Control and Cybernetics
vol. **54** (2025) No. 2
pages: 205-221
DOI: 10.2478/candc-2025-0007

The Two-Constraint Binary Knapsack Problem's average case analysis for constraints with small, moderate and large coefficients*

by

Krzysztof Szkatuła

Systems Research Institute, Polish Academy of Sciences
ul. Newelska 6, 01-447 Warszawa, Poland
Krzysztof.Szkatula@ibspan.waw.pl

Abstract: The paper addresses the Two-Constraint Binary Knapsack Problem. It is assumed that some of the problem coefficients are the realizations of mutually independent random variables. Asymptotic probabilistic properties of selected problem characteristics are investigated for special cases of Lagrange multipliers with small, moderate and mixed values.

Keywords: combinatorial optimization, knapsack problems, probabilistic analysis, Lagrange function, Lagrange multipliers, constraint activity

1. Introduction

Let us consider a Two-Constraint Binary Knapsack Problem in the following formulation:

$$\begin{aligned} z_{OPT}(n) = \max & \sum_{i=1}^{n} c_i \cdot x_i \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{1i} \cdot x_i \leqslant b_1(n) \\ & \sum_{i=1}^{n} a_{2i} \cdot x_i \leqslant b_2(n) \\ \text{where} \quad & x_i = 0 \text{ or } 1,\ i = 1, \ldots, n. \end{aligned} \tag{1}$$

*Submitted: October 2025; Accepted: November 2025.

It is assumed that:

$$c_i > 0,\ a_{ji} > 0,\ 0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji},\ i = 1, \ldots, n,\ j = 1, 2.$$

Without restricting the generality of considerations, it may also be assumed that:

$$b_1(n) \leqslant b_2(n).$$

The goal of the assumptions that $c_i,\ a_{ji} > 0,\ 0 < b_j(n) \leqslant \sum_{i=1}^{n} a_{ji}$, $i = 1, \ldots, n,\ j = 1, 2$, is to avoid the trivial and degenerated problems. More precisely, the interpretation of the cases with $a_{ji} = 0$ or $c_i = 0$ is by far not obvious. When $b_j(n) > \sum_{i=1}^{n} a_{ji}$, then the corresponding constraint is always fulfilled and therefore it may be removed from the problem formulation, otherwise, if $b_j(n) = 0$, then (1) has only the trivial solution i.e. $x_i = 0$, $i = 1, \ldots, n$ and $z_{OPT}(n) = 0$.

Modeling different scenarios can be achieved, in particular, by using the functional dependence of the constraints' right-hand side values upon the size of the problem (n). The respective scope of possibilities encompasses situations, in which the right-hand side values are asymptotically very small, including being equal to a constant or even tending to zero, as well as very large ones, up to $n/2$, and all cases in between.

The classical one-constraint binary knapsack problem is one of the important problems in combinatorial optimization. It may be formulated as follows: Given a set of items, each with a weight and a value, determine which items to include in the collection so that the total weight is less than or equal to a given limit and the total value is as large as possible. The multi-constraint knapsack problem involves two or more constraints that may be associated with the availability of different resources, concerning such features as weight, size, financial limitations, deadlines, and many others, meant to model practical problems more accurately. A decision variable that is equal to 1 or 0 is used to model the choice or refusal regarding a definite item.

The Two-Constraint Binary Knapsack Problem is a special case of the binary multiconstraint knapsack problem, also known as m-constraint knapsack problem, see Nemhauser and Wolsey (1988) and Martello and Toth (1990), where, in general case, there is an arbitrary number m of constraints, i.e. $b_j(n)$, $j = 1, \ldots, m$. Another important special case of the multiconstraint knapsack problem is the classical (single constraint) or, in other words, Binary Knapsack Problem, which has only one constraint, i.e. $m = 1$, see Martello and Toth, (1990) or Szkatuła (2021). In the Szkatuła's papers, Szkatuła (1994, 1997), probabilistic analysis results of the different cases of the binary multiconstraint knapsack problem were presented. Moreover, the full case of the classical (single constraint) Binary Knapsack Problem was considered in the paper Szkatuła (1997).

The Multi-Constraint Knapsack Problem is well known to be $\mathcal{NP}$ hard and moreover, when $m \geqslant 2$, it is $\mathcal{NP}$ hard in the strong sense (see Garey and Johnson, 1979). This means that the Two-Constraint Binary Knapsack Problem (1) is also $\mathcal{NP}$ hard in the strong sense.

The Classical (one-constraint) Binary Knapsack Problem is a combinatorial optimization problem that is $\mathcal{NP}$ hard combinatorial optimization problem, but not in a strong sense.

The papers by Frieze and Clarke (1984), Mamer and Schilling (1990), Schilling (1990, 1994) investigated the asymptotic value of $z_{OPT}(n)$ for the random model of Multi-Constraint Knapsack Problem, where $b_j(n) = 1$, $j = 1, \ldots, m$. Papers by Szkatuła (1994, 1997) were dealing with the random model of the Multi-Constraint Knapsack Problem, where $b_j(n)$ are not restricted to be equal to 1. Papers by Meanti, Rinnooy Kan, Stougie and Vercellis (1990), Lee and Oh (1997) consider more general random models of Multi-Constraint Knapsack Problem, but only for $j = 1, 2$ some partial analytical results, describing the growth of $z_{OPT}(n)$ were obtained.

