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On the existence of proportional-integral observer for the state estimation of linear time-invariant systems

by

Konstadinos H. Kiritsis

Hellenic Air Force Academy, Department of Aeronautical Sciences,
Division of Automatic Control, Dekelia Air Base,
PC 13671, Acharnes, Attikis,Tatoi, Greece
konstantinos.kyritsis@hafa.haf.gr

Abstract: In this paper, the explicit necessary and sufficient conditions are established for the existence of proportional-integral observer for the state estimation of linear time-invariant continuous-time systems. In particular, it is proven that for a given linear time-invariant continuous-time system of order n, having m inputs and p linearly independent outputs, a proportional-integral observer of order n can be constructed if and only if the given system is detectable. Furthermore a simple procedure is given for the construction of proportional-integral observer. Our approach is based on properties of real and polynomial matrices.

Keywords: proportional-integral observer, necessary and sufficient conditions

1. Introduction

In 1963, Luenberger initiated the theory of observers for the state estimation of linear time-invariant continuous-time systems, Luenberger (1963); see also Luenberger (1964, 1966, 1971). Then, in Luenberger (1971) he proposed the full order observer. Seven years later, Wojciechowski (1978) added an additional term to Luenberger's full order observer for the state estimation of single-input single-output linear time-invariant systems. This term is proportional to the integral of the output estimation error. The resulting new observer was called proportional-integral observer and has also a long and rich history. The main result of Wojciechowski (1978) was later generalized to multivariable linear time-varying systems (see Kaczorek, 1979; Shafai and Carroll, 1985). Furthermore, Shafai and Carroll (1985) first considered a reduced order proportional-integral observer. Then, Beale and Shafai (1989) studied the robustness property of feedback control systems using a proportional-integral observer. Niemann et al. (1995) derived the necessary and sufficient conditions, under which the proportional-integral observer achieves Exact Loop Transfer Recovery for linear time-invariant continuous-time systems, and similar results have been obtained by Shafai et al. (1996) for linear time-invariant discrete-time systems. It was proven by Söffker, Yu and Muller (1995) that the proportional-integral observer can estimate the state not only of linear time-invariant systems, but also of systems with arbitrary external input, which appears as unknown input, non-linearity or unmodelled dynamics.

It was shown by Busawon and Kabore (2000) that, for some classes of systems, the proportional-integral observer has the ability to completely decouple the modeling uncertainties, while keeping satisfactory convergence properties. Furthermore, a comparison of classical proportional observer with proportional-integral observer was given, using a simulation example. A parametric eigen-structure assignment design approach for proportional-integral observers of multivariable linear systems was proposed in Duan, Liu and Thompson (2001, 2003) and Wu, Duan and Liu (2012). In Bakhshande and Söffker (2017) a proportional-integral observer based backstepping controller was proposed for systems with model uncertainties and measurement noise. Bakhshande, Bach and Söffker (2020) proposed a proportional-integral observer based sliding mode controller for nonlinear hydraulic differential cylinder systems, affected by uncertainties. Białoń, Pasko and Niestrój (2020) studied the state reconstruction problem of an induction motor, using proportional-integral observer and it is stated by the authors of that paper that the proportional-integral observer provides better state reconstruction quality in comparison with the proportional Luenberger observer of Luenberger (1971). Proportional-integral observer-based approaches for fault detection were developed in Hu et al. (2022), Duan and Wu (2006), Khedher et al. (2009), Hamdi et al. (2012), Gao, Cecati and Ding (2015), Shafai and Saif (2015), Shafai and Moradmand (2020) and Lin (2022), and in the references, provided therein. The proportional-integral observer literature is extremely rich; for more complete references, we refer the reader to Bakhshande, Bach and Söffker (2020) and to Bakhshande and Söffker (2015), Liu (2011) and Bakhshande (2018).

To the best of our knowledge, the problem of existence of proportional-integral observer for the state estimation of linear time-invariant continuous-time systems is still an open problem. This motivates the present study. In this paper, using basic notions and basic results from linear systems and control theory and the theory of matrices, the explicit necessary and sufficient conditions for the existence of proportional-integral observer for the state estimation of linear time-invariant systems are established. In particular, it is proven that for a given linear time-invariant continuous-time system of order n, having m inputs and p linearly independent outputs, a proportional-integral observer of order n can be constructed if and only if the given system is detectable. Furthermore, a simple procedure is given for the construction of the proportional-integral observer.

2. Problem statement

Consider a linear time-invariant continuous-time system, described by the following state-space equations:

$$\dot{x}\,(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t) \tag{1}$$

$$\mathbf{y}(t) = \mathbf{Cx}(t) \tag{2}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are real matrices of size (nxn), (nxm) and (pxn), respectively, $\mathbf{x}(t)$ is the state vector of dimensions $(nx1)$, $\mathbf{u}(t)$ is the vector of inputs of size $(mx1)$ and $\mathbf{y}(t)$ is the vector of outputs of size $(px1)$. Without any loss of generality we assume that

$$rank\,[\mathbf{C}] = p. \tag{3}$$

Let us consider a linear time-invariant continuous-time system, described by the equations

$$\dot{\hat{x}}(t) = (\mathbf{A} - \mathbf{LC})\hat{x}(t) + \mathbf{Ly}(t) + \mathbf{Bu}(t) + \mathbf{Fw}(t) \tag{4}$$

$$\dot{w}(t) = \mathbf{G}[\mathbf{y}(t) - \mathbf{C}\hat{x}(t)] \tag{5}$$

where $\hat{x}(t)$ is the state vector of system (4), (5) of dimensions $(nx1)$, $\mathbf{w}(t)$ is a vector of dimensions $(kx1)$, and $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$ are real matrices of size (nxp), (nxk) and (kxp), respectively. The linear time-invariant system (4), (5) is a proportional-integral observer of order n for the system (1), (2), if and only if for arbitrary initial conditions $\hat{x}(0)$, $\mathbf{x}(0)$ and any input $\mathbf{u}(t)$, the following relationships hold (see Kaczorek, 1979; Shafai and Carroll, 1985; Beale and Shafai, 1989; Niemann et al., 1995; Söffker, Yu and Muller, 1995; Duan, Liu and Thompson, 2001):

$$\lim_{t\mapsto+\infty} \mathbf{e}(t) = 0 \tag{6}$$

$$\lim_{t\mapsto+\infty} \mathbf{w}\,(t) = 0 \tag{7}$$

where $\mathbf{e}(t) = [\hat{x}(t) - \mathbf{x}(t)]$ is the state estimation error, $\hat{x}(t)$ is an estimate of the state vector $\mathbf{x}(t)$, and $\mathbf{w}(t)$ is a vector representing the integral of the weighted output estimation error, see Duan, Liu and Thompson (2001). The relationships (6) and (7) are simultaneously satisfied if and only if the following matrix

