \underline{N_1(\mathcal{B},1/6)}(\vartheta_2) = \emptyset,\ \underline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},1/6)}(\vartheta_2) = \emptyset,$$
$$\overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},$$
$$\overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cup \overline{N_1(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},$$
$$\underline{N_1(\mathcal{B},2/3)}(\vartheta_1) \cap \underline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \emptyset,\ \underline{N_1(\mathcal{B},2/3)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \emptyset,$$
$$\overline{N_1(\mathcal{B},2/3)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_1\},$$
$$\overline{N_1(\mathcal{B},2/3)}(\vartheta_1) \cup \overline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_1,\alpha_2,\alpha_3\},$$
$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cap \underline{N_1(\mathcal{B},1/2)}(\vartheta_2) = \emptyset,\ \underline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},1/2)}(\vartheta_2) = \emptyset,$$
$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/2)}(\vartheta_2) = \{\alpha_5\},\ \overline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cup \overline{N_1(\mathcal{B},1/2)}(\vartheta_2) = \{\alpha_4,\alpha_5,1\},$$

With these implied information, we recognize that all items of Proposition 3 are satisfy such as the following results. Also, we show that the converses of some parts of Proposition 3 cannot be satisfied:

$(10): \ \overline{N_1(\mathcal{B},1/5)}(\vartheta_1 \cap \vartheta_2) \subseteq \overline{N_1(\mathcal{B},1/5)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/5)}(\vartheta_2) \Longrightarrow \{0,\alpha_1\} \subseteq \{0,\alpha_1\},$

$(10): \ \overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cap \vartheta_2) \subseteq \overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/6)}(\vartheta_2) \Longrightarrow$
$\{\alpha_1\} \subseteq \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\}, \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\} \nsubseteq \{\alpha_1\},$

$(12): \ \underline{N_1(\mathcal{B},2/3)}(\vartheta_1 \cap \vartheta_2) = \underline{N_1(\mathcal{B},2/3)}(\vartheta_1) \cap \underline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \emptyset,$

$(13): \ \underline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},1/2)}(\vartheta_2) \subseteq \underline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cup \vartheta_2) \Longrightarrow$
$\emptyset \subseteq \{\alpha_4,\alpha_5,\alpha_6,1\}, \ \{\alpha_4,\alpha_5,\alpha_6,1\} \nsubseteq \emptyset$

$(13): \ \overline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cup \vartheta_2) \nsubseteq \overline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cup \overline{N_1(\mathcal{B},1/2)}(\vartheta_2) \Longrightarrow \{\alpha_4,\alpha_5,\alpha_6,1\} \nsubseteq \emptyset.$

$(10): \ \overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \cap \overline{N_1(\mathcal{B},1/6)}(\vartheta_2) \nsubseteq \overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \cap \vartheta_2)$
$\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\} \nsubseteq \emptyset.$

$(13): \ \underline{N_1(\mathcal{B},1/2)}(\vartheta_1 \cup \vartheta_2) \nsubseteq \underline{N_1(\mathcal{B},1/2)}(\vartheta_1) \cup \underline{N_1(\mathcal{B},1/2)}(\vartheta_2) \Longrightarrow \{\alpha_4,\alpha_5,\alpha_6,1\} \nsubseteq \emptyset.$

In this part of the example by Examples 1 and 2, we achieve for $r = 2$ and $r = 3$ of an NAS (or nearness approximation space):

$$\mu_1 \cap \mu_2(a) = \begin{cases} 1/5 & \text{if } a = 0 \\ 1/6 & \text{if } a = \alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1 \end{cases}$$

$$\mu_2 \cap \mu_3(a) = \begin{cases} 2/3 & \text{if } a = 0,\alpha_2 \\ 1/6 & \text{if } a = \alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1 \end{cases}$$

$$\mu_3 \cap \mu_2(a) = \begin{cases} 1/5 & \text{if } a = 0,\alpha_1 \\ 1/6 & \text{if } a = \alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1 \end{cases}$$

Now, we consider $\mu_1' := \mu_1 \cap \mu_2$, $\mu_2' := \mu_2 \cap \mu_3$ and $\mu_3' := \mu_3 \cap \mu_2$. Additionally, $\mathcal{B} = \{\mu_1', \mu_2', \mu_3'\}$ is resulted and then we have:

$N_2(\mathcal{B},1/5) = \{[0]_{\mu_1'},[0]_{\mu_3'}\}, N_2(\mathcal{B},1/6) = \{[\alpha_1]_{\mu_1'},[\alpha_1]_{\mu_2'},[\alpha_2]_{\mu_3'}\}, N_2(\mathcal{B},2/3) = \{[0]_{\mu_2'}\},$
$\overline{N_2(\mathcal{B},1/5)}(\vartheta_1) = \{0,\alpha_1\}, \ \overline{N_2(\mathcal{B},1/6)}(\vartheta_1) = \{\alpha_1,\alpha_3,\alpha_5,1\}, \ \overline{N_2(\mathcal{B},2/3)}(\vartheta_1) = \{0,\alpha_2\},$
$\underline{N_2(\mathcal{B},1/5)}(\vartheta_1) = \{0,\alpha_1\}, \ \underline{N_2(\mathcal{B},1/6)}(\vartheta_1) = \emptyset, \ \underline{N_2(\mathcal{B},2/3)}(\vartheta_1) = \{0,\alpha_2\},$
$\overline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\}, \ \overline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \emptyset, \ \overline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \{0\},$
$\underline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\}, \ \underline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_4,\alpha_6\}, \ \underline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \emptyset.$
$\overline{N_2(\mathcal{B},1/5)}(\vartheta_1) \cap \overline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\}, \ \overline{N_2(\mathcal{B},1/5)}(\vartheta_1) \cup \overline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},$
$\overline{N_2(\mathcal{B},1/6)}(\vartheta_1) \cap \overline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \emptyset, \ \overline{N_2(\mathcal{B},1/6)}(\vartheta_1) \cup \overline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_3,\alpha_5,1\},$
$\overline{N_2(\mathcal{B},2/3)}(\vartheta_1) \cap \overline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \{0\}, \ \overline{N_2(\mathcal{B},2/3)}(\vartheta_1) \cup \overline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_2\},$
$\underline{N_2(\mathcal{B},1/5)}(\vartheta_1) \cap \underline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,x_1\}, \ \underline{N_2(\mathcal{B},1/5)}(\vartheta_1) \cup \underline{N_2(\mathcal{B},1/5)}(\vartheta_2) = \{0,\alpha_1\},$
$\underline{N_2(\mathcal{B},1/6)}(\vartheta_1) \cap \underline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \emptyset, \ \underline{N_2(\mathcal{B},1/6)}(\vartheta_1) \cup \underline{N_2(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_4,\alpha_6\},$
$\underline{N_2(\mathcal{B},2/3)}(\vartheta_1) \cap \underline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \emptyset, \ \underline{N_2(\mathcal{B},2/3)}(\vartheta_1) \cup \underline{N_2(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_2\},$