The objective of this paper is to analyze the growth of the asymptotic value of $z_{OPT}(n)$ for the class of random Two-Constraint Binary Knapsack Problems (1) with possibly full spectrum of the constraints' right-hand sides values when Lagrange multipliers have values, that range from small to moderate and mixed values. The outcome of the probabilistic analysis of this important problem may allow for describing the asymptotic behavior of $z_{OPT}(n)$ for almost all combinations of values of $b_1(n)$ and $b_2(n)$, as well as other coefficients of the problem (considered as realizations of random variables). Those results may help to better understand a number of theoretical issues, associated with Two-Constraint Binary Knapsack Problems, as well as enable the construction of more efficient algorithms for solving the practical instances of (1).

2. Definitions

For further presentation purposes, the following definitions are necessary:

Definition 1 *We denote $V_n \approx Y_n$, where $n \to \infty$, if*

$$Y_n \cdot (1 - o_n(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o_n(1))$$

when V_n, Y_n are sequences of numbers, or

$$\lim_{n\to\infty} P\{Y_n \cdot (1 - o_n(1)) \leqslant V_n \leqslant Y_n \cdot (1 + o_n(1))\} = 1$$

when V_n *is a sequence of random variables and* Y_n *is a sequence of numbers or random variables, where* $o_n(1)$ *is function fulfilling:* $o_n(1) \geq 0$ *and* $\lim_{n\to\infty} o_n(1) = 0$.

Definition 2 *We denote* $V_n \preceq Y_n (V_n \succeq W_n)$ *if*

$$V_n \leqslant (1 + o(1)) \cdot Y_n \ (V_n \geqslant (1 - o(1)) \cdot W_n)$$

when V_n, Y_n *(*W_n*) are sequences of numbers, or*

$$\lim_{n\to\infty} P\{V_n \leqslant (1 + o(1)) \cdot Y_n\} = 1 \ (\lim_{n\to\infty} P\{V_n \geqslant (1 - o(1)) \cdot W_n\} = 1)$$

when V_n *is a sequence of random variables and* Y_n *(*W_n*) is a sequence of numbers or random variables, where* $\lim_{n\to\infty} o(1) = 0$.

Definition 3 *We denote* $V_n \approxeq Y_n$ *if there exist constants* $c'' \geqslant c' > 0$ *such that*

$$c' \cdot Y_n \preceq V_n \preceq c'' \cdot Y_n$$

where Y_n, V_n *are sequences of numbers or random variables.*

The following random model of (1) will be considered in the paper:

- $n \to \infty$, $i = 1, \ldots, n$, $j = 1, 2$.
- c_i, a_{ji} are realizations of mutually independent random variables and, moreover, c_i, a_{ji} are uniformly distributed over $(0, 1]$.
- $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n + 1)$, for every $n \geqslant 1$ and all $b_j(n)$, $j = 1, 2$, are deterministic, where δ is a constant.

Under the assumptions made with regard to c_i, a_{ji} and $b_j(n)$, the following always hold

$$0 \leqslant z_{OPT}(n) \leqslant \sum_{i=1}^{n} c_i \leqslant n, \quad \delta \leqslant b_j(n) \leqslant \sum_{i=1}^{n} a_{ji} \leqslant n, \quad j = 1, 2. \tag{2}$$

Moreover, from the strong law of large numbers it follows that

$$\sum_{i=1}^{n} c_i \approx E(c_1) \cdot n = n/2, \ \sum_{i=1}^{n} a_{ji} \approx E(a_{11}) \cdot n = n/2.$$

Therefore, it is justified to enhance formula (2) in the following way:

$$0 \leqslant z_{OPT}(n) \preceq n/2, \ 0 < \delta \leqslant b_1(n) \leqslant b_2(n) \preceq n/2 \tag{3}$$

Formula (3) shows that the random model of the Two-Constraint Binary Knapsack Problem (1) is complete in the sense that nearly all possible instances of the problem are considered. In this respect the model, in which

$b_1(n) = b_2(n) = 1$, is just a very special case. Taking into account the fact that $\sum_{i=1}^{n} a_{ji} \approx n/2$, the assumption that $b_j(n) \leqslant b_j(n+1)$, $j = 1, 2$, for all $n \geqslant 1$, is quite logical.

The growth of $z_{OPT}(n)$, i.e. the value of the optimal solution of problem (1) may be influenced by the problem coefficients, namely:

$$n,\ c_i,\ a_{ji},\ b_1(n),\ b_2(n),\ \text{where } i = 1, \ldots, n,\ j = 1, 2.$$

It is assumed that c_i, a_{ji} are realizations of random variables and therefore their impact on the $z_{OPT}(n)$ growth is in this case indirect. Moreover, we have assumed that $n \to \infty$. The aim of the probabilistic analysis is to investigate asymptotic behavior of $z_{OPT}(n)$ when $n \to \infty$. The impact of the right-hand-side values - $b_1(n), b_2(n)$ - is well illustrated by the Lagrange function and the problem dual to (1), see Averbakh (1994), Meanti, Rinnooy Kan, Stougie and Vercellis (1990), Szkatuła (1994, 1997). Due to the very complicated formulas, impossible to handle efficiently in the general case, the papers by Szkatuła (1994, 1997) investigate only two important special cases of values of constraints right hand sides in the case of Multi-Constraint Knapsack Problem, with two or more constraints.

