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On dual problems of second order for (η, ξ)-bonvex interval-valued control problems*

by

Cristina-Florentina Pîrje[1], Savin Treanţă[1,2,3], Yating Guo[4], Balendu Bhooshan Upadhyay[5] and Fangfang Shi[4]

[1]Department of Applied Mathematics, National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania
[2]Academy of Romanian Scientists, 54 Splaiul Independentei, 050094 Bucharest, Romania
[3]Fundamental Sciences Applied in Engineering - Research Center (SFAI), National University of Science and Technology Politehnica Bucharest, 060042 Bucharest, Romania
[4]College of Science, Hohai University, Nanjing 210098, China
[5]Department of Mathematics, Indian Institute of Technology Patna, Patna, India.
Corresponding author: savin.treanta@upb.ro (Savin Treanţă)

Abstract: The paper studies the dual problems of second order, associated with a new class of (η, ξ)-bonvex interval-valued variational control problems. More precisely, by considering the corresponding necessary optimality conditions, we prove the associated duality (weak, strong, strictly converse) results under the new (η, ξ)-bonvexity assumptions of the involved functionals. In addition, illustrative examples are provided in order to highlight the theoretical elements established in the paper.

Keywords: control; interval-valued optimization problem; dual problem

1. Introduction

Calculus of Variations is the branch that still retains the essence of natural philosophy that permeated mathematical analysis before the 20^{th} century. Natural philosophy was the precursor to modern science and encompassed a wide range of disciplines, including physics, astronomy, chemistry, and mathematics.

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During this period, there was a deep-seated belief that the laws governing the physical world could be discovered through logical reasoning, observation, and mathematical analysis. Variational calculus is a mathematical framework that emerged during this era and it played a significant role in preserving and advancing this spirit of natural philosophy. Variational calculus focuses on finding solutions to optimization problems, particularly those involving the minimization or maximization of certain physical quantities. One of the most famous early applications of variational calculus is the Bernoulli problem of identifying the brachistochrone.

Real-world optimization problems are often characterized by ambiguous or uncertain information, due to the complexity of the environment, in which they arise. It is inherently unpredictable, and there are numerous factors that can introduce uncertainties into the problem at hand. These uncertainties can arise from various sources, including measurement errors, variability in system behavior, limited knowledge about the problem domain, and external factors that affect the problem parameters. To overcome this shortcoming, using fuzzy numbers to represent data and thus build fuzzy optimization models has been a popular research direction in recent years. For more excellent results on fuzzy optimization, the interested reader is referred to Al-Awami et al. (2017), Babazadeh (2019), Guo et al. (2021, 2022, 2023). On the other hand, to address uncertainty in an optimization problem, interval-valued and robust optimization are also growing areas of applied mathematics and can provide an alternative choice for considering uncertainty. Over time, several researchers have been involved in obtaining solution procedures in robust control and interval analysis (see, for instance, Moore, 1979, Ishibuch and Tanaka, 1990, Ahmad et al., 2019, Treanţă, 2021b,c, 2022b).

Duality theory is an effective method for examining (primal) optimization problems. It offers a complementary perspective through the dual problem, allowing us to derive insights, obtain bounds, and solve optimization problems more effectively, particularly in the context of control programming and under convexity assumptions. In this regard, the duality theory for a class of control programming problems has been established under convexity assumptions by Mond and Hanson (1968). Beck and Ben-Tal (2009) demonstrated that the primal worst is equivalent to the dual best and reached a robust optimization solution via duality. Jeyakumar, Li and Lee (2012) established the robust duality results under data uncertainty for generalized convex problems. In recent years, the utilization of optimization problems has been witnessed in a wide range of fields such as artificial intelligence (see Koopialipoor and Noorbakhsh, 2020), orbit optimization (see Esmaelzadeh, 2014), and other diverse areas. Treanţă (2021a) studied a novel class of multi-objective optimization problems with a duality analysis incorporating the interval-valued ratio vector components. Based on Becor, Chandra and Husain (1984), and Bector and Husain

(1992), recent advances in this field of research can be seen in Upadhyay et al. (2022), and Upadhyay, Gosh and Treanţă (2023). Recently, Dhingra and Kailey (2022) studied optimality conditions and duality theory for second-order interval-valued variational problems. More precisely, they formulated the necessary optimality conditions and, under the assumptions of η-bonvexity, they established the duality results (that is, weak and strong duality models) to relate the values of the two problems.

In this paper, motivated by the results stated in Dhingra and Kailey (2022), and following Treanţă (2022a), we introduce and investigate a novel class of dual problems of second order, associated with a new class of (η, ξ)-bonvex interval-valued variational control problems. The main merit of this study is the presence of the control variable, implying the introduction of the new concept of (η, ξ)-bonvexity for a controlled real-valued integral-type functional, which is the key element for proving the main results derived in this paper. We notice that the considered classes of extremal optimization and control problems extend the problems investigated by Ahmad et al. (2019) and Bector, Chandra and Husain (1984) and Bector and Husain (1992) (see Remark 2). Also, the presence of mixed-type constraints (equality and inequality constraints) is another element of contribution relative to the study carried out by Dhingra and Kailey (2022). More specifically, by considering the corresponding optimality necessary conditions, we prove the associated duality (weak, strong, strictly converse) results under the new (η, ξ)-bonvexity assumptions of the involved functionals. In addition, illustrative examples are provided in order to highlight the theoretical elements established in the paper. Consequently, because the families of variational models considered in this article cover more general cases (compared to previous studies) of optimization problems, and due to the applicability of control in various branches of science, the present work represents a study of real importance for the academic community.