$$\begin{aligned}
&\vartheta_1 \cap \vartheta_2 = \{0, \alpha_1\},\ \vartheta_1 \cup \vartheta_2 = \{0, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\},\\
\text{then}\quad &\overline{N_2(\mathcal{B}, 1/5)}(\vartheta_1 \cap \vartheta_2) = \{0, \alpha_1\},\ \overline{N_2(\mathcal{B}, 1/6)}(\vartheta_1 \cap \vartheta_2) = \emptyset,\ \overline{N_2(\mathcal{B}, 2/3)}(\vartheta_1 \cap \vartheta_2) = \emptyset,\\
&\overline{N_2(\mathcal{B}, 1/5)}(\vartheta_1 \cup \vartheta_2) = \{0, \alpha_1\},\ \overline{N_2(\mathcal{B}, 1/6)}(\vartheta_1 \cup \vartheta_2) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\},\\
&\overline{N_2(\mathcal{B}, 2/3)}(\vartheta_1 \cup \vartheta_2) = \{0\},\\
&\underline{N_2(\mathcal{B}, 1/5)}(\vartheta_1 \cap \vartheta_2) = \{0, \alpha_1\},\ \underline{N_2(\mathcal{B}, 1/6)}(\vartheta_1 \cap \vartheta_2) = \emptyset,\ \underline{N_2(\mathcal{B}, 2/3)}(\vartheta_1 \cap \vartheta_2) = \emptyset,\\
&\underline{N_2(\mathcal{B}, 1/5)}(\vartheta_1 \cup \vartheta_2) = \{0, \alpha_1\},\ \underline{N_2(\mathcal{B}, 1/6)}(\vartheta_1 \cup \vartheta_2) = \emptyset,\ \underline{N_2(\mathcal{B}, 2/3)}(\vartheta_1 \cup \vartheta_2) = \emptyset.
\end{aligned}$$

$$\mu_1 \cap \mu_2 \cap \mu_3(a) = \begin{cases} 1/5 & \text{if } a = 0 \\ 1/6 & \text{if } a = \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1 \end{cases}$$

Now, we consider $\mu'' := \mu_1 \cap \mu_2 \cap \mu_3$. Additionally, $\mathcal{B} = \{\mu''\}$ is resulted and then we have:

$$\begin{aligned}
&N_3(\mathcal{B}, 1/5) = \{[0]_{\mu''}\},\ N_3(\mathcal{B}, 1/6) = \{[\alpha_1]_{\mu''}\},\\
&\overline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) = \{0\},\ \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) = \{\alpha_1, \alpha_3, \alpha_5, 1\},\\
&\underline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) = \{0\},\ \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) = \emptyset,\\
&\overline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\ \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \{\alpha_1, \alpha_4, \alpha_6\},\\
&\underline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\ \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \emptyset,\\
&\overline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) \cap \overline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\ \overline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) \cup \overline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\\
&\overline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) \cap \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \{\alpha_1\},\\
&\overline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) \cup \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\},\\
&\underline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) \cap \underline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\ \underline{N_3(\mathcal{B}, 1/5)}(\vartheta_1) \cup \underline{N_3(\mathcal{B}, 1/5)}(\vartheta_2) = \{0\},\\
&\underline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) \cap \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \emptyset,\ \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_1) \cup \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_2) = \emptyset,\\
\text{then}\quad &\overline{N_3(\mathcal{B}, 1/5)}(\vartheta_1 \cap \vartheta_2) = \{0\},\ \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_1 \cap \vartheta_2) = \{\alpha_1\},\\
&\overline{N_3(\mathcal{B}, 1/5)}(\vartheta_1 \cup \vartheta_2) = \{0\},\ \overline{N_3(\mathcal{B}, 1/6)}(\vartheta_1 \cup \vartheta_2) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, 1\},\\
&\underline{N_3(\mathcal{B}, 1/5)}(\vartheta_1 \cap \vartheta_2) = \{0\},\ \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_1 \cap \vartheta_2) = \emptyset,\\
&\underline{N_3(\mathcal{B}, 1/5)}(\vartheta_1 \cup \vartheta_2) = \{0\},\ \underline{N_3(\mathcal{B}, 1/6)}(\vartheta_1 \cup \vartheta_2) = \emptyset.
\end{aligned}$$

Again, we show that the converses of parts 13 and 10 of Proposition 3 cannot be satisfied for $r = 2$ and $r = 3$, too:

$$(10):\quad \overline{N_2(\mathcal{B}, 2/3)}(\vartheta_1) \cap \overline{N_2(\mathcal{B}, 2/3)}(\vartheta_2) \nsubseteq \overline{N_2(\mathcal{B}, 2/3)}(\vartheta_1 \cap \vartheta_2) \Longrightarrow \{0\} \nsubseteq \emptyset.$$

Proposition 4. *If $(M, \mathbb{F})$ constitutes a nearness approximation space, and M_1 and M_2 represent non-empty sub-MV-algebras of M with $t \in [0, 1]$, then the following set inclusion holds:*

$$\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1 \oplus M_2) \subseteq \Big(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)\Big) \oplus \Big(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)\Big).$$

Proof. Suppose that z is an arbitrary element of $\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1 \oplus M_2)$, implying that $[z]_{(\mathcal{B}_r, t)} \cap (M_1 \oplus M_2) \neq \emptyset$ for some $\mathcal{B}_r \subseteq \mathcal{B}$. Consequently, there exists an element

$a \in M$, such that $a \in [z]_{(\mathcal{B}_r,t)} \cap (M_1 \oplus M_2)$. This implies $a \in [z]_{(\mathcal{B}_r,t)}$ and $a \in (M_1 \oplus M_2)$, leading to $a = \nu_1 \oplus \nu_2$ with $\nu_1 \in M_1$ and $\nu_2 \in M_2$. Given that $[z]_{(\mathcal{B}_r,t)}$ is an equivalence relation and $a \in [z]_{(\mathcal{B}_r,t)}$, it follows that $z \in [a]_{(\mathcal{B}_r,t)} = [\nu_1 \oplus \nu_2]_{(\mathcal{B}_r,t)} = [\nu_1]_{(\mathcal{B}_r,t)}[\nu_2]_{(\mathcal{B}_r,t)}$. Therefore, $z = r \oplus s$ where $r \in [\nu_1]_{(\mathcal{B}_r,t)}$ and $s \in [\nu_2]_{(\mathcal{B}_r,t)}$. Hence, $\nu_1 \in [r]_{(\mathcal{B}_r,t)}$, resulting in $\nu_1 \in [r]_{(\mathcal{B}_r,t)} \cap M_1$, which further implies $r \in \overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$. Likewise, $s \in \overline{\mathcal{N}_M(\mathcal{B},t)}(M_2)$, leading to $z = r \oplus s \in \left(\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\right) \oplus \left(\overline{\mathcal{N}_M(\mathcal{B},t)}(M_2)\right)$. Thus, we deduce that $\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1 \oplus M_2) \subseteq \left(\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\right) \oplus \left(\overline{\mathcal{N}_M(\mathcal{B},t)}(M_2)\right)$. $\square$