3. Lagrange and dual estimations

When the general knapsack type problem, with one or many constraints, is considered, then Lagrange function and the corresponding dual problems are very useful tools to perform various kind of analyses of the original problem, see Averbakh (1994), Meanti, Rinnooy Kan, Stougie and Vercellis (1990), Szkatuła (1994, 1997). In the particular case of the Two-Constraint Binary Knapsack Problem, Lagrange function of the problem (1) may be formulated as follows:

$$\begin{aligned} L_n(x) &= \sum_{i=1}^{n} c_i \cdot x_i + \sum_{j=1}^{2} \lambda_j \cdot \left(b_j(n) - \sum_{i=1}^{n} a_{ji} \cdot x_i \right) = \\ &= \lambda_1 \cdot b_1(n) + \lambda_2 \cdot b_2(n) + \sum_{i=1}^{n} (c_i - \lambda_1 \cdot a_{1i} - \lambda_2 \cdot a_{2i}) \cdot x_i \end{aligned}$$

where $x = [x_1, \ldots, x_n]$ and $\Lambda = [\lambda_1, \lambda_2]$ - vector of Lagrange multipliers. Moreover, let for every Λ, $\lambda_j \geq 0$, $j = 1, 2$:

$$\begin{aligned} \phi_n(\Lambda) &= \max_{x \in \{0,1\}^n} L_n(x, \Lambda) \\ &= \max_{x \in \{0,1\}^n} \left\{ \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) x_i \right\}. \end{aligned}$$

Using the following notation:

$$x_i(\Lambda) = \begin{cases} 1 & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$c_i(\Lambda) = \begin{cases} c_i & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ji}(\Lambda) = \begin{cases} a_{ji} & \text{if } c_i - \sum\limits_{j=1}^{2} \lambda_j \cdot a_{ji} > 0 \\ 0 & \text{otherwise} \end{cases}$$

we have for every Λ, $\lambda_j \geq 0$, $j = 1, 2$:

$$\begin{aligned} \phi_n(\Lambda) &= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i - \sum_{j=1}^{2} \lambda_j \cdot a_{ji} \right) \cdot x_i(\Lambda) = \\ &= \sum_{j=1}^{2} \lambda_j \cdot b_j(n) + \sum_{i=1}^{n} \left(c_i(\Lambda) - \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda) \right). \end{aligned}$$

Obviously, for $i = 1, \ldots, n,\ j = 1, 2,$

$$c_i(\Lambda) = c_i \cdot x_i(\Lambda), \quad a_{ji}(\Lambda) = a_{ji} \cdot x_i(\Lambda).$$

The dual problem to Two-Constraint Binary Knapsack Problem (1) may be formulated as follows:

$$\Phi_n^* = \min_{\Lambda \geq 0} \phi_n(\Lambda). \tag{5}$$

For every $\Lambda \geq 0$ the following holds:

$$z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + \sum_{j=1}^{2} \lambda_j (b_j(n) - s_j(\Lambda)). \tag{6}$$

Let us denote:

$$z_n(\Lambda) = \sum_{i=1}^{n} c_i \cdot x_i(\Lambda) = \sum_{i=1}^{n} c_i(\Lambda),\ s_j(\Lambda) = \sum_{i=1}^{n} a_{ji} \cdot x_i(\Lambda) = \sum_{i=1}^{n} a_{ji}(\Lambda),$$

$$S_n(\Lambda) = \sum_{j=1}^{2} \lambda_j \cdot s_j(\Lambda),\ B(\Lambda) = \sum_{j=1}^{2} \lambda_j \cdot b_j(n).$$

By definition of $c_i(\Lambda)$ and $a_{ji}(\Lambda)$, see (4), we have:

$$c_i(\Lambda) \geq \sum_{j=1}^{2} \lambda_j \cdot a_{ji}(\Lambda),\ i = 1, \ldots, n,$$

and therefore

$$z_n(\Lambda) \geq S_n(\Lambda). \tag{7}$$

Some Λ, $x_i(\Lambda)$ given by (4), may provide feasible solution of (1), i.e.:

$$s_1(\Lambda) \leq b_1(n) \quad \text{and} \quad s_2(\Lambda) \leq b_2(n). \tag{8}$$

If the above holds, then:

$$z_n(\Lambda) \leq z_{OPT}(n) \leq \Phi_n^* \leq \phi_n(\Lambda) = z_n(\Lambda) + B(\Lambda) - S_n(\Lambda). \tag{9}$$

So, if (8) holds, then the inequality below also holds:

$$B(\Lambda) - S_n(\Lambda) \geq 0.$$

From (7) we get:

$$\frac{\phi_n(\Lambda)}{z_n(\Lambda)} = \frac{z_n(\Lambda)}{z_n(\Lambda)} + \frac{B(\Lambda) - S_n(\Lambda)}{z_n(\Lambda)} \leq 1 + \frac{B(\Lambda) - S_n(\Lambda)}{S_n(\Lambda)}.$$