The following outline presents the structure of this article. Section 2 introduces important preliminaries, addresses the concept of (η, ξ)-bonvexity for a controlled real-valued integral-type functional, and formulates the variational control problem under study. In Section 3, the dual problem of second-order associated with the primal control problem is formulated, and strong, weak, and strictly converse duality results are established. Section 4 includes an illustrative example, associated with the theoretical results, established in this paper, and Section 5 concludes the paper.

2. Preliminary results and notions

In the following, we consider $\mathcal{B} = [t_0, t_1] \subset \mathbb{R}$ and $\mathbb{R}^n$ be the classical Euclidean space. Also, let $\mathcal{A}$ be the space of piecewise differentiable state functions γ :

$\mathcal{B} \to \mathbb{R}^n$, with the norm

$$\|\gamma\| = \|\gamma\|_\infty + \|\dot{\gamma}\|_\infty,$$

and consider $\mathcal{U}$ to be the space of piecewise continuous control functions $u : \mathcal{B} \to \mathbb{R}^l$, equipped with the uniform norm. Moreover, for notational convenience, we represent $\gamma(t)$ and $\dot{\gamma}(t)$ by γ and $\dot{\gamma}$, respectively. Denote by q_γ and q_u the gradient vectors of $q : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathbb{R}$, $q = q(t, \gamma, \dot{\gamma}, u)$, with respect to γ and u, that is:

$$q_\gamma = \begin{pmatrix} \frac{\partial q}{\partial \gamma_1} \\ \frac{\partial q}{\partial \gamma_2} \\ \vdots \\ \frac{\partial q}{\partial \gamma_n} \end{pmatrix} \quad \text{and} \quad q_u = \begin{pmatrix} \frac{\partial q}{\partial u_1} \\ \frac{\partial q}{\partial u_2} \\ \vdots \\ \frac{\partial q}{\partial u_l} \end{pmatrix},$$

and $q_{\gamma\gamma}$ is the Hessian matrix, associated with q, with respect to γ (similarly, $q_{\dot{\gamma}\gamma}, q_{\gamma\dot{\gamma}}$ and $q_{\dot{\gamma}\dot{\gamma}}$):

$$q_{\gamma\gamma} = \begin{pmatrix} \frac{\partial^2 q}{\partial \gamma_1 \partial \gamma_1} & \frac{\partial^2 q}{\partial \gamma_1 \partial \gamma_2} & \cdots & \frac{\partial^2 q}{\partial \gamma_1 \partial \gamma_n} \\ \frac{\partial^2 q}{\partial \gamma_2 \partial \gamma_1} & \frac{\partial^2 q}{\partial \gamma_2 \partial \gamma_2} & \cdots & \frac{\partial^2 q}{\partial \gamma_2 \partial \gamma_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 q}{\partial \gamma_n \partial \gamma_1} & \frac{\partial^2 q}{\partial \gamma_n \partial \gamma_2} & \cdots & \frac{\partial^2 q}{\partial \gamma_n \partial \gamma_n} \end{pmatrix}.$$

Let $\mathcal{I}$ be the class of bounded and closed real intervals. If $S \in \mathcal{I}$, we use the notation $S = \left[s^L, s^U\right] = \left\{s : s^L \leq s \leq s^U\right\}$. If $s^L = s^U = s \in \mathbb{R}$, then $S = [s, s] = s$. For $S = \left[s^L, s^U\right]$ and $W = \left[w^L, w^U\right] \in \mathcal{I}$, we consider:

(i) $S = W \Longrightarrow s^L = w^L$ and $s^U = w^U$,

(ii) $S + W = \{s + w : s \in S \text{ and } w \in W\} = \left[s^L + w^L, s^U + w^U\right]$,

(iii) $-S = \{-s : s \in S\} = \left[-s^U, -s^L\right]$.

In consequence, $S - W = S + (-W) = \left[s^L - w^U, s^U - w^L\right]$. Also, for $c \in \mathbb{R}$, we consider:

(iv) $c + S = \{c + s : s \in S\} = \left[c + s^L, c + s^U\right]$

(v) $cS = \{cs : s \in S\} = \begin{cases} \left[cs^L, cs^U\right] & \text{if } c \geq 0 \\ \left[cs^U, cs^L\right] & \text{if } c < 0. \end{cases}$

In interval analysis, some partial orderings are defined in Moore (1979), Ishibuch and Tanaka (1990), and Treanţă (2022a). In this paper, by using the notations "$\preceq_{LU}$" and "$\prec_{LU}$", the following rules are assumed:

$$S \preceq_{LU} W \text{ if and only if } s^L \leq w^L \text{ and } s^U \leq w^U,$$

and

$$S \prec_{LU} W \text{ if and only if } S \preceq_{LU} W \text{ and } S \neq W,$$

or, equivalently,

$$\begin{aligned} s^L < w^L,\ s^U < w^U, \\ \text{or, } s^L \leq w^L,\ s^U < w^U, \\ \text{or, } s^L < w^L,\ s^U \leq w^U. \end{aligned}$$

Now, we formulate the definition of (η, ξ)-bonvexity for a controlled real-valued integral-type functional.