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{GC} & \mathbf{0} \end{bmatrix} \tag{8}$$

is Hurwitz stable (i.e., all of its eigenvalues have negative real parts) (see Kaczorek, 1979; Shafai and Carroll, 1985; Beale and Shafai, 1989; Niemann et al., 1995; Söffker, Yu and Muller, 1995; Duan, Liu and Thompson, 2001). Thus, the problem of existence of the proportional-integral observer of order n can be stated as follows: Do there exist real matrices $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$ such that the matrix $\mathbf{R}$ of appropriate dimensions, given by (8), is Hurwitz stable? If so, give the conditions for the existence and a procedure for the calculation of the matrices $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$.

3. Basic concepts and preliminary results

This section contains lemmas, which are needed to prove the main results of this paper and some basic notions from linear systems and control theory that are used throughout the paper. And so, let R be the field of real numbers. Also let $R[s]$ be the ring of polynomials with coefficients in R. Further, let C be the field of complex numbers, and let C^+ be the set of all complex numbers λ with $Re(\lambda) \geq 0$. All nonzero finite real numbers are called units of $R[s]$ (Mc Duffee, 1946). A matrix, whose elements are polynomials over $R[s]$, is termed a polynomial matrix. A polynomial matrix $\mathbf{U}(s)$ over $R[s]$ of dimensions (kxk) is said to be unimodular if and only if $det[\mathbf{U}(s)]$ is a unit of $R[s]$ (Mc Duffee, 1946).Every polynomial matrix $\mathbf{W}(s)$ of size (mxp) with $rank[\mathbf{W}(s)] = r$, can be expressed as (see Kucera, 1991):

$$\mathbf{U}_1(s)\mathbf{W}(s)\mathbf{U}_2(s) = \mathbf{M}(s). \tag{9}$$

The polynomial matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the matrix $\mathbf{M}(s)$ is given by

$$\mathbf{M}(s) \begin{bmatrix} \mathbf{M}_r(s) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{10}$$

The non-singular polynomial matrix $\mathbf{M}_r(s)$ of size (rxr) in (10) is given by

$$\mathbf{M}_r(s) = diag[a_1(s), a_2(s), \ldots, a_r(s)]. \tag{11}$$

The nonzero polynomials $a_i(s)$ for $i = 1, 2, ..., r$ are termed invariant polynomials of $\mathbf{W}(s)$ and have the following property

$$a_i(s) \textit{ divides } a_{i+1}(s), \text{ for } i = 1, 2, \ldots, r\text{-}1. \tag{12}$$

The relationship (9) with $\mathbf{M}(s)$, given by (10), is called Smith-McMillan form of $\mathbf{W}$(s) over $R[s]$. Since the matrices $\mathbf{U}_1(s)$ and $\mathbf{U}_2(s)$ are unimodular and the polynomial matrix $\mathbf{M}_r(s)$, given by (11), is non-singular, from (9) and (10) it follows that

$$rank[\mathbf{W}(s)] = rank[\mathbf{M}_r(s)] = r. \tag{13}$$

Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $R[s]$ of appropriate dimensions. If there is a matrix $\mathbf{Q}(s)$ over $R[s]$ of appropriate dimensions, such that

$$\mathbf{A}(s) = \mathbf{B}(s)\mathbf{Q}(s), \tag{14}$$

then we say that the matrix $\mathbf{Q}(s)$ is a right divisor of the matrix $\mathbf{A}(s)$ (Wolowich, 1974). Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be matrices over $R[s]$ of appropriate dimensions. If there are matrices $\mathbf{D}$(s), $\mathbf{A}_1(s)$ and $\mathbf{B}_1(s)$ over $R[s]$ of appropriate dimensions, such that

$$\mathbf{A}(s) = \mathbf{A}_1(s)\mathbf{D}(s),\ \mathbf{B}(s) = \mathbf{B}_1(s)\mathbf{D}(s) \tag{15}$$

then the polynomial matrix $\mathbf{D}(s)$ is called the common right divisor of polynomial matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ (Wolowich, 1974). A greatest common right divisor of two polynomial matrices $\mathbf{A}$(s) and $\mathbf{B}$(s) is a common right divisor which is a left multiple of every common right divisor.

Let $\mathbf{A}$ and $\mathbf{C}$ be matrices over R of size (nxn) and (pxn), respectively, and $\mathbf{C}$ not zero. Then, there always exists a unimodular matrix $\mathbf{U}(s)$ over $R[s]$ such that (Wolowich, 1974)

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix}. \tag{16}$$

The non-singular polynomial matrix $\mathbf{V}(s)$ of size (nxn) is a greatest common right divisor of the polynomial matrices $[\mathbf{I}_n s - \mathbf{A}]$ and $\mathbf{C}$ (Wolowich, 1974). Since the polynomial matrix $\mathbf{U}(s)$ is unimodular, from (13) and (16) it follows that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = rank \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} = rank\,[\mathbf{V}(s)] = n. \tag{17}$$

DEFINITION 1 *The nonzero polynomial $c(s)$ over $R[s]$ is said to be strictly Hurwitz if and only if $c(s) \neq 0$, $\forall s \in C^+$.*