EXAMPLE 3. Regarding to Examples 1 and 2, we have the condition of Proposition 4 as following results:

$$\begin{aligned}
&\overline{N_1(\mathcal{B},2/3)}(\vartheta_1) = \{0,\alpha_1,\alpha_2,\alpha_3\},\ \overline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_1\}\\
&\overline{N_1(\mathcal{B},2/3)}(\vartheta_1) \oplus \overline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \{0,\alpha_1,\alpha_2,\alpha_3\}\\
&\vartheta_1 \oplus \vartheta_2 = \{0,\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\ \overline{N_1(\mathcal{B},2/3)}(\vartheta_1 \oplus \vartheta_2) = \{0,\alpha_1,\alpha_2,\alpha_3\}\\
&\overline{N_1(\mathcal{B},2/3)}(\vartheta_1 \oplus \vartheta_2) \subseteq \overline{N_1(\mathcal{B},2/3)}(\vartheta_1) \oplus \overline{N_1(\mathcal{B},2/3)}(\vartheta_2)
\end{aligned}$$

Now, we show that the converse state of Proposition 4 is not satisfied and respectively for $r = 2$ and $r = 3$:

$$\begin{aligned}
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_1) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\\
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_2) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\}\\
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \oplus \overline{N_1(\mathcal{B},2/3)}(\vartheta_2) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\\
&\vartheta_1 \oplus \vartheta_2 = \{0,\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\},\ \overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \oplus \vartheta_2) = \{\alpha_1,\alpha_3,\alpha_4,\alpha_5,\alpha_6,1\}\\
&\overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \oplus \vartheta_2) \subseteq \overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \oplus \overline{N_1(\mathcal{B},1/6)}(\vartheta_2)\\
\text{but}\quad &\overline{N_1(\mathcal{B},1/6)}(\vartheta_1) \oplus \overline{N_1(\mathcal{B},1/6)}(\vartheta_2) \nsubseteq \overline{N_1(\mathcal{B},1/6)}(\vartheta_1 \oplus \vartheta_2).
\end{aligned}$$

Lemma 8. *Let $(M,\mathbb{F})$ is an approximation space of nearness and $t \in [0,1]$. Let M_1 be a non-empty sub-MV-algebra of M. Then*

$$[\eta_1]_{(\mathcal{B}_r,t)} \oplus [\eta_2]_{(\mathcal{B}_{r'},t)} = [\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'},t)}$$

where $\eta_1,\eta_2 \in M$ and $(\mathcal{B}_r,t),(\mathcal{B}_{r'},t) \subseteq \mathcal{B}$.

Proof. Assume $\nu_1 \in [\eta_1]_{(\mathcal{B}_r,t)} \oplus [\eta_2]_{(\mathcal{B}_{r'},t)}$. There exist $\nu_2 \in [\eta_1]_{(\mathcal{B}_r,t)}$ and $\nu_3 \in [\eta_2]_{(\mathcal{B}_{r'},t)}$ such that $\nu_1 = \nu_2 \oplus \nu_3$. As $(\nu_2,\eta_1) \in U(\mathcal{B}_r,t)$, $(\nu_3,\eta_2) \in U(\mathcal{B}_{r'},t)$, and $(\mathcal{B}_r,t),(\mathcal{B}_{r'},t)$ are fuzzy ideals, we have:

$$(\mathcal{B}_r,t)(\nu_2 \oplus \eta_1^*) \geq t, \quad (\mathcal{B}_{r'},t)(\nu_3 \oplus \eta_2^*) \geq t.$$

Let's say $\eta_3 = \nu_2 \oplus \eta_2$ for an element $\eta_3 \in M$. Then $\nu_2 = \eta_3 \oplus \eta_2^* \Rightarrow \nu_2^* = \eta_2 \oplus \eta_3^*$. Consequently:

$$(1)\quad (\mathcal{B}_r,t)(\eta_1 \oplus \eta_2 \oplus \eta_3^*) = (\mathcal{B}_r,t)(\eta_1 \oplus \nu_2^*) \geqslant t.$$

Due to $(\mathcal{B}_{r'}, t)$ being a fuzzy normal subgroup, we can deduce:

$$\begin{aligned}(2)\quad (\mathcal{B}_{r'}, t)(\eta_2 \oplus \nu_3^*) &= (\mathcal{B}_{r'}, t)(\nu_3^* \oplus \eta_2) = (\mathcal{B}_{r'}, t)(\nu_3^* \oplus \nu_2^* \oplus \nu_2 \oplus \eta_2)\\ &= (\mathcal{B}_{r'}, t)(\nu_2 \oplus \eta_2 \oplus (\nu_2 \oplus \nu_3)^*) = (\mathcal{B}_{r'}, t)(\eta_3 \oplus \nu_1^*) \geqslant t.\end{aligned}$$

Combining (1) and (2) yields:

$$(\mathcal{B}_r, t)(\eta_1 \oplus \eta_2 \oplus \eta_3^*) \geq t \text{ and } (\mathcal{B}_{r'}, t)(\eta_3 \oplus \nu_1^*) \geq t.$$