Definition 1 *The functional* $\mathfrak{G}(\gamma, u) = \displaystyle\int_{t_0}^{t_1} q(t, \gamma, \dot{\gamma}, u)dt$ *is said to be bonvex at* $(z, v) \in \mathcal{A} \times \mathcal{U}$ *with respect to the* C^1*-class functions* $\eta : \mathcal{B} \times (\mathcal{A} \times \mathcal{U})^2 \to \mathbb{R}^n$, $\xi : \mathcal{B} \times (\mathcal{A} \times \mathcal{U})^2 \to \mathbb{R}^l$ *(or,* (η, ξ)*-bonvex at* $(z, v) \in \mathcal{A} \times \mathcal{U}$*), where* $\eta|_{t=t_0,t_1} = \xi|_{t=t_0,t_1} = 0$*, if for all* $(\gamma, u) \in \mathcal{A} \times \mathcal{U}$ *and* $b(t) \in \mathbb{R}^n$*, the following inequality is satisfied:*

$$\begin{aligned} &\int_{t_0}^{t_1} \left\{ q(t, \gamma, \dot{\gamma}, u) - q(t, z, \dot{z}, v) + \frac{1}{2} b^T \mathrm{G} b \right\} dt \\ &\quad \geq \int_{t_0}^{t_1} \eta(t, \gamma, u, z, v) \Big\{ q_\gamma(t, z, \dot{z}, v) - D q_{\dot{\gamma}}(t, z, \dot{z}, v) \\ &+ \mathrm{G} b \Big\} dt + \int_{t_0}^{t_1} \xi(t, \gamma, u, z, v) q_u(t, z, \dot{z}, v) dt, \end{aligned}$$

where $\mathrm{G} := (q_{\gamma\gamma} + q_{u\gamma})(t, z, \dot{z}, v) - D(q_{\gamma\dot{\gamma}} + q_{u\dot{\gamma}})(t, z, \dot{z}, v)$ *and* $D := \frac{d}{dt}$.

If the previous inequality is a strict inequality for $(\gamma, u) \in \mathcal{A} \times \mathcal{U}$, $(\gamma, u) \neq (z, v)$, then the functional is named *strictly* (η, ξ)*-bonvex at* (z, v).

Remark 1 *For* $b = 0$*, the previous definition becomes the definition of* (η, ξ)*-invex controlled integral-type functionals.*

Example 1 *Consider* $\mathcal{B} = [1, 4]$ *and define* $\mathcal{A}$ *as the family of piecewise differentiable state functions* $\gamma : \mathcal{B} \to [0, \infty)$*. Also, consider* $\mathcal{U}$ *as the space of piecewise continuous control functions* $u : \mathcal{B} \to \mathbb{R}$*. Define* $q : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathbb{R}$ *as* $q(t, \gamma, \dot{\gamma}, u) = \gamma^2 - \gamma$*,* $\eta, \xi : \mathcal{B} \times (\mathcal{A} \times \mathcal{U})^2 \to \mathbb{R}$ *let be defined as* $\eta(t, \gamma, u, z, v) = \xi(t, \gamma, u, z, v) = z - \gamma$*, and* $b : \mathcal{B} \to \mathbb{R}$ *as* $b(t) = (t + 1)^2$*. Thus,* $\mathrm{G} = (q_{\gamma\gamma} + q_{u\gamma})(t, z, \dot{z}, v) - D(q_{\gamma\dot{\gamma}} + q_{u\dot{\gamma}})(t, z, \dot{z}, v) = 2 - 0 = 2$*. Since the* *inequality*

$$\begin{aligned}
&\int_1^4 \left(q(t,\gamma,\dot{\gamma},u) - q(t,z,\dot{z},v) + \frac{1}{2}b^T \mathrm{G} b\right) dt \\
&\qquad - \int_1^4 \left(\eta(t,\gamma,u,z,v)\left\{q_\gamma(t,z,\dot{z},v) - Dq_{\dot{\gamma}}(t,z,\dot{z},v) + \mathrm{G}b\right\}\right) dt \\
&\qquad - \int_1^4 \xi(t,\gamma,u,z,v) q_u(t,z,\dot{z},v) dt \\
&= \int_1^4 \left(\gamma^2 - \gamma - \left(z^2 - z\right) + \frac{1}{2}(t+1)^2 2(t+1)^2\right) \\
&\qquad - \left((z-\gamma)\left\{2z - 1 + 2(t+1)^2\right\}\right) dt \\
&= \int_1^4 \left(\gamma^2 - \gamma + (t+1)^4 - \left((z-\gamma)\left\{2z-1+2(t+1)^2\right\}\right) - \left(z^2 - z\right)\right) dt \\
&= \int_1^4 \left(\gamma^2 + 2\gamma\left((t+1)^2 - 1\right) + (t+1)^4\right) dt \geq 0,
\end{aligned}$$

is satisfied, for $(\gamma,u) \in \mathcal{A} \times \mathcal{U}$ *and* $(z,v) = (0,0)$*, therefore, the functional* $\int_1^4 q(t,\gamma,\dot{\gamma},u)dt$ *is* (η,ξ)*-bonvex at* $(0,0)$*. But, it is not an* (η,ξ)*-invex function at* $(0,0)$ *as:*

$$\begin{aligned}
&\int_1^4 (q(t,\gamma,\dot{\gamma},u) - q(t,z,\dot{z},v)) - \left(\eta(t,\gamma,u,z,v)\left\{q_\gamma(t,z,\dot{z},v)\right.\right. \\
&\qquad \left.\left. - Dq_{\dot{\gamma}}(t,z,\dot{z},v)\right\}\right) dt - \int_1^4 \xi(t,\gamma,u,z,v) q_u(t,z,\dot{z},v) dt \\
&= \int_1^4 \left(\gamma^2 - \gamma - \left(z^2 - z\right)\right) - ((z-\gamma)\{2z-1\}) dt \\
&= \int_1^4 \left(\gamma^2 - \gamma + \gamma(-1)\right) dt \\
&= \int_1^4 \left(\gamma^2 - 2\gamma\right) dt \ngeq 0,
\end{aligned}$$

for $(\gamma,u) \in \mathcal{A} \times \mathcal{U}$ *and* $(z,v) = (0,0)$.