REMARK 1 *From the Definition 1 it follows that the set of all strictly Hurwitz polynomials over R[s] consists of all units of R[s] and all polynomials $c(s)$ over R[s] with $deg[c(s)] \geq 1$ (by $deg[c(s)]$ we denote the degree of $c(s)$), whose roots are all in the open left-half complex plane.*

DEFINITION 2 *Let $\mathbf{V}(s)$ be a non-singular matrix over R[s], of size (nxn).Also let $c_i(s)$ for $i = 1, 2, ..., n$ be the invariant polynomials of polynomial matrix $\mathbf{V}(s)$. The polynomial matrix $\mathbf{V}(s)$ is said to be strictly Hurwitz if and only if the polynomials $c_i(s)$ are strictly Hurwitz for every $i = 1, 2, ..., n$, or, alternatively, if and only if $det[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial.*

Definition 3 *The matrix* $\mathbf{A}$ *over* R *of size (nxn) is said to be Hurwitz stable if and only if all eigenvalues of the matrix* $\mathbf{A}$ *have negative real parts or, alternatively, if and only if the characteristic polynomial of the matrix* $\mathbf{A}$ *is a strictly Hurwitz polynomial.*

Definition 4 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* R *of sizes (nxn) and (p x n), respectively. Then, the pair* $\mathbf{A}$, $\mathbf{C}$) *is said to be detectable if and only if there exists a matrix* $\mathbf{K}$ *over* R *of size (nxp) such that the matrix* $[\mathbf{A}+\mathbf{KC}]$ *is Hurwitz stable (Wonham, 1968).*

Definition 5 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* R *of sizes (nxn) and (pxn), respectively, and* $\mathbf{C}$ *not zero. Then, an eigenvalue* λ *of the matrix* $\mathbf{A}$ *is said to be observable (Trentelman, Stoorvogel and Hautus, 2001), if and only if the following condition holds:*

$$rank\begin{bmatrix}\mathbf{C}\\ \mathbf{I}_n\lambda-\mathbf{A}\end{bmatrix}=n.$$

Let $\mathbf{A}$ be a real matrix of size (nxn). The spectrum of the matrix $\mathbf{A}$ is the set of all of its eigenvalues and is denoted by $\sigma(\mathbf{A})$. An eigenvalue λ of $\mathbf{A}$ is called a stable eigenvalue if and only if $Re(\lambda)<0$. Otherwise, the eigenvalue λ of the matrix $\mathbf{A}$ is said to be unstable. The following Lemma is taken from Zhou, Doyle and Glover (1996).

Lemma 1 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over* R *of size (n x n) and (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let* $\sigma(\boldsymbol{A})$ *be the spectrum of the matrix* $\mathbf{A}$*. The pair* $(\mathbf{A},\mathbf{C})$ *is detectable if, and only if, one of the following equivalent conditions holds:*

(a) $rank\begin{bmatrix}\mathbf{C}\\ \mathbf{I}_n s-\mathbf{A}\end{bmatrix}=n\,,\ \forall s\in C^+$

$(b)\,\forall\ \lambda\in\sigma(\boldsymbol{A})$ *with* $Re(\lambda)\geq 0$ *and for all* $\boldsymbol{x}$ *such that* $\mathbf{Ax}=\lambda\mathbf{x}$, $\mathbf{Cx}\neq\mathbf{0}$, *or, alternatively,* $rank\begin{bmatrix}\mathbf{C}\\ \mathbf{I}_n\lambda-\mathbf{A}\end{bmatrix}=n\,,\forall\lambda\in\sigma(\boldsymbol{A})$ *with* $Re(\lambda)\geq 0$.

Remark 2 *From condition (b) of Lemma 1 it follows that the pair* $(\mathbf{A},\mathbf{C})$ *is detectable if and only if all unstable eigenvalues of the matrix* $\mathbf{A}$ *are observable, see Zhou, Doyle and Glover (1996).*

Lemma 2 *Let* $\mathbf{V}(s)$ *be a non-singular polynomial matrix over* $R[s]$*, of size (n x n). Also let* $c\ _i(s)$ *for* $i=1,2,...,n$*, be the invariant polynomials of the polynomial matrix* $\mathbf{V}(s)$*. The polynomial matrix* $\mathbf{V}(s)$ *is strictly Hurwitz if and only if the following condition holds:*

(a) $rank[\mathbf{V}(s)]=n,\quad\forall s\in C^+.$

Proof Let $\mathbf{V}(s)$ be a non-singular and strictly Hurwitz polynomial matrix of size (nxn) with invariant polynomials $c_i(s)$ for $i = 1, 2, ..., n$. From Definition 2 it follows that the polynomials c_i(s) are strictly Hurwitz for every $i = 1, 2, ..., n$, and therefore, from Definition 1, it follows that

$$c_i(s) \neq 0,\ \forall s \in C^+,\ \forall i = 1,2,\ \ldots,n. \tag{18}$$

We define the polynomial matrix

$$\mathbf{V}_n(s) = diag[c_1(s), c_2(s),\ \ldots, c_n(s)]. \tag{19}$$

From (18) and (19) it follows that

$$rank[\mathbf{V}_n(s)] = rank\{[diag[c_1(s), c_2(s), ..., c_n(s)]\} = n,\ \ \forall s \in C^+. \tag{20}$$

The Smith-McMillan form of the polynomial matrix $\mathbf{V}(s)$ over $R[s]$ is given by

$$\mathbf{K}(s)\ \mathbf{V}(s)\ \mathbf{L}(s) = \mathbf{V}_n(s), \tag{21}$$

where $\mathbf{K}$(s) and $\mathbf{L}$(s) are unimodular matrices. Since the matrices $\mathbf{K}$(s), $\mathbf{L}$(s) are unimodular, from (13), (20) and (21) we have that

$$rank\,[\mathbf{V}(s)] = rank[\mathbf{V}_n(s)] = n,\ \forall \text{s} \in C^+. \tag{22}$$