Therefore, if (8) holds, then the following inequality also holds:

$$1 \leq \frac{z_{OPT}(n)}{z_n(\Lambda)} \leq \frac{\Phi_n^*}{z_n(\Lambda)} \leq \frac{\phi_n(\Lambda)}{z_n(\Lambda)} \leq \frac{B(\Lambda)}{S_n(\Lambda)}. \tag{10}$$

Formulas (8) and (10) may allow for providing the asymptotical approximation of the optimal solution value $z_{OPT}(n)$ of the problem (1). Namely, if there exists such a set of Lagrange multipliers $\Lambda(n)$ that asymptotically fulfil the formulas (8) and (10), then the following conjecture holds:

$$\text{If } \lim_{n\to\infty} \frac{B(\Lambda(n))}{S_n(\Lambda(n))} = 1 \text{ and (8) holds then } \lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1. \tag{11}$$

Therefore, if (11) holds, then $x_i(\Lambda(n))$, $i = 1, \ldots, n$, given by (4), provides the asymptotically suboptimal solution of the Two-Constraint Binary Knapsack Problem (1). Moreover, the value of $z_n(\Lambda(n))$ is an asymptotical approximation of the optimal solution value of the Two-Constraint Binary Knapsack Problem i.e. $z_{OPT}(n)$.

4. Probabilistic analysis

4.1. The setting

The present section of the paper investigates some probabilistic properties of the Two-Constraint Binary Knapsack Problem (1). It is assumed that c_i, a_{ji} $i = 1, \ldots, n$, $j = 1, 2$ are realizations of mutually independent random variables and, moreover, c_i, a_{ji} are uniformly distributed over $(0, 1]$. In addition, it is assumed that $0 < \delta \leqslant b_1(n) \leqslant b_2(n) \leqslant n/2$, $b_j(n) \leqslant b_j(n+1)$, as well as that the Lagrange multipliers λ_1 and λ_2, $\Lambda = (\lambda_1, \lambda_2)$ are deterministic (not random variables). Monotonicity of the constraint right hand sides, $b_1(n) \leqslant b_2(n)$, in this case implies monotonicity of the Lagrange multipliers, i.e. $\lambda_2 \leq \lambda_1$. Such a probabilistic model of general knapsack problems is often used in the literature, and it also fits very well the Two-Constraint Binary Knapsack Problem (1).

Let us first observe that, due to the assumptions made, the following holds for $i = 1, \ldots, n$, $j = 1, 2$:

$$P(a_{ji} < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}, \; P(c_i < x) = \begin{cases} 0 & \text{when } x \leqslant 0 \\ x & \text{when } 0 < x \leqslant 1 \\ 1 & \text{when } x \geqslant 1 \end{cases}. \tag{12}$$

The probabilistic analysis of the Two-Constraint Binary Knapsack Problem (1) requires considering the probabilistic distribution of the following random variables:

$$\sum_{j=1}^{k} \lambda_j \cdot a_{ji}, \; k = 1 \text{ or } 2.$$

Let

$$(x)_+ = \frac{|x| + x}{2} = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}, \; j^* = \begin{cases} 2 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \end{cases},$$

Then for $i = 1, \ldots, n$, $j = 1, 2$, the following holds:

$$F_1(x, \lambda_j) = P\{\lambda_j \cdot a_{ji} < x\} = \frac{1}{\lambda_j}((x)_+ - (x - \lambda_j)_+), \; j = 1, 2,$$

$$\begin{aligned} F_2(x, \Lambda) = P\{\lambda_1 \cdot a_{1i} + \lambda_2 \cdot a_{2i} < x\} = \frac{1}{\lambda_j} \int_0^1 F_1(x - \lambda_{j^*} t, \lambda_j) dt = \\ = \frac{1}{\lambda_1 \cdot \lambda_2} \left((x)_+^2 - (x - \lambda_1)_+^2 - (x - \lambda_2)_+^2 + (x - \lambda_1 - \lambda_2)_+^2 \right). \end{aligned} \tag{13}$$

The distribution functions for random variables $a_{ji}(\Lambda),\ c_i(\Lambda),\ i=1,\ldots,n$, $j=1,2$ are:

$$\begin{aligned} G_{ji}(x,\Lambda) &= P\left\{a_{ji}(\Lambda)<x\right\} = \\ &= P\left\{a_{ji}<x \bigcup a_{ji}\geq x \bigcap \sum_{k=1}^{2}\lambda_k\cdot a_{ik}\geq c_i\right\} = \\ &= 1-\int\limits_{x}^{1}\int\limits_{0}^{1}F_1(r-\lambda_j\cdot t,\lambda_{j^*})drdt \end{aligned} \tag{14}$$

$$\begin{aligned} H_i(x,\Lambda) &= P\left\{c_i(\Lambda)<x\right\} = \\ &= P\left\{c_i<x \bigcup c_i\geq x \bigcap \sum_{k=1}^{2}\lambda_k\cdot a_{ik}\geq c_i\right\} = \\ &= 1-\int\limits_{x}^{1}F_2(t,\Lambda)dt, \end{aligned} \tag{15}$$