DEFINITION 2 *A functional* $\mathcal{F} : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathcal{I}$*, delineated by*

$$\mathcal{F}(t,\gamma,\dot{\gamma},u) = \left[\mathfrak{f}^L(t,\gamma,\dot{\gamma},u), \mathfrak{f}^U(t,\gamma,\dot{\gamma},u)\right],$$

where $\mathfrak{f}^L(t,\gamma,\dot{\gamma},u) \leq \mathfrak{f}^U(t,\gamma,\dot{\gamma},u)$, $\forall t \in \mathcal{B}$*, is called an interval-valued controlled integral-type functional.*

Next, we introduce the following problem of minimizing an interval-valued controlled integral-type functional:

$$(P) \quad \min_{(\gamma,u)} C(\gamma, u) = \left[\int_{t_0}^{t_1} \mathfrak{q}^L(t,\gamma,\dot{\gamma},u)dt, \int_{t_0}^{t_1} \mathfrak{q}^U(t,\gamma,\dot{\gamma},u)dt\right]$$

subject to

$$\gamma(t_0) = \alpha, \quad \gamma(t_1) = \beta, \tag{1}$$

$$h(t,\gamma,\dot{\gamma},u) \leq 0, \quad t \in \mathcal{B}, \tag{2}$$

$$f(t,\gamma,u) = \dot{\gamma}, \quad t \in \mathcal{B}, \tag{3}$$

where $\mathfrak{q} : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathcal{I}$ is an interval-valued functional, and $\mathfrak{q}^L, \mathfrak{q}^U : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathbb{R}$, $h : \mathcal{B} \times \mathcal{A} \times \mathcal{A} \times \mathcal{U} \to \mathbb{R}^m$ and $f : \mathcal{B} \times \mathcal{A} \times \mathcal{U} \to \mathbb{R}^n$ are continuously differentiable functionals.

Let us denote the class of feasible points for (P) as:

$$Z = \Big\{(\gamma, u) \in \mathcal{A} \times \mathcal{U} : \gamma(t_0) = \alpha, \gamma(t_1) = \beta, h(t,\gamma,\dot{\gamma},u) \leq 0,$$

$$f(t,\gamma,u) = \dot{\gamma}, t \in \mathcal{B}\Big\}.$$

Definition 3 *The pair $(\gamma, u) \in Z$ is an LU-optimal solution for (P) if $C(\bar{\gamma}, \bar{u}) \prec_{LU} C(\gamma, u)$ is not fulfilled, for all $(\bar{\gamma}, \bar{u}) \in Z$.*

Now, in accordance with Ahmad et al. (2019) and Treanţă (2021b), we formulate the necessary optimality conditions for (P).

Theorem 1 (Necessary optimality conditions) *Let $(\bar{\gamma}, \bar{u})$ be an LU-optimal solution of (P). Then there exist $\iota : \mathcal{B} \to \mathbb{R}^2, \iota = \left(\iota^L, \iota^U\right) \geq 0, \iota \neq 0$, $\zeta : \mathcal{B} \to \mathbb{R}^m, \zeta \geq 0$, and $\nu : \mathcal{B} \to \mathbb{R}^n$, such that the following relations*

(i) $\iota^L \mathfrak{q}^L_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \iota^U \mathfrak{q}^U_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \zeta^T h_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \nu^T f_\gamma(t,\bar{\gamma},\bar{u})$
$- D\left[\iota^L \mathfrak{q}^L_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \iota^U \mathfrak{q}^U_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) \ + \zeta^T h_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) - \nu^T\right] = 0,$

(ii) $\iota^L \mathfrak{q}^L_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \iota^U \mathfrak{q}^U_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \zeta^T h_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) + \nu^T f_u(t,\bar{\gamma},\bar{u}) = 0,$

(iii) $\zeta^T h(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u}) = 0, \quad t \in \mathcal{B},$

are fulfilled, except at discontinuity points, where $D := \frac{d}{dt}$.

Proof The proof associated with this result can be inferred from Ahmad et al. (2019) (for instance) and so we omitted it. ■

3. Second-order dual problems associated with (P)

In the following, we formulate the *dual problem of second-order* associated with the primal control problem (P):

$$\text{(Dual)} \quad \max_{(z,v)} J(z,v,\iota,\zeta,\nu,b)$$
$$= \Big[\int_{t_0}^{t_1}\{\mathfrak{q}^L(t,z,\dot z,v) - \frac{1}{2}b^T\left((\mathfrak{q}^L_{\gamma\gamma}+\mathfrak{q}^L_{u\gamma})(t,z,\dot z,v) - D(\mathfrak{q}^L_{\gamma\dot\gamma}+\mathfrak{q}^L_{u\dot\gamma})(t,z,\dot z,v)\right)b\}dt\,,$$
$$\int_{t_0}^{t_1}\{\mathfrak{q}^U(t,z,\dot z,v) - \frac{1}{2}b^T\left((\mathfrak{q}^U_{\gamma\gamma}+\mathfrak{q}^U_{u\gamma})(t,z,\dot z,v) - D(\mathfrak{q}^U_{\gamma\dot\gamma}+\mathfrak{q}^U_{u\dot\gamma})(t,z,\dot z,v)\right)b\}dt\Big]$$

subject to

$$z(t_0)=\alpha, \quad z(t_1)=\beta \tag{4}$$

$$\iota^L\mathfrak{q}^L_\gamma(t,z,\dot z,v)+\iota^U\mathfrak{q}^U_\gamma(t,z,\dot z,v)+\zeta^T h_\gamma(t,z,\dot z,v)+\nu^T f_\gamma(t,z,v)$$
$$-D\left(\iota^L\mathfrak{q}^L_{\dot\gamma}(t,z,\dot z,v)+\iota^U\mathfrak{q}^U_{\dot\gamma}(t,z,\dot z,v)+\zeta^T h_{\dot\gamma}(t,z,\dot z,v)-\nu^T\right)$$
$$+(G+H+F)b=0, \quad t\in\mathcal{B}, \tag{5}$$