This is condition (a) of the Lemma. In order to prove sufficiency, we assume that condition (a) holds. Using (13) from (19) and (21), we obtain that

$$rank[\mathbf{V}\,(s)\,] = rank[\mathbf{V}_n(s)] = rank\{\,diag[c_1(s), c_2(s), \ldots, c_n(s)]\} = n. \tag{23}$$

Since, by assumption, condition (a) holds, we have that

$$rank[\mathbf{V}(s)] = n, \forall s \in C^+. \tag{24}$$

Relationships (23) and (24) imply

$$rank\,[\mathbf{V}_n(s)] = rank\{\,diag[c_1(s), c_2(s), \ldots, c_n(s)]\} = n,\ \forall s \in C^+. \tag{25}$$

From (25) it follows that

$$c_i(s)) \neq 0,\ \forall s \in C^+,\ \forall i = 1, 2, ..., n. \tag{26}$$

Relationship (26) and Definition 1 imply that polynomials $c_i(s)$ are strictly Hurwitz for every $i = 1, 2, ..., n$, and therefore, according to Definition 2, the non-singular polynomial matrix $\mathbf{V}(s)$ over $R[s]$, is strictly Hurwitz. This completes the proof. ∎

Lemma 3 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over R of sizes (n x n) and (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let* $\mathbf{V}(s)$ *be a greatest common right divisor of polynomial matrices* $[\mathbf{I}_n s - \mathbf{A}]$ *and* $\mathbf{C}$ *of size (n x n). The pair* $(\mathbf{A}, \mathbf{C})$ *is detectable if and only if the following condition holds:*

(a) The polynomial matrix $\mathbf{V}(s)$ *is strictly Hurwitz.*

Proof Let the pair $(\mathbf{A}, \mathbf{C})$ be detectable. Then, from Lemma 1, it follows that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = n \ \ \forall s \in C^+. \tag{27}$$

Since, by assumption, the polynomial matrix $\mathbf{V}$(s) is the greatest common right divisor of the polynomial matrices [$\mathbf{I}_n s$-$\mathbf{A}$] and $\mathbf{C}$, from (16) it follows that there exists a unimodular matrix $\mathbf{U}$(s), such that

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = \mathbf{U}(s) \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix}. \tag{28}$$

Since the polynomial matrix $\mathbf{U}(s)$ is unimodular, then from (17) and (28) it follows that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = rank \begin{bmatrix} \mathbf{V}(s) \\ \mathbf{0} \end{bmatrix} = rank[\mathbf{V}(s)]. \tag{29}$$

From relationships (27) and (29) we have that

$$rank[\mathbf{V}(s)] = n, \ \forall s \in C^+. \tag{30}$$

Relationship (30) and Lemma 2 imply that the polynomial matrix $\mathbf{V}$(s) is strictly Hurwitz. This is the condition (a) of the Lemma. To prove sufficiency, we assume that the polynomial matrix $\mathbf{V}$(s) is strictly Hurwitz. Then, from Lemma 2, it follows that

$$rank[\mathbf{V}(s)] = n, \ \forall \mathrm{s} \in C^+. \tag{31}$$

Since, by assumption, the polynomial matrix $\mathbf{V}$(s) is the greatest common right divisor of polynomial matrices [$\mathbf{I}_n$s-$\mathbf{A}$] and $\mathbf{C}$, from (31) and (29) it follows that

$$rank \begin{bmatrix} \mathbf{C} \\ \mathbf{I}_n s - \mathbf{A} \end{bmatrix} = n\,, \ \forall s \in C^+. \tag{32}$$

Lemma 1 and relationship (32) imply that the pair $(\mathbf{A}, \mathbf{C})$ is detectable. This completes the proof. ■

The following lemma is taken from Kucera (1991).

LEMMA 4 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over R of size (n x n) and (p x n), respectively. Then, the pair* $(\mathbf{A}, \mathbf{C})$ *is observable if and only if for every monic polynomial* $c(s)$ *over* $R[s]$ *of degree n there exists a matrix* $\mathbf{K}$ *over R of size (n x p), such that the matrix* $[\mathbf{A}+\mathbf{KC}]$ *has characteristic polynomial* $c(s)$.

The standard decomposition of unobservable systems, given in the following Lemma, was first published by Kalman (1963) and can be also found in any standard text of linear systems theory.

LEMMA 5 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over R of size (n x n) and (p x n), respectively. Further, let the pair* $(\mathbf{A}, \mathbf{C})$ *be unobservable and* $\mathbf{C}$ *not zero. Then, there exists a non-singular matrix* $\mathbf{T}$ *of size (n x n) such that*

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$
$$\mathbf{CT} = [\mathbf{C}_1, \mathbf{0}].$$

The pair $(\mathbf{A}_{11}, \mathbf{C}_1)$ *is observable and the eigenvalues of the matrix* $\mathbf{A}_{22}$ *are the unobservable eigenvalues of the pair* $(\mathbf{A}, \mathbf{C})$.

The following lemma is taken from Zhou, Doyle and Glover (1996).

LEMMA 6 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over R of sizes (n x n), (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let*

$$\boldsymbol{A} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}, \quad \boldsymbol{C} = [\mathbf{C}_1, \mathbf{0}]\mathbf{T}^{-1}.$$

with $(\mathbf{A}_{11}, \mathbf{C}_1)$ *observable. If the pair* $(\mathbf{A}, \mathbf{C})$ *is detectable then the matrix* $\mathbf{A}_{22}$ *is Hurwitz stable.*

The proof of the following Lemma is based on the results of Kucera (1991).

LEMMA 7 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices over R of sizes (n x n), (p x n), respectively, and* $\mathbf{C}$ *not zero. Further, let the pair* $(\mathbf{A}, \mathbf{C})$ *be detectable. Then, there exists a matrix* $\mathbf{K}$ *over R of size (n x p), such that the matrix* $[\mathbf{A}+\mathbf{KC}]$ *is Hurwitz stable.*

PROOF Let the pair $(\mathbf{A}, \mathbf{C})$ be detectable. Detectability of the pair $(\mathbf{A}, \mathbf{C})$ implies that the pair $(\mathbf{A}, \mathbf{C})$ is either observable or is unobservable with stable unobservable eigenvalues.