From the above equation, we get $(\eta_1 \oplus \eta_2, \eta_3) \in U(\mathcal{B}_r, t)$ and $(\eta_3, \nu_1) \in U(\mathcal{B}_{r'}, t)$. Thus, $(\eta_1 \oplus \eta_2, \eta_3) \in U(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)$. Hence, $\nu_1 \in [\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)}$. Therefore, $[\eta_1]_{(\mathcal{B}_r, t)} \oplus [\eta_2]_{(\mathcal{B}_{r'}, t)} \subseteq [\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)}$. On the contrary, if $\nu_1 \in [\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)}$, then $(\eta_1 \oplus \eta_2, \eta_3) \in U(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)$. Thus, there exists $\eta_3 \in M$ such that $(\eta_1 \oplus \eta_2, \eta_3) \in U(\mathcal{B}_r, t)$ and $(\eta_3, \nu_1) \in U(\mathcal{B}_{r'}, t)$. Let $\nu_2 = \eta_3 \oplus \eta_2^*$, then $\nu_2 \in M$ and $\eta_3 = \nu_2 \oplus \eta_2$. Thus, $(\mathcal{B}_r, t)(\eta_1 \oplus \eta_2 \oplus \eta_3^*) \geq t$. As $\mathcal{B}_{r'}$ is a fuzzy ideal, we infer that $(\mathcal{B}_{r'}, t)(\eta_3 \oplus \nu_1^*) = (\mathcal{B}_{r'}, t)(\nu_2 \oplus \eta_2 \oplus \nu_1^*) \geq t$. Consider $\nu_3 = \nu_2^* \oplus \nu_1$, then $\nu_3^* = \nu_1^* \oplus \nu_2 \in M$. Consequently, $(\mathcal{B}_{r'}, t)(\eta_2 \oplus \nu_3^*) \geq t$. Thus, we have $(\eta_1, \nu_2) \in U(\mathcal{B}_r, t)$ and $(\eta_2, \nu_3) \in U(\mathcal{B}_{r'}, t)$. This implies $\nu_2 \in [\eta_1]_{(\mathcal{B}_r, t)}$ and $\nu_3 \in [\eta_2]_{(\mathcal{B}_{r'}, t)}$. Therefore, $\nu_1 = \nu_2 \oplus \nu_3 \in [\eta_1]_{(\mathcal{B}_r, t)} \oplus [\eta_2]_{(\mathcal{B}_{r'}, t)}$. Therefore, $[\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)} \subseteq [\eta_1]_{(\mathcal{B}_r, t)} \oplus [\eta_2]_{(\mathcal{B}_{r'}, t)}$. □

Proposition 5. *If $(M, \mathbb{F})$ is an approximation space of nearness, and M_1 and M_2 are non-empty sub-MV-algebra of M and $t \in [0, 1]$. Then*

$$\left(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)\right) \oplus \left(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)\right) \subseteq \left(\overline{\mathcal{N}_M(\sum_{(\mathcal{B}_r, t) \subseteq (\mathcal{B}, t)} (\mathcal{B}_r, t))}\right)(M_1 \oplus M_2).$$

Proof. Let $z \in (\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)) \oplus (\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2))$. This implies that $z = \nu_1 \oplus \nu_2$, where $\nu_1 \in \overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)$ and $\nu_2 \in \overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)$. Therefore, there exist $\mathcal{B}_r, \mathcal{B}_{r'} \subseteq \mathcal{B}$ and $t \in [0, 1]$ such that $[\nu_1]_{(\mathcal{B}_r, t)} \cap M_1$ and $[\nu_2]_{(\mathcal{B}_{r'}, t)} \cap M_2$ are non-empty. Consequently, there are elements a and b in M such that $a \in [\nu_1]_{(\mathcal{B}_r, t)} \cap M_1$, $a \in M_1$, $b \in [\nu_2]_{(\mathcal{B}_{r'}, t)}$, and $b \in M_2$. By Lemma 8, we have $a \oplus b \in [\nu_1]_{(\mathcal{B}_r, t)} \oplus [\nu_2]_{(\mathcal{B}_{r'}, t)} = [\nu_1 \oplus \nu_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)}$. This implies that $a \oplus b \in (M_1 \oplus M_2)$. Therefore, $a \oplus b \in [\nu_1 \oplus \nu_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)} \cap (M_1 \oplus M_2) = [z]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'}, t)} \cap (M_1 \oplus M_2)$, and thus $z \in (\overline{\mathcal{N}_r(\mathcal{B}_r \oplus \mathcal{B}_{r'})}(M_1 \oplus M_2))$. Consequently, we have $(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)) \oplus (\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)) \subseteq (\overline{\mathcal{N}_r(\mathcal{B}_r \oplus \mathcal{B}_{r'})}(M_1 \oplus M_2))$ and

$$\left(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_1)\right) \oplus \left(\overline{\mathcal{N}_M(\mathcal{B}, t)}(M_2)\right) \subseteq \left(\overline{\mathcal{N}_M(\sum_{(\mathcal{B}_r, t) \subseteq (\mathcal{B}, t)} (\mathcal{B}_r, t))}\right)(M_1 \oplus M_2).$$

□

EXAMPLE 4. Let $(A, \oplus, *, 0)$ an MV-algebra and $A \cong S_1 \times S_1$. Let $I_1 = \{0, \alpha_1\}$ and $I_2 = \{0, \alpha_2\}$ be two ideals of A. $(\alpha_1 \oplus \alpha_1 = \alpha_1,\ \alpha_2 \oplus \alpha_2 = \alpha_2.)$

$$I_1 = \{(0, 0), (0, 1)\}, I_2 = \{(0, 0), (1, 0)\}.$$

$$A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

In addition, $\oplus$ and $*$ are defined in Tables 4 and 5.

$*$	0	α_1	α_2	1
	1	α_2	α_1	0

Table 4. The unary operation * in this MV-algebra.

$\oplus$	0	α_1	α_2	1
0	0	α_1	α_2	1
α_1	α_1	α_1	1	1
α_2	α_2	1	α_2	1
1	1	1	1	1

Table 5. The binary operation $\oplus$ in this MV-algebra.

Now, we introduce Fuzzy ideals of MV-algebra A such that $\mu_1, \mu_2 : A \to [0,1]$ and for every $a \in A$ as follow:

$$\mu_1(a) = \begin{cases} 1/2 & \text{if } a = 0, \alpha_1 \\ 1/3 & \text{if } a = \alpha_2 \end{cases}$$

$$\mu_2(a) = \begin{cases} 1/3 & \text{if } a = 0, \alpha_2 \\ 1/4 & \text{if } a = \alpha_1 \end{cases}$$