$$\iota^L\mathfrak{q}^L_u(t,z,\dot z,v)+\iota^U\mathfrak{q}^U_u(t,z,\dot z,v)+\zeta^T h_u(t,z,\dot z,v)+\nu^T f_u(t,z,v)=0, \quad t\in\mathcal{B}, \tag{6}$$

$$\int_{t_0}^{t_1}\{\zeta^T h(t,z,\dot z,v)-\frac{1}{2}b^T Hb\}dt\geq 0, \quad t\in\mathcal{B}, \tag{7}$$

$$\int_{t_0}^{t_1}\{\nu^T[f(t,z,v)-\dot z]-\frac{1}{2}b^T Fb\}dt\geq 0, \quad t\in\mathcal{B}, \tag{8}$$

$$\iota=\left(\iota^L,\ \iota^U\right)\geq 0,\ \iota\neq 0,\ \zeta\geq 0, \tag{9}$$

where

$b:\mathcal{B}\to\mathbb{R}^n,\ \zeta:\mathcal{B}\to\mathbb{R}^m,\ \iota:\mathcal{B}\to\mathbb{R}^2,\ \nu:\mathcal{B}\to\mathbb{R}^n,$

$G=\iota^L\mathfrak{q}^L_{\gamma\gamma}(t,z,\dot z,v)+\iota^L\mathfrak{q}^L_{u\gamma}(t,z,\dot z,v)+\iota^U\mathfrak{q}^U_{\gamma\gamma}(t,z,\dot z,v)+\iota^U\mathfrak{q}^U_{u\gamma}(t,z,\dot z,v)$
$-D\left(\iota^L\mathfrak{q}^L_{\gamma\dot\gamma}+\iota^L\mathfrak{q}^L_{u\dot\gamma}+\iota^U\mathfrak{q}^U_{\gamma\dot\gamma}+\iota^U\mathfrak{q}^U_{u\dot\gamma}\right)(t,z,\dot z,v),$

$H=\zeta^T h_{\gamma\gamma}(t,z,\dot z,v)+\zeta^T h_{u\gamma}(t,z,\dot z,v)-D\left(\zeta^T h_{\gamma\dot\gamma}(t,z,\dot z,v)+\zeta^T h_{u\dot\gamma}(t,z,\dot z,v)\right),$

$F=\nu^T f_{\gamma\gamma}(t,z,\dot z,v)+\nu^T f_{u\gamma}(t,z,\dot z,v)-D\left(\nu^T f_{\gamma\dot\gamma}(t,z,\dot z,v)+\nu^T f_{u\dot\gamma}(t,z,\dot z,v)\right).$

The class of feasible points for (Dual) is:

$$Z_1=\{(z,v,\iota,\zeta,\nu,b)\text{ satisfying (4)-(9)}\}.$$

REMARK 2 *(i) For $b = 0$ and if we remove the control variable, then (P) and (Dual) become the problems investigated by Ahmad et al. (2019).*

(ii) For $\mathfrak{q}^L = \mathfrak{q}^U, \iota^L = \iota^U$, $b = 0$, and if we remove the control variable, then (P) and (Dual) become the problems investigated in Bector, Chandra and Husain (1984) and Bector and Husain (1992).

REMARK 3 *If we take $\mathfrak{q}^L_\gamma(t,\gamma,\dot\gamma,u)+\mathfrak{q}^L_u(t,\gamma,\dot\gamma,u)-\frac{d}{dt}\mathfrak{q}^L_{\dot\gamma}(t,\gamma,\dot\gamma,u) = L^L(t,\gamma,\dot\gamma,u)$ and by using the observations:*

$$\begin{aligned}
&D\mathfrak{q}^L_{\dot\gamma} = \mathfrak{q}^L_{\dot\gamma t} + \mathfrak{q}^L_{\dot\gamma\gamma}\dot\gamma + \mathfrak{q}^L_{\dot\gamma\dot\gamma}\ddot\gamma + \mathfrak{q}^L_{\dot\gamma u}\dot u,\\
&\frac{\partial}{\partial\gamma}D\mathfrak{q}^L_{\dot\gamma} = D\mathfrak{q}^L_{\dot\gamma\gamma},\\
&\frac{\partial}{\partial\dot\gamma}D\mathfrak{q}^L_{\dot\gamma} = D\mathfrak{q}^L_{\dot\gamma\dot\gamma} + \mathfrak{q}^L_{\dot\gamma\gamma},\\
&\frac{\partial}{\partial\ddot\gamma}D\mathfrak{q}^L_{\dot\gamma} = \mathfrak{q}^L_{\dot\gamma\dot\gamma} \ \textit{where}\ D := \frac{d}{dt},
\end{aligned}$$