If the pair $(\mathbf{A}, \mathbf{C})$ is observable, then from Lemma 4 it follows that there exists a matrix $\mathbf{K}$ over R of appropriate dimensions, such that

$$det[\mathbf{I}_n s - \mathbf{A} - \mathbf{KC}] = \det[\mathbf{I}_n s - \mathbf{A} \; - \mathbf{KC}] = c(s) \tag{33}$$

where $c\,(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $R[s]$ of degree n. Since the notion of observability is a dual of controllability (i.e., observability of the pair ($\mathbf{A}$, $\mathbf{C}$) implies controllability of the pair($\mathbf{A}^T$, $\mathbf{C}^T$)), see Kucera (1991), the matrix $\mathbf{K}$ can be calculated using known methods for the solution of pole assignment problem by state feedback, see also Kucera (1991). Since $c(s)$ is the characteristic polynomial of the matrix [$\mathbf{A}$+$\mathbf{KC}$], from Definition 3 and (33) it follows that the matrix [$\mathbf{A}$+$\mathbf{KC}$] is Hurwitz stable.

If the pair ($\mathbf{A}$, $\mathbf{C}$) is unobservable with stable unobservable eigenvalues, then from Lemma 5 and Lemma 6 it follows that there exists a non-singular matrix $\mathbf{T}$ such that

$$\mathbf{A} = \mathbf{T}\begin{bmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}, \;\; \mathbf{C} = \;[\mathbf{C}_1, \mathbf{0}]\mathbf{T}^{-1}. \tag{34}$$

The pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) is observable and the matrix $\mathbf{A}_{22}$ is Hurwitz stable. Hurwitz stability of the matrix $\mathbf{A}_{22}$ and Definition 3 imply that the polynomial $\chi\,(s)$, given by

$$\det\,[\mathbf{I}\text{s}-\mathbf{A}_{22}] = \chi\,(s) \tag{35}$$

is a strictly Hurwitz polynomial.

Observability of the pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) and Lemma 4 imply the existence of a matrix $\mathbf{K}_1$ over R of appropriate dimensions, such that

$$det[\mathbf{I}\text{s} - \mathbf{A}_{11}-\mathbf{K}_1\mathbf{C}_1]=\varphi(s), \tag{36}$$

where $\varphi(s)$ is an arbitrary monic, strictly Hurwitz polynomial over $R[s]$ of appropriate degree. Since the notion of observability is a dual of controllability (i.e., observability of the pair ($\mathbf{A}_{11}$, $\mathbf{C}_1$) implies controllability of the pair ($\mathbf{A}_{11}^T$, $\mathbf{C}_1^T$)), the matrix $\mathbf{K}_1$ can be calculated using known methods for the solution of pole assignment problem by state feedback, see Kucera (1991). Let

$$\mathbf{K} \;= \mathbf{T}\begin{bmatrix} \mathbf{K}_1 \\ \mathbf{0} \end{bmatrix}. \tag{37}$$

Using (34) and (37), we obtain that

$$\mathbf{A} + \mathbf{KC} \;=\; \mathbf{T}\begin{bmatrix} \mathbf{A}_{11}+\mathbf{K}_1\mathbf{C}_1 & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}\mathbf{T}^{-1}. \tag{38}$$

From (35), (36) and (38), we deduce that

$$det[(\mathbf{I}_n\text{s} - \mathbf{A} - \mathbf{KC}]=\varphi(s)\chi(s). \tag{39}$$

Since by (35) and (36) the polynomials $\chi(s)$ and $\varphi(s)$ are strictly Hurwitz, the polynomial $\varphi(s)\chi(\mathrm{s})$ is also a strictly Hurwitz polynomial. Since $\varphi(s)\chi(\mathrm{s})$ is the characteristic polynomial of the matrix $[\mathbf{A}{+}\mathbf{KC}]$, from Definition 3 and (39) it follows that the matrix $[\mathbf{A}{+}\mathbf{KC}]$ is Hurwitz stable. This completes the proof. ■

Lemma 8 *Let* $\mathbf{A}$ *be a Hurwitz stable matrix over R of size (n x n). Then, the following condition holds:*

(a) The matrix $\mathbf{A}$ *is non-singular.*

Proof Let $\mathbf{A}$ be a Hurwitz stable matrix over R of size (nxn). The characteristic polynomial $a(s)$ of the matrix $\mathbf{A}$ is given by (see Lay, 2005; Gantmacher, 1959)

$$det[(\mathbf{I}_n s-\mathbf{A}] = a(s)\,. \tag{40}$$

From Definition 3 it follows that the polynomial $a(s)$ is a strictly Hurwitz polynomial over $R[s]$ of degree n. Let ξ_i for $i = 1,2,\ldots,n$, be the roots of $a(s)$. Then

$$a(\xi_i) = 0\,,\ \forall i = 1,2,...,n. \tag{41}$$

Since $a(s)$ is a strictly Hurwitz polynomial of degree n, from Remark 1 it follows that all roots of $a(s)$ lie in the open left-half complex plane, that is

$$Re(\xi_i) < 0,\ \ \forall i = 1,2,...,n. \tag{42}$$

From (42) it follows that

$$\xi_i \neq 0,\ \ \forall i = 1,2,....,n. \tag{43}$$

Since the polynomial $a(s)$ in (40) is the characteristic polynomial of the matrix $\mathbf{A}$, the complex numbers ξ_i for $i = 1,2,...,n$, which satisfy (41), are the eigenvalues of the matrix $\mathbf{A}$ (Lay, 2005; Gantmacher, 1959); therefore, from (43) it follows that the matrix $\mathbf{A}$ is non-singular (Lay, 2005; Gantmacher, 1959). This is condition (a) of the Lemma and the proof is complete. ■

4. Main results

The following Theorems 1 and 2, along with Corollary 1, are the main results of this paper.