Using the above formulas, (14) and (15), expectations of $a_{ji}(\Lambda),\ c_i(\Lambda)$ can be expressed as follows:

$$\begin{aligned} E(a_{ji}(\Lambda)) &= \int\limits_{0}^{1}xdG_{ji}(x,\Lambda) = \int\limits_{0}^{1}x\int\limits_{0}^{1}F_1(r-\lambda_j\cdot x,\lambda_{j^*})drdx = \\ &= \frac{1}{\lambda_{j^*}}\left(\int\limits_{0}^{1}x\int\limits_{0}^{1}((r-x\cdot\lambda_j)_+-(r-x\cdot\lambda_j-\lambda_{j^*})_+)drdx\right) \end{aligned} \tag{16}$$

$$\begin{aligned} E(c_i(\Lambda)) &= \int\limits_{0}^{1}xdH_i(x,\Lambda) = \int\limits_{0}^{1}x\cdot F_2(x,\Lambda)dx = \\ &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\int\limits_{0}^{1}x\cdot\left((x)_+-(x-\lambda_1)_+^2-(x-\lambda_2)_+^2+(x-\lambda_1-\lambda_2)_+^2\right)dx = \\ &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4}-\int\limits_{0}^{1}x\cdot\left((x-\lambda_1)_+^2+(x-\lambda_2)_+^2-(x-\lambda_1-\lambda_2)_+^2\right)dx\right). \end{aligned} \tag{17}$$

The average case, or, in other words, the relevant probabilistic analysis, consists of determining the following Lagrange multipliers: $\lambda_1(n),\ \lambda_2(n)$ that when $n \to \infty$, $x_i(\Lambda(n))$, $i = 1, \ldots, n$, defined by (4), will provide solutions of the Two-Constraint Binary Knapsack Problem (1), which are, in the sense of convergence in probability, see Loéve (1977), asymptotically feasible, i.e. $s_j(\Lambda(n))$ is satisfying (8) and, moreover, if $S_n(\Lambda(n))$ is fulfilling (11) then, due to (10), $\lim_{n\to\infty} \frac{z_{OPT}(n)}{z_n(\Lambda(n))} = 1$ and $z_n(\Lambda(n))$ is a suboptimal solution of (1), and yet, in addition

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx E(z_n(\Lambda(n))).$$

The goal mentioned above can be achieved by determining $\Lambda(n)$ as the solution of the following system of equations:

$$E(s_1(\Lambda(n))) = b'_1(n), \quad E(s_2(\Lambda(n))) = b'_2(n), \tag{18}$$

where $b'_j(n) = b_j(n) - \epsilon_j(n)$, $\epsilon_j(n) = o(b_j(n))$ and $s_j(\Lambda(n)) \approx b'_j(n) \approx b_j(n)$, $s_j(\Lambda(n)) \leqslant b_j(n)$, $j = 1, 2$ (in the sense of convergence in probability) and therefore $\Lambda(n)$ is fulfilling both (8) and (11).

It may also happen that the system of equations (18) has no solutions, e.g. when the difference between $b_1(n)$ and $b_2(n)$ (asymptotical divergence) is too large. In this case $\lambda_1(n) > 0$ and $\lambda_2(n) = 0$, which means that the second constraint in the Two-Constraint Binary Knapsack Problem (1) formulation is excessive and could be removed. This means that in this situation problem (1) reduces to the classical single constraint knapsack problem. In the paper by Szkatuła (1997) the following formulas, summarizing the behavior of the optimal solution value, were presented:

$$z_{OPT}(n) \approx z_n(\Lambda(n)) \approx z'_{OPT}(n) \approx z'_n(\Lambda(n)) \approx E(z'_n(\Lambda(n)))$$

and

$$E(z'_n(\Lambda(n))) = \begin{cases} \sqrt{\frac{2\cdot n\cdot b'_1(n)}{3}} & \text{if } \delta \le b'_1(n) \le \frac{n}{6}, \\ \frac{1}{4}\cdot\left(\frac{n}{2} + 6\cdot b'_1(n)\cdot\left(1 - \frac{b'_1(n)}{n}\right)\right) & \text{if } \frac{n}{6} \le b'_1(n) \le \frac{n}{2}. \end{cases} \tag{19}$$

The subsequent part of the paper provides an overview of the asymptotic probabilistic properties of selected problem characteristics of Lagrange multipliers when their values are small, moderate, or mixed. It is assumed that $c_i, a_{ji}\ i = 1, \ldots, n,\ j = 1, 2$, are realizations of mutually independent random variables uniformly distributed over $(0, 1]$.

Lemma 1 *If there does not exist* $0 < \lambda_2(n) \le \lambda_1(n)$, *being a solution of (18) for all* $n \ge 1$, *then the second constraint (1) is excessive in the sense that it is* *always fulfilled if the first one holds. In this case $\lambda_2(n) = 0$ and the problem is equivalent to the single constraint knapsack problem, and*

$$z_{OPT}(n) \approx \begin{cases} \sqrt{\frac{2 \cdot n \cdot b_1'(n)}{3}} & \text{if } \delta \leq b_1'(n) \leq \frac{n}{6}, \\ \frac{1}{4} \cdot \left(\frac{n}{2} + 6 \cdot b_1'(n) \cdot \left(1 - \frac{b_1'(n)}{n} \right) \right) & \text{if } \frac{n}{6} \leq b_1'(n) \leq \frac{n}{2}. \end{cases} \tag{20}$$

Proof In this case the system of equations (18) has no solution and therefore the Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, see Szkatuła (1997); (20) follows immediately from (19). ■

This case is in depth discussed in Szkatuła (2021).