we obtain

$$\begin{aligned}
G^L &= L^L_\gamma(t,\gamma,\dot\gamma,u) - \frac{d}{dt}L^L_{\dot\gamma}(t,\gamma,\dot\gamma,u) + \frac{d^2}{dt^2}L^L_{\ddot\gamma}(t,\gamma,\dot\gamma,u)\\
&= \frac{\partial}{\partial\gamma}\left(\mathfrak{q}^L_\gamma(t,\gamma,\dot\gamma,u) + \mathfrak{q}^L_u(t,\gamma,\dot\gamma,u) - \frac{d}{dt}\mathfrak{q}^L_{\dot\gamma}(t,\gamma,\dot\gamma,u)\right)\\
&- \frac{d}{dt}\left(\frac{\partial}{\partial\dot\gamma}\left(\mathfrak{q}^L_\gamma(t,\gamma,\dot\gamma,u) + \mathfrak{q}^L_u(t,\gamma,\dot\gamma,u) - \frac{d}{dt}\mathfrak{q}^L_{\dot\gamma}(t,\gamma,\dot\gamma,u)\right)\right)\\
&+ \frac{d^2}{dt^2}\left(\frac{\partial}{\partial\ddot\gamma}\left(\mathfrak{q}^L_\gamma(t,\gamma,\dot\gamma,u) + \mathfrak{q}^L_u(t,\gamma,\dot\gamma,u) - \frac{d}{dt}\mathfrak{q}^L_{\dot\gamma}(t,\gamma,\dot\gamma,u)\right)\right)\\
&= \mathfrak{q}^L_{\gamma\gamma} + \mathfrak{q}^L_{u\gamma} - \frac{d}{dt}\mathfrak{q}^L_{\dot\gamma\gamma}\\
&- \frac{d}{dt}\mathfrak{q}^L_{\gamma\dot\gamma} - \frac{d}{dt}\mathfrak{q}^L_{u\dot\gamma} + \frac{d^2}{dt^2}\mathfrak{q}^L_{\dot\gamma\dot\gamma} + \frac{d}{dt}\mathfrak{q}^L_{\dot\gamma\gamma}\\
&- \frac{d^2}{dt^2}\mathfrak{q}^L_{\dot\gamma\dot\gamma}\\
&= \mathfrak{q}^L_{\gamma\gamma} + \mathfrak{q}^L_{u\gamma} - \frac{d}{dt}\mathfrak{q}^L_{\gamma\dot\gamma} - \frac{d}{dt}\mathfrak{q}^L_{u\dot\gamma}.
\end{aligned}$$

Similarly, we obtain $G^U = \mathfrak{q}^U_{\gamma\gamma} + \mathfrak{q}^U_{u\gamma} - \frac{d}{dt}\mathfrak{q}^U_{\gamma\dot\gamma} - \frac{d}{dt}\mathfrak{q}^U_{u\dot\gamma}$, and as $G = \iota^L G^L + \iota^U G^U$, this implies

$$G = \iota^L\mathfrak{q}^L_{\gamma\gamma} + \iota^L\mathfrak{q}^L_{u\gamma} + \iota^U\mathfrak{q}^U_{\gamma\gamma} + \iota^U\mathfrak{q}^U_{u\gamma} - D\left(\iota^L\mathfrak{q}^L_{\gamma\dot\gamma} + \iota^L\mathfrak{q}^L_{u\dot\gamma} + \iota^U\mathfrak{q}^U_{\gamma\dot\gamma} + \iota^U\mathfrak{q}^U_{u\dot\gamma}\right).$$

In the same manner, we compute H and F, as well.

Now, we will formulate and prove a weak-type duality theorem to connect the values of (P) and (Dual).

Theorem 2 *Consider* $(\gamma, u) \in Z$ *and* $(z, v, \iota, \zeta, \nu, b) \in Z_1$ *and assume that:*

(i) $\iota = \left(\iota^L, \iota^U\right) > 0$;

(ii) $\int_{t_0}^{t_1} \left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right)(t, \gamma, \dot{\gamma}, u) dt$ *is* (η, ξ)*-bonvex at* (z, v);

(iii) $\int_{t_0}^{t_1} \zeta^T h(t, \gamma, \dot{\gamma}, u) dt$ *is* (η, ξ)*-bonvex at* (z, v);

(iv) $\int_{t_0}^{t_1} \nu^T [f(t, \gamma, u) - \dot{\gamma}] dt$ *is* (η, ξ)*-bonvex at* (z, v).

Then, the following relation is fulfilled

$$C(\gamma, u) \not\prec_{LU} J(z, v, \iota, \zeta, \nu, b).$$

Proof Let us consider, contrary to the results, that

$$C(\gamma, u) \prec_{LU} J(z, v, \iota, \zeta, \nu, b)$$

is satisfied. This implies

$$\begin{cases} \int_{t_0}^{t_1} \mathfrak{q}^L(t, \gamma, \dot{\gamma}, u) dt < \int_{t_0}^{t_1} \left(\mathfrak{q}^L(t, z, \dot{z}, v) - \frac{1}{2} b^T G^L(t, z, \dot{z}, v) b\right) dt, \\ \int_{t_0}^{t_1} \mathfrak{q}^U(t, \gamma, \dot{\gamma}, u) dt < \int_{t_0}^{t_1} \left(\mathfrak{q}^U(t, z, \dot{z}, v) - \frac{1}{2} b^T G^U(t, z, \dot{z}, v) b\right) dt, \end{cases}$$

or

$$\begin{cases} \int_{t_0}^{t_1} \mathfrak{q}^L(t, \gamma, \dot{\gamma}, u) dt \leq \int_{t_0}^{t_1} \left(\mathfrak{q}^L(t, z, \dot{z}, v) - \frac{1}{2} b^T G^L(t, z, \dot{z}, v) b\right) dt, \\ \int_{t_0}^{t_1} \mathfrak{q}^U(t, \gamma, \dot{\gamma}, u) dt < \int_{t_0}^{t_1} \left(\mathfrak{q}^U(t, z, \dot{z}, v) - \frac{1}{2} b^T G^U(t, z, \dot{z}, v) b\right) dt, \end{cases}$$

or

$$\begin{cases} \int_{t_0}^{t_1} \mathfrak{q}^L(t, \gamma, \dot{\gamma}, u) dt < \int_{t_0}^{t_1} \left(\mathfrak{q}^L(t, z, \dot{z}, v) - \frac{1}{2} b^T G^L(t, z, \dot{z}, v) b\right) dt, \\ \int_{t_0}^{t_1} \mathfrak{q}^U(t, \gamma, \dot{\gamma}, u) dt \leq \int_{t_0}^{t_1} \left(\mathfrak{q}^U(t, z, \dot{z}, v) - \frac{1}{2} b^T G^U(t, z, \dot{z}, v) b\right) dt. \end{cases}$$