Theorem 1 *The system (4), (5) is a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs, only if the following condition holds:*

(a) $rank[\mathbf{G}] = k$.

Proof Let the system (4), (5) be a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs. Then, the matrix $\mathbf{R}$ of size $((n+k)x(n+k))$, given by (8), is Hurwitz stable. Hurwitz stability of the matrix $\mathbf{R}$ and Lemma 8 imply that the matrix $\mathbf{R}$ is non-singular; therefore

$$rank[\mathbf{R}] = rank\begin{bmatrix} \mathbf{A}-\mathbf{LC} & \mathbf{F} \\ -\mathbf{GC} & \mathbf{0} \end{bmatrix} = (n+k). \tag{44}$$

Since, by (44), the rows of the matrix $\mathbf{R}$ are linearly independent over R, a subset of these rows consisting of the last k rows must be also linearly independent over R; therefore

$$rank[-\mathbf{GC}, \mathbf{0}] = rank[\mathbf{GC}] = k \tag{45}$$

From (3) it follows that there exists a non-singular matrix $\mathbf{T}$ of size (nxn) such that

$$\mathbf{C} \;= [\mathbf{I}_{\mathrm{p}}, \mathbf{0}]\mathbf{T}. \tag{46}$$

By substituting (46) to (45) we obtain that

$$rank[\mathbf{GC}] = \; rank\{[\mathbf{G},\; \mathbf{0}]\mathbf{T}\} = k. \tag{47}$$

Since the matrix $\mathbf{T}$ is non-singular and $rank[\mathbf{G}, \mathbf{0}] = rank[\mathbf{G}]$, condition (a) of Theorem 1 follows from (47) and the proof is complete. ■

Corollary 1 *The system (4), (5) with k integrators is a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs, only if the following condition holds:*

$(a) k \leq p.$

Proof Let the system (4), (5) with k integrators be a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs. Then, from Theorem 1, it follows that

$$rank[\mathbf{G}] = k. \tag{48}$$

Since the matrix $\mathbf{G}$ is of size (kxp), condition (a) of Corollary 1 follows from (48), and hence the proof is complete. ■

Theorem 2 *The system (4), (5) with $rank[\mathbf{G}] = k$, is a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs, if and only if the following condition holds:*

(a) The pair $(\mathbf{A}, \mathbf{C})$ is detectable.

Proof Let the system (4), (5) with $rank[\mathbf{G}] = k$, be a proportional-integral observer of order n of system (1), (2) with p linearly independent outputs. Then, the matrix $\mathbf{R}$ of size $((n+k)x(n+k))$, given by (8), is Hurwitz stable. The characteristic polynomial $c(s)$ of degree $(n+k)$ of the matrix $\mathbf{R}$ is given by

$$det\begin{bmatrix} \mathbf{I}_n\mathrm{s} - \mathbf{A}{+}\mathbf{LC} & -\mathbf{F} \\ \mathbf{GC} & \mathbf{I}_k\mathrm{s} \end{bmatrix} = c(s). \quad (49)$$

Hurwitz stability of the matrix $\mathbf{R}$ and Definition 3 imply that $c\,(s)$ is a strictly Hurwitz polynomial over $R[s]$. Let $\mathbf{V}(s)$ be a greatest common right divisor of polynomial matrices $[\mathbf{I}_n s\text{-}\,\mathbf{A}]$ and $\mathbf{C}$ of size (nxn). Then, from (15) it follows that

$$[\mathbf{I}_n s - \mathbf{A}] = \mathbf{X}(s)\mathbf{V}(s) \quad (50)$$

$$\mathbf{C} = \mathbf{Y}(s)\mathbf{V}(s). \quad (51)$$

for polynomial matrices $\mathbf{X}$(s) and $\mathbf{Y}$(s) over R[s] of appropriate dimensions. Using (50) and (51) and after simple algebraic manipulations, the relationship (49) can be rewritten as follows

$$det\begin{bmatrix} \mathbf{I}_n\mathrm{s} - \mathbf{A}{+}\mathbf{LC} & -\mathbf{F} \\ \mathbf{GC} & \mathbf{I}_k\mathrm{s} \end{bmatrix} =$$

$$det\{\begin{bmatrix} \mathbf{X}(s){+}\mathbf{LY}(s) & -\mathbf{F} \\ \mathbf{GY}(s) & \mathbf{I}_k\mathrm{s} \end{bmatrix} diag[\mathbf{V}(s), \mathbf{I}_k]\} =$$

$$det\begin{bmatrix} \mathbf{X}(s){+}\mathbf{LY}(s) & -\mathbf{F} \\ \mathbf{GY}(s) & \mathbf{I}_k\mathrm{s} \end{bmatrix} det[\mathbf{V}(s)] = c(s). \quad (52)$$

From the relationship (52) it follows that

$$det[\mathbf{V}(s)] \ divides\ [c\,(s)\,]. \quad (53)$$

Since the polynomial $c\,(s)$ is a strictly Hurwitz polynomial over $R[s]$, from (53) it follows that $det[\mathbf{V}(s)]$ is a strictly Hurwitz polynomial over $R[s]$; therefore, by Definition 2, the polynomial matrix $\mathbf{V}$(s) is strictly Hurwitz. Since the matrix $\mathbf{V}$(s) is strictly Hurwitz, from Lemma 3 it follows that the pair ($\mathbf{A}$, $\mathbf{C}$) is detectable. This is the condition (a) of the Theorem.

In order to prove sufficiency, we assume that condition (a) holds. Detectability of the pair ($\mathbf{A}$, $\mathbf{C}$) and Lemma 7 imply the existence of matrix $\mathbf{K}$ over R of size (nxp), such that the matrix $[\mathbf{A}{+}\mathbf{KC}]$ is Hurwitz stable, that is

$$det[\mathbf{I}_n\mathrm{s} - \mathbf{A}{-}\mathbf{KC}] = \pi(s), \quad (54)$$

where $\pi(s)$ is a strictly Hurwitz polynomial over $R[s]$ of degree n. The matrix $\mathbf{K}$ in (54) can be calculated as in the proof of Lemma 7.