According to Theorem 1 μ_1, μ_2 are fuzzy ideals of A for every $t \in [0,1]$ and then $(\mu_1)_t = \{0, \alpha_1\}$ and $(\mu_2)_t = \{0, \alpha_2\}$ are ideals of A for every $t \in [0,1]$. Furthermore, let $\mathcal{B} = \{\mu_1, \mu_2\}$ and $\vartheta_1 = \{0, \alpha_1, \alpha_2\}$ and $\vartheta_2 = \{0\}$ are sub-MV-algebras of A. Then

$$\begin{aligned}
&N_1(\mathcal{B}, 1/2) = \{[\alpha]_{\mu_1,1/2}, \text{for each } \alpha \in A\} = \{[0]_{\mu_1}\},\\
&N_1(\mathcal{B}, 1/3) = \{[\alpha]_{\mu_1,1/3}, [\alpha]_{\mu_2,1/3}, \text{for each } \alpha \in A\} = \{[\alpha_2]_{\mu_1}, [0]_{\mu_2}\},\\
&N_1(\mathcal{B}, 1/4) = \{[\alpha]_{\mu_2,1/4}, \text{for each } \alpha \in A\} = \{[\alpha_1]_{\mu_2}\}.\\
&\underline{N_1(\mathcal{B}, 1/2)}(\vartheta_1) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \subseteq \vartheta_1\} = \{0, \alpha_1\},\\
&\overline{N_1(\mathcal{B}, 1/2)}(\vartheta_1) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \cap \vartheta_1 \neq \emptyset\} = \{0, \alpha_1\},\\
&\underline{N_1(\mathcal{B}, 1/2)}(\vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \subseteq \vartheta_2\} = \emptyset,\\
&\overline{N_1(\mathcal{B}, 1/2)}(\vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/2)} \cap \vartheta_2 \neq \emptyset\} = \{0\},\\
&\underline{N_1(\mathcal{B}, 1/3)}(\vartheta_1) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/3)} \subseteq \vartheta_1\} = \{0, \alpha_2\},\\
&\overline{N_1(\mathcal{B}, 1/3)}(\vartheta_1) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/3)} \cap \vartheta_1 \neq \emptyset\} = \{0, \alpha_2\},\\
&\underline{N_1(\mathcal{B}, 1/3)}(\vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/3)} \subseteq \vartheta_2\} = \emptyset,\\
&\overline{N_1(\mathcal{B}, 1/3)}(\vartheta_2) = \cup\{\alpha \in A : [\alpha]_{(\mathcal{B}_1,1/3)} \cap \vartheta_2 \neq \emptyset\} = \{0\},
\end{aligned}$$

$$\underline{N_1(\mathcal{B},1/4)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\subseteq\vartheta_1\}=\{\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/4)}(\vartheta_1)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\cap\vartheta_1\neq\emptyset\}=\{\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\subseteq\vartheta_2\}=\emptyset,$$
$$\overline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\cap\vartheta_2\neq\emptyset\}=\emptyset.$$

Also, we obtain $\vartheta_1\cap\vartheta_2=\{0\}$, $\vartheta_1\cup\vartheta_2=\vartheta_2=\{0,\alpha_1,\alpha_2\}$,

$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\subseteq\vartheta_1\cap\vartheta_2\}=\emptyset,$$
$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\cap\vartheta_1\cap\vartheta_2\neq\emptyset\}=\{0\},$$
$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\subseteq\vartheta_1\cup\vartheta_2\}=\{0,\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/2)}\cap\vartheta_1\cup\vartheta_2\neq\emptyset\}=\{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/3)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/3)}\subseteq\vartheta_1\cap\vartheta_2\}=\emptyset,$$
$$\overline{N_1(\mathcal{B},1/3)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/3)}\cap\vartheta_1\cap\vartheta_2\neq\emptyset\}=\{0\},$$
$$\underline{N_1(\mathcal{B},1/3)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/3)}\subseteq\vartheta_1\cup\vartheta_2\}=\{0,\alpha_2\},$$
$$\overline{N_1(\mathcal{B},1/3)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/3)}\cap\vartheta_1\cup\vartheta_2\neq\emptyset\}=\{0,\alpha_2\},$$
$$\underline{N_1(\mathcal{B},1/4)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\subseteq\vartheta_1\cap\vartheta_2\}=\emptyset,$$
$$\overline{N_1(\mathcal{B},1/4)}(\vartheta_1\cap\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\cap\vartheta_1\cap\vartheta_2\neq\emptyset\}=\emptyset,$$
$$\underline{N_1(\mathcal{B},1/4)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\subseteq\vartheta_1\cup\vartheta_2\}=\{\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/4)}(\vartheta_1\cup\vartheta_2)=\cup\{\alpha\in A:[\alpha]_{(\mathcal{B}_1,1/4)}\cap\vartheta_1\cup\vartheta_2\neq\emptyset\}=\{\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/2)}(\vartheta_1)\cap\underline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\emptyset,\ \underline{N_1(\mathcal{B},1/2)}(\vartheta_1)\cup\underline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\{0,\alpha_1\},$$

$$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1)\cap\overline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\{0\},\ \overline{N_1(\mathcal{B},1/2)}(\vartheta_1)\cup\overline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\{0,\alpha_1\},$$
$$\underline{N_1(\mathcal{B},1/3)}(\vartheta_1)\cap\underline{N_1(\mathcal{B},1/3)}(\vartheta_2)=\emptyset,\ \underline{N_1(\mathcal{B},1/3)}(\vartheta_1)\cup\underline{N_1(\mathcal{B},1/3)}(\vartheta_2)=\{0,\alpha_2\},$$
$$\overline{N_1(\mathcal{B},1/3)}(\vartheta_1)\cap\overline{N_1(\mathcal{B},1/3)}(\vartheta_2)=\{0\},\ \overline{N_1(\mathcal{B},1/3)}(\vartheta_1)\cup\overline{N_1(\mathcal{B},1/3)}(\vartheta_2)=\{0,\alpha_2\},$$
$$\underline{N_1(\mathcal{B},1/4)}(\vartheta_1)\cap\underline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\emptyset,\ \underline{N_1(\mathcal{B},1/4)}(\vartheta_1)\cup\underline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\{\alpha_1\},$$
$$\overline{N_1(\mathcal{B},1/4)}(\vartheta_1)\cap\overline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\emptyset,\ \overline{N_1(\mathcal{B},1/4)}(\vartheta_1)\cup\overline{N_1(\mathcal{B},1/4)}(\vartheta_2)=\{\alpha_1\}.$$

Now, as we want to obtain a counterexample, by this example, for the converse of Proposition 5, we then obtain as following steps:

$$\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t)=(\mu_1)_t\oplus(\mu_2)_t=\{0,\alpha_1,\alpha_2,1\}=A,\vartheta_1\oplus\vartheta_2=\{0,\alpha_1,\alpha_2\}$$

$$\Big(\overline{\mathcal{N}_1(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t))}\Big)(\vartheta_1\oplus\vartheta_2)=\overline{\mathcal{N}_1(\mathcal{B},A)}(\vartheta_1\oplus\vartheta_2)=\{0,\alpha_1,\alpha_2\},$$

$\overline{N_1(\mathcal{B},1/2)}(\vartheta_1)\oplus\overline{N_1(\mathcal{B},1/2)}(\vartheta_2)=\{0\}\oplus\{0,\alpha_1\}=\{0,\alpha_1\}$ then according to Proposition 5 we infer that the converse of it is not satisfied:

$$\Big(\overline{\mathcal{N}_M(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t))}\Big)(\vartheta_1\oplus\vartheta_2)\nsubseteq\Big(\overline{\mathcal{N}_M(\mathcal{B},t)}(\vartheta_1)\Big)\oplus\Big(\overline{\mathcal{N}_M(\mathcal{B},t)}(\vartheta_2)\Big).$$