Let us observe that $E(s_j(\Lambda(n))) = n \cdot E(a_{j1}(\Lambda(n)))$, $E(z_n(\Lambda(n))) = n \cdot E(c_1(\Lambda(n)))$. It is easy to note that the above formulas, (16) and (17), may take different formulations, depending on the mutual relations between λ_1, λ_2 and x, r, since some items of the above formulas may remain strongly positive or become 0, due to the properties of function $(.)_+$. In general, four specific cases could be distinguished for $i = 1, \ldots, n$, $j = 1, 2$. Each of those cases, in which $0 \leq \lambda_2(n) \leq \lambda_1(n)$, is considered below.

4.2. Case of "large" values of the Lagrange multipliers

In this case $1 \leq \lambda_2 \leq \lambda_1$ and:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^*}} \int_0^{1/\lambda_j} x \int_{x \cdot \lambda_j}^1 (r - x \cdot \lambda_j) dr dx = \frac{1}{24 \cdot \lambda_j^2 \cdot \lambda_{j^*}} \tag{21}$$

$$E(c_i(\Lambda)) = \frac{1}{2 \cdot \lambda_1 \cdot \lambda_2} \int_0^1 x^3 dx = \frac{1}{8 \cdot \lambda_1 \cdot \lambda_2}.$$

In this case both constraints are active, $\lambda_1(n) \geq \lambda_2(n) > 0$.

Lemma 2 *If there exists a constant $\delta > 0$ such that $\delta \leq b_1'(n) \leq b_2'(n) \leq \frac{n}{24}$ then*

$$\lambda_1(n) = \frac{1}{b_1'(n)} \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}}, \quad \lambda_2(n) = \frac{1}{b_2'(n)} \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}}, \tag{22}$$

is the solution of (18) and

$$E(z_n'(\Lambda(n))) = 3 \cdot \sqrt[3]{\frac{n \cdot b_1'(n) \cdot b_2'(n)}{24}}. \tag{23}$$

Proof The formulas above follow immediately from (21) and (18). From the assumptions of Lemma 2 and from (22) it follows that $1 \leq \lambda_2(n) \leq \lambda_1(n)$. ■

In this case, $E(z_n(\Lambda(n))) = \frac{n}{8}$ is the maximum value of $E(z_n(\Lambda(n)))$.

4.3. Case of "mixed" values of the Lagrange multipliers

In this case $\lambda_2 < 1 \leq \lambda_1$ and:

$$E(a_{1i}(\Lambda)) = \frac{1}{\lambda_2}\left(\int_0^{1/\lambda_1} x \int_{x\cdot\lambda_1}^{1} (r - x\cdot\lambda_1)drdx - \right. \tag{24}$$

$$\left. - \int_0^{\frac{1-\lambda_2}{\lambda_1}} x \int_{(x\cdot\lambda_1+\lambda_2)}^{1} (r - x\cdot\lambda_1 - \lambda_2)drdx\right) =$$

$$= \frac{1}{24}\frac{-\lambda_2^3 + 4\lambda_2^2 - 6\lambda_2 + 4}{\lambda_1^2},$$

$$E(a_{2i}(\Lambda)) = \frac{1}{\lambda_1}\left(\int_0^1 x \int_{x\cdot\lambda_2}^1 (r - x\cdot\lambda_2)drdx\right) = \frac{1}{24}\frac{3\lambda_2^2 - 8\lambda_2 + 6}{\lambda_1},$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2}\left(\frac{1}{4} - \int\limits_{\lambda_2}^{1} x\cdot(x-\lambda_2)^2dx\right) =$$

$$= \frac{1}{24\lambda_1}\left(\lambda_2^3 - 6\lambda_2 + 8\right).$$

In this case only the first constraint is active: $\lambda_1(n) > 0$ and $\lambda_2(n) = 0$.

Lemma 3 *If there exists a constant $\delta > 0$ such that $\delta \leq b_1'(n) \leq \frac{n}{24} < b_2'(n) \leq \frac{n}{6}$ then (18) has no solution, $\lambda_2(n) = 0$, and*

$$z_{OPT}(n) \approx \sqrt{\frac{2\cdot n\cdot b_1'(n)}{3}}. \tag{25}$$

Proof In this case (18) has no solution, $\lambda_2(n) = 0$, and from (24) we obtain

$$\lambda_1(n) = \sqrt{\frac{n}{6\cdot b_1'(n)}}.$$

(25) follows from (24), but also from Lemma 1. ■

In this case $E(z_n(\Lambda(n))) = \frac{n}{6}$ is the maximum value of $E(z_n(\Lambda(n)))$.