From (3) it follows that there exists a non-singular matrix $\mathbf{T}$ of size (nxn) such that

$$\mathbf{C} \;=[\mathbf{I}_p, \mathbf{0}]\mathbf{T}. \tag{55}$$

Let $\mathbf{M}$ be an arbitrary non-singular matrix over R of size (pxp). Since, by assumption, $rank[\mathbf{G}] = k$, we put

$$\mathbf{G} \;=[\mathbf{I}_k, \mathbf{0}]\mathbf{M}. \tag{56}$$

From (56) it follows that the matrix $\mathbf{G}$, of size (kxp), has full row rank. Let Φ be an arbitrary Hurwitz stable matrix over R of size (kxk). Further, let $\mathbf{X}$ be a matrix over R of size (nxk), given by

$$\mathbf{X} = \mathbf{T}^{-1}diag[\mathrm{M}^{-1},\ \mathbf{I}_{n-p}]\begin{bmatrix} (-\mathbf{\Phi}) \\ \mathbf{\Lambda} \end{bmatrix} \tag{57}$$

where $\mathbf{\Lambda}$ is an arbitrary matrix over R of size $((n-k)xk)$. From (55), (56) and (57), we obtain:

$$-\mathbf{GCX} \;= -[\mathbf{I}_k, \mathbf{0}]\mathbf{M}[\mathbf{I}_p, \mathbf{0}]\mathbf{T}\mathbf{T}^{-1}diag[\mathrm{M}^{-1},\ \mathbf{I}_{n-p}]\begin{bmatrix} (-\mathbf{\Phi}) \\ \mathbf{\Lambda} \end{bmatrix} =$$

$$= -[\mathbf{I}_k, \mathbf{0}]\begin{bmatrix} (-\mathbf{\Phi}) \\ \mathbf{\Lambda} \end{bmatrix} = \mathbf{\Phi}. \tag{58}$$

Hurwitz stability of the matrix Φ and Definition 3 imply that the polynomial $\rho(s)$, given by

$$det[\mathbf{I}_k \mathrm{s} - \Phi] = \rho(s) \tag{59}$$

is a strictly Hurwitz polynomial over $R[s]$ of degree k. Now, we form the matrix $\mathbf{Q}$ over R of size $((n+k)x(n+k))$, see Takahashi (1996),

$$\mathbf{Q} \;= \begin{bmatrix} \mathbf{I}_n & \mathbf{X} \\ \mathbf{0} & \mathbf{I}_k \end{bmatrix}. \tag{60}$$

The matrix $\mathbf{Q}$ over R, given by (60), is non-singular and its inverse is given by

$$\mathbf{Q}^{-1} = \begin{bmatrix} \mathbf{I}_n & -\mathbf{X} \\ \mathbf{0} & \mathbf{I}_k \end{bmatrix}. \tag{61}$$

We have:

$$\mathbf{Q}^{-1}\mathbf{R}\mathbf{Q} = \mathbf{Q}^{-1}\begin{bmatrix} \mathbf{A}-\mathbf{LC} & \mathbf{F} \\ -\mathbf{GC} & \mathbf{0} \end{bmatrix}\mathbf{Q}$$

$$= \begin{bmatrix} \mathbf{A} + (-\mathbf{L}+\mathbf{XG})\mathbf{C} & (\mathbf{A}-\mathbf{LC})\,\mathbf{X}+\mathbf{F}+\mathbf{XGCX} \\ -\mathbf{GC} & -\mathbf{GCX} \end{bmatrix}. \tag{62}$$

We put

$$\mathbf{L} \ = \ \mathbf{XG} \ - \mathbf{K} \tag{63}$$

$$\mathbf{F} = -[\,(\mathbf{A} - \mathbf{LC})\,\mathbf{X} + \mathbf{XGCX}]. \tag{64}$$

Now, by substituting (58), (63) and (64) into (62), we obtain:

$$\mathbf{Q}^{-1}\mathbf{RQ} = \mathbf{Q}^{-1}\begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{GC} & \mathbf{0} \end{bmatrix}\mathbf{Q} = \begin{bmatrix} \mathbf{A}{+}\mathbf{KC} & \mathbf{0} \\ -\mathbf{GC} & \mathbf{\Phi} \end{bmatrix}. \tag{65}$$

From the relationship (65), by using (54) and (59), we deduce that

$$det\,[\mathbf{I}\mathrm{s} - \mathbf{R}] = det\begin{bmatrix} \mathbf{I}_n\mathrm{s} - \mathbf{A} - \mathbf{KC} & \mathbf{0} \\ \mathbf{GC} & \mathbf{I}_k\mathrm{s} - \mathbf{\Phi} \end{bmatrix} =$$

$$det[\mathbf{I}_n\mathrm{s} - \mathbf{A}{-}\mathbf{KC}]det[\mathbf{I}_n\mathrm{s}{-}\Phi] = \pi(s)\rho(s). \tag{66}$$

The polynomial $[\pi(s)\rho(s)]$ is the characteristic polynomial of the matrix $\mathbf{R}$, given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{A} - \mathbf{LC} & \mathbf{F} \\ -\mathbf{GC} & \mathbf{0} \end{bmatrix}, \tag{67}$$

of order n of system (1), (2). Since, according to (54) and (59), the polynomials $\pi(s)$ and $\rho(s)$ are both strictly Hurwitz, the polynomial $[\pi(s)\rho(s)]$ must be also strictly Hurwitz; therefore, according to Definition 3, the matrix $\mathbf{R}$, given by (67), with $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$ given by (63), (64) and (56), respectively, is Hurwitz stable. Thus, according to (8), the system (4), (5) with $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$ given by (63), (64) and (56), respectively, and $rank[\mathbf{G}]= k$, is a proportional-integral observer of order $n$of system (1), (2). This completes the proof. ■

The sufficiency part of the proof of Theorem 2 provides a construction of the matrices $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$ of proportional-integral observer of order n for system (1), (2). The major steps of this construction are given below.