Proposition 6. *Assuming $(M,\mathbb{F})$ is an approximation space of nearness, and M_1 and M_2 are non-empty sub-MV-algebras of M, with $t\in[0,1]$, we then have the following inclusion:*

$$\Big(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\Big)\oplus\Big(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_2)\Big)\subseteq\Big(\underline{\mathcal{N}_M(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t))}\Big)(M_1\oplus M_2).$$

Proof. Let z be an element in $(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1))\oplus(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_2))$. This means z can be expressed as $z=\nu_1\oplus\nu_2$, where ν_1 belongs to $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$ and ν_2 belongs to $\underline{\mathcal{N}_M(\mathcal{B},t)}(M_2)$. Thus, for some $\mathcal{B}_r,\mathcal{B}_{r'}\subseteq\mathcal{B}$ and $t\in[0,1]$, we have $[\nu_1]_{(\mathcal{B}_r,t)}\subseteq M_1$ and $[\nu_2]_{(\mathcal{B}_{r'},t)}\subseteq M_2$. By utilizing Lemma 8, we know that $[\nu_1]_{(\mathcal{B}_r,t)}\oplus[\nu_2]_{(\mathcal{B}_{r'},t)}=[\nu_1\oplus\nu_2]_{(\mathcal{B}_r\oplus\mathcal{B}_{r'},t)}=[z]_{(\mathcal{B}_r\oplus\mathcal{B}_{r'},t)}\cap M_1\oplus M_2$. Consequently, z must be a part of $(\underline{\mathcal{N}_r(\mathcal{B}_r\oplus\mathcal{B}_{r'})}(M_1\oplus M_2))$. Hence, we can deduce that $(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1))\oplus(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_2))\subseteq(\underline{\mathcal{N}_r(\mathcal{B}_r\oplus\mathcal{B}_{r'})}(M_1\oplus M_2))$, leading us to the following equation:

$$\Big(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\Big)\oplus\Big(\underline{\mathcal{N}_M(\mathcal{B},t)}(M_2)\Big)\subseteq\Big(\underline{\mathcal{N}_M(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t))}\Big)(M_1\oplus M_2).$$

□

Proposition 7. *Suppose that $(M,\mathbb{F})$ denotes a near approximation space and suppose $t\in[0,1]$. Consider an arbitrary non-empty subset M_1 of M, then the following inclusion holds:*

$$\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\subseteq\big(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t)\big)(M_1).$$

Proof. For any $z\in\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)$, there exists $\mathcal{B}_r\subseteq\mathcal{B}$ such that $[z]_{(\mathcal{B}_r,t)}\cap M_1\neq\emptyset$. This implies the existence of $\nu_1\in M$ where $\nu_1\in[z]_{(\mathcal{B}_r,t)}$ and $\nu_1\in M_1$. Since $z\in[\nu_1]_{(\mathcal{B}_r,t)}$ and since $\mathcal{B}_r(e)\geq\mathcal{B}_r(\nu_2)$ for all $\nu_2\in M$ where e is an identity element in M, it follows that $z\in[\nu_1]_{\sum_{\mathcal{B}\subseteq\mathcal{B}}}\mathcal{B}$. Given that $\nu_1\in M_1$, we deduce that $z\in\big(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t)\big)(M_1)$. Thus, the relationship is established as:

$$\overline{\mathcal{N}_M(\mathcal{B},t)}(M_1)\subseteq\big(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)}(\mathcal{B}_r,t)\big)(M_1).$$

□

6. Applied examples

Example 5. Fuzzy MV-Algebra Related Near Approximation in Medical Diagnosis:

Fuzzy MV-algebra provides a mathematical structure to handle uncertainty and vagueness in various fields, including medical diagnosis. This paper illustrates the application of fuzzy MV-algebra related near approximation for diagnosing a patient based on symptoms that exhibit degrees of presence.

Consider the symptom "Cough," which can be categorized into three fuzzy sets based on severity:

- Low Cough: $\mu_L(x)$ (e.g., membership degree from 0 to 0.35)
- Medium Cough: $\mu_M(x)$ (e.g., membership degree from 0.3 to 0.65)
- High Cough: $\mu_H(x)$ (e.g., membership degree from 0.6 to 1.0)

Using MV-Algebra for Near Approximation: Suppose a patient's cough severity is rated as 0.55 based on observations. This degree indicates some presence of both "Medium Cough" and "Low Cough."

Application of Near Approximation: Using the fuzzy MV-algebra principles, we can determine how close this patient's symptom profile is to the "High Cough" fuzzy set. The approximate membership can be defined as:

$$\text{Approximate Degree} = \max(\mu_M(0.55), \mu_L(0.55))$$

Evaluation: Performing the calculations might yield:

$$\mu_M(0.5) = 0.55, \quad \mu_L(0.55) = 0.35$$

Thus,

$$\text{Approximate Degree} = \max(0.55, 0.35) = 0.55$$

This result provides a measure of the patients symptoms classified as medium severity with an uncertainty quantification.

Result: Fuzzy MV-algebra related near approximation allows health care professionals to assess patient symptoms with a nuanced understanding of their severity. By modelling the uncertainty inherent in clinical observations, this approach can lead to improved diagnosis and treatment recommendations.

Example 6. Fuzzy MV-Algebra Related Near Approximation in Customer Satisfaction Assessment:

The concepts of fuzzy MV-algebra and near approximation are important tools for analyzing imprecise information. This example demonstrates their application in assessing customer satisfaction in the marketing domain.

Consider a product's customer satisfaction measured on a scale from 0 to 100. We can define three fuzzy sets based on various satisfaction levels:

- Low Satisfaction: $\mu_L(x)$ (e.g., membership degree from 0 to 35)
- Medium Satisfaction: $\mu_M(x)$ (e.g., membership degree from 30 to 65)
- High Satisfaction: $\mu_H(x)$ (e.g., membership degree from 60 to 100)

Using MV-Algebra for Near Approximation: Suppose a recent customer survey rates a particular customer's satisfaction as 55. This score indicates a degree of satisfaction that falls into the "Medium" and "High Satisfaction" categories.

Application of Near Approximation: We would like to determine how closely these customers satisfaction resembles the "High Satisfaction" fuzzy set. We can define the approximate membership degree as:

$$\text{Approximate Degree} = \max(\mu_M(55), \mu_H(55))$$

Evaluation: Assuming that:

$$\mu_M(55) = 0.8, \quad \mu_H(55) = 0.55$$

We can find the approximate membership degree:

$$\text{Approximate Degree} = \max(0.8, 0.55) = 0.8$$

This outcome indicates that the customer has a high degree of satisfaction with the product.