4.4. Case of "moderate" values of the Lagrange multipliers

In this case $0 \le \lambda_2 \le \lambda_1 \le 1,\ \lambda_2 + \lambda_1 \ge 1$ and:

$$E(a_{ji}(\Lambda)) = \frac{1}{\lambda_{j^*}} \left(\int_0^1 x \int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j) dr dx - \right. \tag{26}$$

$$\left. - \int_0^{(1-\lambda_{j^*})/\lambda_j} x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^1 (r - x\cdot\lambda_j - \lambda_{j^*}) dr dx \right) =$$

$$= \frac{1}{24} \frac{3\lambda_j^4 - 8\lambda_j^3 + 6\lambda_j^2 - 6\lambda_{j^*}^2 + 4\lambda_{j^*} - \lambda_{j^*}^4 + 4\lambda_{j^*}^3 - 1}{\lambda_j^2 \lambda_{j^*}},$$

$$E(c_i(\Lambda)) = \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^1 x\cdot(x-\lambda_1)^2 dx - \int_{\lambda_2}^1 x\cdot(x-\lambda_2)^2 dx \right)$$

$$= \frac{1}{24\lambda_1\lambda_2} \left(\lambda_1^4 - 6\lambda_1^2 + 8\lambda_1 + \lambda_2^4 - 6\lambda_2^2 + 8\lambda_2 - 3 \right).$$

In this case conditions $\lambda_2 \le \lambda_1 \le 1,\ \lambda_2 + \lambda_1 \ge 1$ imply that $\frac{1}{2} \le \lambda_1 \le 1$ and $1 - \lambda_1 \le \lambda_2 \le \lambda_1$. If $\lambda_1 = \frac{1}{2}$, then $\lambda_2 = \frac{1}{2}$ and:

$$\frac{n}{24} \le b_1'(n) \le \frac{5\cdot n}{24},\ b_1'(n) \le b_2'(n),\ b_2'(n) \le \frac{n}{4}.$$

In this case

$$\frac{n}{8} \le E(z_n(\Lambda(n))) \le \frac{17\cdot n}{48};$$

moreover

$$E(z_n(\Lambda(n))) = \frac{n}{8} \text{ when } b_1'(n) = b_2'(n) = \frac{n}{24};$$
$$E(z_n(\Lambda(n))) = \frac{17\cdot n}{48} \text{ when } b_1'(n) = b_2'(n) = \frac{5\cdot n}{24}.$$

4.5. Case of "small" values of the Lagrange multipliers

In this case conditions $\lambda_2 \le \lambda_1 \le 1$, $\lambda_2 + \lambda_1 \le 1$ imply that $\lambda_2 \le \lambda_1 \le 1 - \lambda_2$ and $0 \le \lambda_2 \le \frac{1}{2}$. If $\lambda_2 = \frac{1}{2}$, then $\lambda_1 = \frac{1}{2}$ and:

$$\begin{aligned}
E(a_{ji}(\Lambda)) &= \frac{1}{\lambda_{j^*}} \left(\int_0^1 x \int_{x\cdot\lambda_j}^1 (r - x\cdot\lambda_j) dr dx - \right. \\
&\left. - \int_0^1 x \int_{(x\cdot\lambda_j+\lambda_{j^*})}^1 (r - x\cdot\lambda_j - \lambda_{j^*}) dr dx \right) = \\
&= \frac{1}{2} - \frac{1}{3}\lambda_j - \frac{1}{4}\lambda_{j^*}, \\
E(c_i(\Lambda)) &= \frac{1}{2\cdot\lambda_1\cdot\lambda_2} \left(\frac{1}{4} - \int_{\lambda_1}^{1} x\cdot(x-\lambda_1)^2 dx - \int_{\lambda_2}^{1} x\cdot(x-\lambda_2)^2 dx + \right. \\
&\left. + \int_{\lambda_1+\lambda_2}^{1} x\cdot(x-\lambda_1-\lambda_2)^2 dx) \right) = \\
&= \frac{1}{2} - \frac{1}{6}\lambda_1^2 - \frac{1}{4}\lambda_1\lambda_2 - \frac{1}{6}\lambda_2^2.
\end{aligned} \tag{27}$$

In this case both constraints are active when $0 < \lambda_2(n) \le \lambda_1(n) \le 1$ and $\lambda_2 + \lambda_1 \le 1$.

Lemma 4 *If*

$$\frac{n}{6} < b_1'(n) \le b_2'(n) \le \frac{n}{2} \quad \textit{and} \quad \frac{3}{4}\cdot b_1'(n) + \frac{5\cdot n}{96} \le b_2'(n) \le \frac{3}{4}\cdot b_1'(n) + \frac{n}{8}$$

then

$$\begin{aligned}
\lambda_1(n) &= \frac{6}{7}\cdot\left(1 - 8\cdot\frac{b_1'(n)}{n} + 6\cdot\frac{b_2'(n)}{n}\right), \\
\lambda_2(n) &= \frac{6}{7}\cdot\left(1 + 6\cdot\frac{b_1'(n)}{n} - 8\cdot\frac{b_2'(n)}{n}\right),
\end{aligned} \tag{28}$$

is the solution of (18) and

$$\begin{aligned}
E(z_n(\Lambda(n))) = \frac{1}{7}\cdot\Big(&\frac{n}{2} + 6\cdot b_1'(n) + 6\cdot b_2'(n) + \\
&+ 36\cdot\frac{b_1'(n)\cdot b_2'(n)}{n} - 24\cdot\frac{(b_1'(n))^2}{n} - 24\cdot\frac{(b_2'(n))^2}{n}\Big).
\end{aligned} \tag{29}$$

PROOF From (27) it can be deduced that $\lambda_1(n)$ and $\lambda_2(n)$, given by formula (28), are solving the equation (18) and fulfilling the condition $\lambda_2(n)+\lambda_1(n) \leq 1$. ■

Finally, there is also the situation, in which $0 < \lambda_1 \leq 1$ and $\lambda_2 = 0$:

LEMMA 5 *If*

$$\frac{n}{6} < b_1'(n) \leq b_2'(n) \leq \frac{n}{2} \quad \textit{and}$$

$$\textit{either} \quad b_2'(n) > \frac{3}{4} \cdot b_1'(n) + \frac{n}{8} \quad \textit{or} \quad b_2'(n) < \frac{3}{4} \cdot b_1'(n) + \frac{5 \cdot n}{96}$$

then

$$\lambda_1(n) = 3 \cdot \left(\frac{1}{2} - \frac{b_1'(n)}{n}\right), \quad \lambda_2(n) = 0$$

is the optimal set of the Lagrange multipliers and

$$E(z_n(\Lambda(n))) = \frac{1}{4} \cdot \left(\frac{n}{2} + 6 \cdot b_1'(n) \cdot \left(1 - \frac{b_1'(n)}{n}\right)\right).$$

PROOF In this case, the system of equations (18) has either no solution or $\lambda_2(n) > \frac{1}{2}$, and therefore, the Two-Constraint Binary Knapsack Problem (1) is equivalent to the single constraint knapsack problem, as shown in Szkatuła (1997); (20) follows immediately from (19). Condition $\lambda_2(n) + \lambda_1(n) \leq 1$ holds also in this case. ■

In the cases, which are examined in Lemma 4 and Lemma 5, the condition $\lambda_2(n) + \lambda_1(n) \leq 1$ and formula (3) are providing that following right-hand-sides of the constraints:

$$\frac{5 \cdot n}{12} \leq b_1'(n) + b_2'(n) \preceq n,$$

which are fulfilling the assumptions of Lemma 4.

5. Concluding remarks

This paper examines the probabilistic properties of the Two-Constraint Binary Knapsack Problem, formulated here in (1), when constraint right-hand sides range as to their values from small, i.e. when $1 \leq \lambda_2(n) \leq \lambda_1(n)$, to large, i.e. when $\lambda_2(n) + \lambda_1(n) \leq 1$.. Exact values of all relevant problem coefficients such as $\lambda_1(n), \lambda_2(n)$, $E(z_n(\Lambda(n)))$ etc. are provided.

Furthermore, cases with small and moderate values on the right-hand sides of the constraints, i.e. when $\lambda_2(n) \leq 1 \leq \lambda_1(n)$, were also considered.

The case where the second constraint is excessive, i.e. when a two-constraint knapsack problem is equivalent to a single constraint knapsack problem, was identified.

The paper presents the distribution functions of the various random variables that represent important problem characteristics.

The aim of the future research is to examine the remaining cases of moderate, high and mixed constraint coefficient values, the issue of mutual relations between $\lambda_1(n)$ and $\lambda_2(n)$, the feasibility of the obtained solutions and the estimation of behaviour of the Two-Constraint Binary Knapsack Problem (1) optimal solution values $z_{OPT}(n)$, when $n \to \infty$.

References

Averbakh, I. (1994) Probabilistic properties of the dual structure of the multidimensional knapsack problem and fast statistically efficient algorithms. *Mathematical Programming*, 65: 311–330.

Frieze, A. and Clarke, M. (1984) Approximation algorithms for the m-dimensional 0-1 knapsack problem: Worst case and probabilistic analysis. *European Journal of Operational Research*, 15: 100–109.

Garey, M. and Johnson, D. (1970) *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman, San Francisco.

Lee, T.-E. and Oh, G.-T. (1997) The asymptotic value-to-capacity ratio for the multi-class stochastic knapsack problem. *European Journal of Operational Research*, 103: 584–594.

Loéve, M. (1977) *Probability Theory I.* Springer Verlag, New York, Heidelberg, Berlin.

Mamer, J. and Schilling, K. (1990) On the growth of random knapsacks. *Discrete Applied Mathematics*, 28: 223–230.

Martello, S. and Toth, P. (1990) *Knapsack Problems: Algorithms and Computer Implementations*. Wiley & Sons.

Meanti, M., Kan, A. R., Stougie, L. and Vercellis, C. (1990) A probabilistic analysis of the multiknapsack value function. *Mathematical Programming*, 46: 237–247.

Nemhauser, G. and Wolsey, L. (1988) *Integer and Combinatorial Optimization*. John Wiley & Sons Inc., New York.

Schilling, K. (1990) The growth of m-constraint random knapsacks. *European Journal of Operational Research*, 46: 109–112.

Schilling, K. (1994) Random knapsacks with many constraints. *Discrete Applied Mathematics*, 48: 163–174.

Szkatuła, K. (1994) On the growth of multi-constraint random knapsacks with various right-hand sides of the constraints. *European Journal of Operational Reserch*, 73: 199–204.

Szkatuła, K. (1997) The growth of multi-constraint random knapsacks with large right-hand sides of the constraints. *Operations Research Letters*, 21: 25–30.

Szkatuła, K. (2021) When two-constraint binary knapsack problem is equivalent to classical knapsack problem? In: K. T. Atanassov, V. Atanassova, J. Kacprzyk, A. Kałuszko, M. Krawczak, J. W. Owsiński, S. S. Sotirov, E. Sotirova, E. Szmidt, and S. Zadrożny, eds., *Advances and New Developments in Fuzzy Logic and Technology*, Cham, Springer International Publishing, 311–321.