Construction

Given: $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$

Find: $\mathbf{L}$, $\mathbf{F}$ and $\mathbf{G}$, with $rank[\mathbf{G}]= k$

Step 1: Check condition (a) of Theorem 2. If this condition is satisfied, go to *Step 2.* If condition (a) is not satisfied, the construction of a proportional-integral observer of order n is impossible.

Step 2: Detectability of the pair ($\mathbf{A}$, $\mathbf{C}$) and Lemma 7 imply the existence of a matrix $\mathbf{K}$ over R of size (nxp), such that the matrix $[\mathbf{A}{+}\mathbf{KC}]$ is Hurwitz stable. The matrix $\mathbf{K}$ can be calculated as in the proof of Lemma 7.

Step 3: Find a non-singular matrix $\mathbf{T}$ of size (nxn), such that

$\mathbf{C} = [\mathbf{I}_p,\ \mathbf{0}]\mathbf{T}$

Step 4: Let $\mathbf{M}$ be an arbitrary non-singular matrix over R of size (pxp). Further, let $\boldsymbol{\Phi}$ be an arbitrary Hurwitz stable matrix over R of size (kxk). Also let Λ be an arbitrary matrix over R of size $((n-k)xk)$. Put

$$\mathbf{X} = \mathbf{T}^{-1} diag[\mathrm{M}^{-1},\ \mathbf{I}_{n-p}] \begin{bmatrix} (-\boldsymbol{\Phi}) \\ \boldsymbol{\Lambda} \end{bmatrix}$$

$$\mathbf{G} = [\mathbf{I}_k, \mathbf{0}]\mathbf{M}$$

$$\mathbf{L} = \mathbf{XG} - \mathbf{K}$$

$$\mathbf{F} = -[(\mathbf{A} - \mathbf{LC})\,\mathbf{X} + \mathbf{XGCX}].$$

5. Computational examples

In this section, computational examples are provided, illustrating applicability of the results here forwarded.

EXAMPLE 1 Consider a linear system (1), specified by:

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \text{ and } \mathbf{C} = [0, 1]$$

$n = 2, m = 2$ and $p = 1$.

The task consists in finding a proportional-integral observer of order n which estimates the state vector of the given system.

We shall follow the steps of construction, given in the preceding section. To execute step 1, we form the following matrices

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_1 - \mathrm{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_2 - \mathrm{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ -1 & -1 \end{bmatrix}$$

where $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of the matrix $\mathbf{A}$. We have that

$$rank\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\,\lambda_1 - \mathrm{A} \end{bmatrix} = rank\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ -1 & -2 \end{bmatrix} = 2$$

$$rank\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_2\lambda_2 - \mathrm{A} \end{bmatrix} = rank\begin{bmatrix} 0 & 1 \\ 2 & 2 \\ -1 & -1 \end{bmatrix} = 2.$$

The last relationships and the Definition 5 imply that the eigenvalues λ_1 and λ_2 are observable; therefore, the given system is observable, see Trentelman, Stoorvogel and Hautus (2001). Observability of the pair ($\mathbf{A}$, $\mathbf{C}$) implies detectability of the pair ($\mathbf{A}$, $\mathbf{C}$). This fact, and Theorem 2 imply the existence of a proportional-integral observer of order n, which estimates the state vector of the given system.

Since the given system is in observability standard form, Kucera (1991), the matrix $\mathbf{K}$, given by

$$\mathbf{K} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

produces [$\mathbf{A}$+$\mathbf{KC}$] in the companion form, Kucera (1991),

$$[\mathbf{A}+\mathbf{KC}] = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

with characteristic polynomial $c(s) = s^2 + 2s + 1$. Since the roots $s_{1,2} = -1$ of the polynomial $c(s)$ have negative real parts, the polynomial $c(s)$ is strictly Hurwitz and therefore, according to Definition 3, the matrix [$\mathbf{A}$+$\mathbf{KC}$] is Hurwitz stable. This completes step 2.

In order to carry out step 3, set

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

To execute step 4, we form the following matrices

$$\mathbf{\Phi} = -1$$
$$\mathbf{\Lambda} = \mu$$
$$\mathbf{G} = 1$$
$$\mathbf{M} = 1$$
$$\mathbf{X} = \mathbf{T}^{-1} diag[\mathrm{M}^{-1}, \mathbf{I}_{n-k}] \begin{bmatrix} (-\mathbf{\Phi}) \\ \mathbf{\Lambda} \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}$$
$$\mathbf{L} = \mathbf{XG} \;\; \mathbf{K} = \begin{bmatrix} \mu{-}1 \\ 6 \end{bmatrix}$$

$$\mathbf{F} = -[(\mathbf{A} - \mathbf{LC})\,\mathbf{X} + \mathbf{XGCX}] = \begin{bmatrix} 1 \\ 2-\mu \end{bmatrix},$$

where μ is an arbitrary finite real number.

Example 2 *Consider a linear system (1), specified by:*

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$n = 3$, $m = 2$ and $p = 2$.

The task consists in finding a proportional-integral observer of order n, which estimates the state vector of the given system.

We shall follow the steps of construction, given in the last section. To execute step 1, we form the following matrix

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{I}_3 s - \mathbf{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s-2 & 0 & 0 \\ 0 & s-2 & 0 \\ -3 & 0 & s-2 \end{bmatrix}.$$

For $s = 2$ we have that

$$rank \begin{bmatrix} \mathbf{C} \\ 2\mathbf{I}_3 - \mathrm{A} \end{bmatrix} = rank \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix} = 2 < 3.$$

The last relationship and condition (a) of Lemma 1 imply that the given system is not detectable; therefore, according to Theorem 2, the construction of proportional-integral observer of order n is impossible.

6. Conclusions

The explicit necessary and sufficient conditions for the existence of full order proportional-integral observer for the state estimation of linear time-invariant continuous-time systems are established in this paper. The proof of the main results of this paper is constructive and furnishes a simple procedure for the construction of full order proportional-integral observer for the state estimation of linear time-invariant continuous-time systems.

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