Result: Incorporating fuzzy MV-algebra related near approximation in customer satisfaction assessment allows marketers to capture the nuanced feelings of customers. By quantifying degrees of satisfaction, organizations can gain insights into customer experiences, improve products and services, and enhance customer retention strategies.

7. The lower and upper near sub-MV-algebras within an MV-algebra

This section presents the concepts of lower and upper near sub-MV-algebras on a near approximation space and explores several properties associated with them.

Definition 15. Consider $(M, \mathbb{F})$ as a near approximation space with $t \in [0,1]$. Suppose S is a non-empty subset of M. We define S as an upper near sub-MV-algebra of M if $\overline{\mathcal{N}_M(\mathcal{B},t)}(S)$ forms a sub-MV-algebra of M. Likewise, S is termed as a lower near sub-MV-algebra of M if $\underline{\mathcal{N}_M(\mathcal{B},t)}(S)$ is a sub-MV-algebra of M.

Proposition 8. *Suppose that $(M, \mathbb{F})$ is referred to as a near approximation space, and also suppose that S is considered a sub-MV-algebra of M such that $\overline{\mathcal{N}_M(\mathcal{B},t)}(S) = \big(\sum_{(\mathcal{B}_r,t)\subseteq(\mathcal{B},t)} (\mathcal{B}_r,t)\big)(S)$. In this context, S is classified as an upper near sub-MV-algebra of M.*

Proof. Since S is recognized as a sub-MV-algebra, it naturally includes the identity element e of M. Since $\mathcal{B}_r$ is a fuzzy ideal, it ensures $\mathcal{B}_r(e) \geq t$ for $t \in [0,1]$. Additionally, $\mathcal{B}_r(e) = \mathcal{B}_r(ee^*) \geq t$ for $t \in [0,1]$. This implies e belongs to $[e]_{(\mathcal{B}_r,t)}$, which leads to $e \in [e]_{(\mathcal{B}_r,t)} \cap S$. Consequently, $e \in \overline{\mathcal{N}_M(\mathcal{B},t)}(S)$, or the set $\overline{\mathcal{N}_M(\mathcal{B},t)}(S)$ is non-empty. Assuming η_1 and η_2 are elements of $\overline{\mathcal{N}_M(\mathcal{B},t)}(S)$, this means there are elements ν_1 and ν_2 in M such that $\nu_1 \in [\nu_1]_{(\mathcal{B}_r,t)} \cap S$ and $\nu_2 \in [\nu_2]_{(\mathcal{B}_{r'},t)} \cap S$ for certain $\mathcal{B}_r, \mathcal{B}_{r'} \subseteq \mathcal{B}$. Consequently, $\nu_1, \nu_2 \in S$, and $\nu_1 \in [\eta_1]_{(\mathcal{B}_r,t)}$, $\nu_2 \in [\eta_2]_{(\mathcal{B}_{r'},t)}$. Since S is a sub-MV-algebra of M, it can be concluded that $\nu_1 \oplus \nu_2 \in S$. Given by Lemma 1, $\nu_1 \oplus \nu_2 \in [\eta_1]_{(\mathcal{B}_r,t)} \oplus [\eta_2]_{(\mathcal{B}_{r'},t)} = [\eta_1 \oplus \eta_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'},t)}$. Hence, $\nu_1 \oplus \nu_2 \in [\eta_1 \oplus \eta_2]_{\sum_{\mathcal{B}_r \subseteq \mathcal{B}}} \mathcal{B}_r$. This implies $\nu_1 \oplus \nu_2 \in [\eta_1 \oplus \eta_2]_{\sum_{\mathcal{B}_r \subseteq \mathcal{B}}} \mathcal{B}_r \cap S$. Consequently, $\eta_1 \oplus \eta_2 \in \overline{\mathcal{N}_M(\mathcal{B},t)}(S)$. Now, let's consider s as any element of $\overline{\mathcal{N}_M(\mathcal{B},t)}(S)$. In this instance, we can find $\eta_1 \in M$ such that $\eta_1 \in [s]_{(\mathcal{B}_r,t)} \cap S$, particularly $\eta_1 \in [s]_{(\mathcal{B}_r,t)}$ and $\eta_1 \in S$. This indicates that $\mathcal{B}_r(\eta_1 \oplus s^*) = \mathcal{B}_r((\eta_1 \oplus s^*)^*) = \mathcal{B}_r(s \oplus \eta_1^*) = \mathcal{B}_r(\eta_1^* \oplus s) \geq t$. Therefore, $\eta_1^* \in [s^*]_{(\mathcal{B}_r,t)}$. Since S is a sub-MV-algebra of M, η_1^* is a member of S. Thus, $\eta_1^* \in [s^*]_{(\mathcal{B}_r,t)} \cap S$, confirming that $s^* \in \overline{\mathcal{N}_M(\mathcal{B},t)}(S)$. Ultimately, it can be established that $\overline{\mathcal{N}_M(\mathcal{B},t)}(S)$ certainly stands as a sub-MV-algebra of M. $\square$

Proposition 9. *Suppose that $(M, \mathbb{F})$ is considered as a nearness approximation space and $t \in [0,1]$. Suppose S is a sub-MV-algebra of M such that $[e]_{(\mathcal{B}_r,t)} \subseteq S$ for every $\mathcal{B}_r \subseteq \mathcal{B}$. So, it can be concluded that S qualifies as a lower near sub-MV-algebra of M.*

Proof. To demonstrate this, let's look at the case when e belongs to $[e]_{(\mathcal{B}_r,t)}$, which implies that e is contained in $[e]_{(\mathcal{B}_r,t)}$ and consequently in S. This further indicates that $e \in \underline{\mathcal{N}_M(\mathcal{B},t)}(S)$, or in other words, $\underline{\mathcal{N}_M(\mathcal{B},t)}(S)$ is non-empty. Next, if we pick arbitrary elements ν_1 and ν_2 from $\underline{\mathcal{N}_M(\mathcal{B},t)(S)}$, we observe that $[\nu_1]_{(\mathcal{B}_r,t)} \subseteq S$ and $[\nu_2]_{(\mathcal{B}_r,t)} \subseteq S$ for certain $\mathcal{B}_r \subseteq \mathcal{B}$. By applying Lemma 8, it follows that $[\nu_1 \oplus \nu_2]_{(\mathcal{B}_r \oplus \mathcal{B}_{r'},t)} = [\nu_1]_{(\mathcal{B}_r,t)} \oplus [\nu_2]_{(\mathcal{B}_{r'},t)} \subseteq S \oplus S \subseteq S$. Consequently, we can deduce that $\nu_1 \oplus \nu_2 \in \underline{\mathcal{N}_M(\mathcal{B},t)}(S)$.

Continuing with the proof, consider an element ν_1 from $\underline{\mathcal{N}_M(\mathcal{B},t)}(S)$, implying $[\nu_1]_{(\mathcal{B}_r,t)} \subseteq S$. Let η be any element of $[\nu_1^*]_{(\mathcal{B}_r,t)}$. Since $\mathcal{B}_r$ is a fuzzy ideal, it follows that $\mathcal{B}_r(\eta \oplus \nu_1) = \mathcal{B}_r(\eta \oplus \nu_1)^* = \mathcal{B}_r(\nu_1^* \oplus \eta^*) = \mathcal{B}_r(\eta^* \oplus \nu_1^*) \geq t$, indicating $\eta^* \in [\nu_1]_{(\mathcal{B}_r,t)} \subseteq S$. Considering S as a sub-MV-algebra of M, we can deduce that $\eta \in S$, leading to $[\nu_1^*]_{(\mathcal{B}_r,t)} \subseteq S$ and $\nu_1^* \in \underline{\mathcal{N}_M(\mathcal{B},t)}(S)$. Therefore, it can be inferred that $\underline{\mathcal{N}_M(\mathcal{B},t)}(S)$ qualifies as a sub-MV-algebra of M. $\square$

8. Compared approximation in a fuzzy MV-algebra

In this section, we examine the comparison between rough approximations and near approximations within the context of a fuzzy MV-algebra M. Moreover, a nested chain comprising near lower and upper approximations is established.

Proposition 10. *Suppose that $(M, \mathbb{F})$ is a nearness approximation space, and consider a non-empty subset S of M.*

If $|\mathcal{B}| = r$, afterwards $\underline{\mathcal{N}_M(\mathcal{B},t)}(S) = \underline{Apr_{(\mathcal{B},t)}}(S)$ and $\overline{\mathcal{N}_M(\mathcal{B},t)}(S) = \overline{Apr_{(\mathcal{B},t)}}(S)$. Subsequently, if $|\mathcal{B}| \geq r$, we have $\underline{Apr_{(\mathcal{B},t)}}(S) \subseteq \underline{\mathcal{N}_M(\mathcal{B},t)}(S)$ and $\overline{Apr_{(\mathcal{B},t)}}(S) \subseteq \overline{\mathcal{N}_M(\mathcal{B},t)}(S)$.

Proof. This follows from Definition 13, Lemma 7 and Definition 13. $\square$

Proposition 11. *Suppose that $(M, \mathbb{F})$ is a nearness approximation space, and consider a non-empty subset S of M. Suppose $\mathcal{B}_i$ is a subset of $\mathcal{B}_j$ for $1 \leq i \leq j \leq |\mathcal{B}|$. Consequently, it result that*

$$\underline{\mathcal{N}_i(\mathcal{B},t)}(S) \subseteq \underline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq S \subseteq \overline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq \overline{\mathcal{N}_i(\mathcal{B},t)}(S).$$

Proof. Let's consider $x \in \underline{\mathcal{N}_i(\mathcal{B},t)}(S)$. As per Definition 13, $[x]_{(\mathcal{B}_i,t)}$ is contained in S. By Lemma 7, $[x]_{(\mathcal{B}_j,t)}$ is also included in $[x]_{(\mathcal{B}_i,t)}$ for $1 \leq i \leq j$. Therefore, $[x]_{(\mathcal{B}_i,t)} \subseteq S$, which implies $x \in [x]_{(\mathcal{B}_j,t)}$. Consequently, $\underline{\mathcal{N}_i(\mathcal{B},t)}(S) \subseteq \underline{\mathcal{N}_j(\mathcal{B},t)}(S)$.

Moreover, Proposition 3 confirms that $\underline{\mathcal{N}_i(\mathcal{B},t)}(S) \subseteq \underline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq S$.

If we take $x \in \overline{\mathcal{N}_j(\mathcal{B},t)}(S)$, by Definition 13, we have $[x]_{(\mathcal{B}_j,t)} \cap S \neq \emptyset$. By Lemma 7, $[x]_{(\mathcal{B}_j,t)}$ is a subset of $[x]_{(\mathcal{B}_i,t)}$ for $1 \leq i \leq j$. This implies $[x]_{(\mathcal{B}_i,t)} \cap S \neq \emptyset$ and thus $x \in [x]_{(\mathcal{B}_j,t)}$, establishing $\overline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq \overline{\mathcal{N}_i(\mathcal{B},t)}(S)$.

From Proposition 3, we deduce that $S \subseteq \overline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq \overline{\mathcal{N}_i(\mathcal{B},t)}(S)$.

Therefore, the conclusion holds that $\underline{\mathcal{N}_i(\mathcal{B},t)}(S) \subseteq \underline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq S \subseteq \overline{\mathcal{N}_j(\mathcal{B},t)}(S) \subseteq \overline{\mathcal{N}_i(\mathcal{B},t)}(S)$. □

9. Conclusion

In this article, we explored the implications of a fuzzy MV-algebra theory within the realm of near set theory. Within a fuzzy MV-algebra, near approximations serve as generalizations of rough approximations [11]. We examined the characteristics of ideals, a fuzzy MV-sub-algebras, prime ideals, primary ideals, and homomorphisms. Ultimately, we contrasted near approximations with rough approximations in the context of a fuzzy MV-algebras, concluding that specific instances of near approximations can be identified as rough approximations.

Looking ahead, our future research will focus on lattice theory in relation to a fuzzy MV-algebras and the theory of soft sets. Additionally, we plan to conduct a similar examination of a fuzzy BCL-algebras, considering their definitions and unique properties. Furthermore, with more comprehensive studies, we aim to apply the theory of near sets to data science, investigating data transformations and analyzing the performance and behavior of data types across various fields, including medicine and beyond, as a promising avenue for future exploration in this area.

Compliance with Ethical Standards:

- The authors declare that they have no conflict of interest.
- This paper does not contain any studies with human participants or animals performed by any of the authors.
- Informed consent was obtained from all individual participants included in the study.

Author Contributions Statement: Conceptualization, F. J. and B.D.; methodology, F.J. and B.D.; formal analysis, F.J.; investigation, B.D.; resources, F.J. and

B.D.; writing an original draft of this paper, F.J.; writing a review and editing, B.D.; supervision, B.D.; project administration, Both authors have read and agreed to the published version of the manuscript.

Acknowledgments. The authors would like to thank the anonymous referees for their constructive comments.

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Received 31.05.2024, Accepted 16.12.2024