The above inequalities, together with the hypothesis (i), give

$$\begin{aligned} &\int_{t_0}^{t_1} \left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right)(t, \gamma, \dot{\gamma}, u) dt \\ &< \int_{t_0}^{t_1} \left(\left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right)(t, z, \dot{z}, v) - \frac{1}{2} b^T G b\right) dt. \end{aligned} \tag{10}$$

Now, by using the constraints (2), (7) and (9), we have

$$\int_{t_0}^{t_1} \zeta^T h(t, \gamma, \dot{\gamma}, u) dt \leq \int_{t_0}^{t_1} \left(\zeta^T h(t, z, \dot{z}, v) - \frac{1}{2} b^T H b\right) dt. \tag{11}$$

Since $\int_{t_0}^{t_1} \zeta^T h \, dt$ is (η, ξ)-bonvex at (z, v), if we also consider (11), we get

$$\int_{t_0}^{t_1} \eta(t, \gamma, u, z, v) \left\{ \zeta^T h_\gamma(t, z, \dot{z}, v) - D\left(\zeta^T h_{\dot{\gamma}}(t, z, \dot{z}, v)\right) + Hb \right\} dt$$

$$+ \int_{t_0}^{t_1} \xi(t, \gamma, u, z, v) \zeta^T h_u(t, z, \dot{z}, v) dt \leq 0. \tag{12}$$

From the constraints (5) and (6), we have

$$\int_{t_0}^{t_1} \eta(t, \gamma, u, z, v) \Big\{ \iota^L \mathfrak{q}_\gamma^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_\gamma^U(t, z, \dot{z}, v) + \zeta^T h_\gamma(t, z, \dot{z}, v) + \nu^T f_\gamma(t, z, v)$$

$$-D\left(\iota^L \mathfrak{q}_{\dot{\gamma}}^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_{\dot{\gamma}}^U(t, z, \dot{z}, v) + \zeta^T h_{\dot{\gamma}}(t, z, \dot{z}, v) - \nu^T\right)$$

$$+(G + H + F)b \Big\} dt = 0,$$

$$\int_{t_0}^{t_1} \xi(t, \gamma, u, z, v) \Big\{ \iota^L \mathfrak{q}_u^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_u^U(t, z, \dot{z}, v)$$

$$+\zeta^T h_u(t, z, \dot{z}, v) + \nu^T f_u(t, z, v) \Big\} dt = 0,$$

and using (12), we obtain

$$\int_{t_0}^{t_1} \eta(t, \gamma, u, z, v) \Big\{ \iota^L \mathfrak{q}_\gamma^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_\gamma^U(t, z, \dot{z}, v) + \nu^T f_\gamma(t, z, v)$$

$$-D\left(\iota^L \mathfrak{q}_{\dot{\gamma}}^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_{\dot{\gamma}}^U(t, z, \dot{z}, v) - \nu^T\right)$$

$$+(G + F)b \Big\} dt \geq 0,$$

$$\int_{t_0}^{t_1} \xi(t, \gamma, u, z, v) \Big\{ \iota^L \mathfrak{q}_u^L(t, z, \dot{z}, v) + \iota^U \mathfrak{q}_u^U(t, z, \dot{z}, v) + \nu^T f_u(t, z, v) \Big\} dt \geq 0.$$

Using the above equation and the hypothesis that

$$\int_{t_0}^{t_1} \left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right) dt$$

and

$$\int_{t_0}^{t_1} \nu^T [f(t, \gamma, u) - \dot{\gamma}] dt$$

are (η,ξ)-bonvex at (z,v), we have

$$\begin{aligned}&\int_{t_0}^{t_1}\left(\iota^L\mathfrak{q}^L+\iota^U\mathfrak{q}^U\right)(t,\gamma,\dot{\gamma},u)dt\\&\geq\int_{t_0}^{t_1}\left(\left(\iota^L\mathfrak{q}^L+\iota^U\mathfrak{q}^U\right)(t,z,\dot{z},v)-\frac{1}{2}b^TGb\right)dt,\end{aligned}$$

a contradiction to (10). Hence, the result

$$C(\gamma,u)\not\prec_{LU}J(z,v,\iota,\zeta,\nu,b).$$

■

The next theorem formulates a strong-type duality result, associated with the primal optimization problem (P).

Theorem 3 *Consider $(\bar{\gamma},\bar{u})$ is an LU-optimal solution of (P). Then there exist piecewise smooth functions $\bar{\iota}:\mathcal{B}\to\mathbb{R}^2,\bar{\iota}\neq 0$, $\bar{\zeta}:\mathcal{B}\to\mathbb{R}^m,\bar{\zeta}\geq 0$, and $\bar{\nu}:\mathcal{B}\to\mathbb{R}^n$, such that $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$ is a feasible point for (Dual) and the objective values of (P) and (Dual) are equal at $(\bar{\gamma},\bar{u})$ and $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$, respectively. Furthermore, if the hypotheses in the weak duality theorem hold, then $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$ is an LU-optimal solution of the dual problem (Dual).*

Proof Since $(\bar{\gamma},\bar{u})$ is an LU-optimal solution for (P), from Theorem 1 there exist piecewise smooth functions $\bar{\iota}=\left(\bar{\iota}^L,\bar{\iota}^U\right)\geq 0,\bar{\iota}\neq 0$, $\bar{\zeta}:\mathcal{B}\to\mathbb{R}^m,\bar{\zeta}\geq 0$, and $\bar{\nu}:\mathcal{B}\to\mathbb{R}^n$ such that:

(i) $\bar{\iota}^L\mathfrak{q}^L_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\iota}^U\mathfrak{q}^U_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\zeta}^Th_\gamma(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\nu}^Tf_\gamma(t,\bar{\gamma},\bar{u})$
$-D\left[\bar{\iota}^L\mathfrak{q}^L_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\iota}^U\mathfrak{q}^U_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})\ +\bar{\zeta}^Th_{\dot{\gamma}}(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})-\bar{\nu}^T\right]=0,$

(ii) $\bar{\iota}^L\mathfrak{q}^L_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\iota}^U\mathfrak{q}^U_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\zeta}^Th_u(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})+\bar{\nu}^Tf_u(t,\bar{\gamma},\bar{u})=0,$

(iii) $\bar{\zeta}^Th(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})=0,\quad t\in\mathcal{B},$

except at discontinuity points. Hence, $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$ fulfills the constraints of (Dual) and objective values are equal. Further, let us show that $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$ is an LU-optimal solution for (Dual). If not, then there exists $\left(\bar{\gamma}_1,\bar{u}_1,\bar{\iota}_1,\bar{\zeta}_1,\bar{\nu}_1,\bar{b}_1\right)\in Z_1$, satisfying:

$$J\left(\bar{\gamma}_1,\bar{u}_1,\bar{\iota}_1,\bar{\zeta}_1,\bar{\nu}_1,\bar{b}_1\right)\succ_{LU}J(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}).$$

As $\bar{b}=0$, then the above relation implies

$$J\left(\bar{\gamma}_1,\bar{u}_1,\bar{\iota}_1,\bar{\zeta}_1,\bar{\nu}_1,\bar{b}_1\right)\succ_{LU}J(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)=C(\bar{\gamma},\bar{u})$$

which contradicts the weak duality result given by Theorem 2. Consequently, the vector $(\bar{\gamma},\bar{u},\bar{\iota},\bar{\zeta},\bar{\nu},\bar{b}=0)$ is an LU-optimal solution for (Dual). The proof is complete. ■

The next theorem formulates a strict converse-type duality result associated with the primal optimization problem (P).

THEOREM 4 *Consider $(\bar{\gamma},\bar{u})$ and $(\bar{z},\bar{v},\iota,\zeta,\nu,\bar{b})$ be feasible points for (P) and (Dual), respectively, satisfying*

$$C(\bar{\gamma},\bar{u}) = J(\bar{z},\bar{v},\iota,\zeta,\nu,\bar{b}). \tag{13}$$

Moreover, if we have:
(i) $\iota = \left(\iota^L,\iota^U\right) > 0$;
(ii) $\int_{t_0}^{t_1} \left(\iota^L\mathfrak{q}^L + \iota^U\mathfrak{q}^U\right)(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})dt$ is strictly (η,ξ)-bonvex at $(\bar{z},\bar{v})$;
(iii) $\int_{t_0}^{t_1} \zeta^T h(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})dt$ is (η,ξ)-bonvex at $(\bar{z},\bar{v})$;
(iv) $\int_{t_0}^{t_1} \nu^T[f(t,\bar{\gamma},\bar{u}) - \dot{\bar{\gamma}}]dt$ is (η,ξ)-bonvex at $(\bar{z},\bar{v})$,
then the following equality

$$(\bar{\gamma},\bar{u}) = (\bar{z},\bar{v})$$

is fulfilled.

PROOF Suppose, to the contrary, that $(\bar{\gamma},\bar{u}) \neq (\bar{z},\bar{v})$. By considering the relation (13), we have

$$C(\bar{\gamma},\bar{u}) = J(\bar{z},\bar{v},\iota,\zeta,\nu,\bar{b}),$$

which implies

$$\int_{t_0}^{t_1} \left(\iota^L\mathfrak{q}^L + \iota^U\mathfrak{q}^U\right)(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})dt = \int_{t_0}^{t_1} \left(\iota^L\mathfrak{q}^L + \iota^U\mathfrak{q}^U - \frac{1}{2}\bar{b}^T G\bar{b}\right)(t,\bar{z},\dot{\bar{z}},\bar{v})dt. \tag{14}$$

Since $(\bar{z},\bar{v},\iota,\zeta,\nu,\bar{b})$ is a feasible point for (Dual), we obtain

$$\int_{t_0}^{t_1} \eta(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\Big\{\iota^L\mathfrak{q}^L_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\iota^U\mathfrak{q}^U_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\zeta^T h_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\nu^T f_{\bar{\gamma}}(t,\bar{z},\ v)$$
$$-D\left(\iota^L\mathfrak{q}^L_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v}) + \iota^U\mathfrak{q}^U_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v}) + \zeta^T h_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v}) - \nu^T\right)$$
$$+(G+H+F)\bar{b}\Big\}dt = 0, \tag{15}$$

$$\int_{t_0}^{t_1} \xi(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\Big\{\iota^L\mathfrak{q}^L_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v}) + \iota^U\mathfrak{q}^U_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v})$$

$$+\zeta^T h_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\nu^T f_{\bar{u}}(t,\bar{z},\bar{v})\Big\}dt=0. \tag{16}$$

The feasibility of $(\bar{\gamma},\bar{u})$ and $(\bar{z},\bar{v},\iota,\zeta,\nu,\bar{b})$ for (P) and (Dual), respectively, yields

$$\int_{t_0}^{t_1}\zeta^T h(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})dt \le \int_{t_0}^{t_1}\left(\zeta^T h(t,\bar{z},\dot{\bar{z}},\bar{v})-\frac{1}{2}\bar{b}^T H\bar{b}\right)dt. \tag{17}$$

Since $\int_{t_0}^{t_1}\zeta^T h\;dt$ is (η,ξ)-bonvex at $(\bar{z},\bar{v})$ and by considering (17), we get

$$\int_{t_0}^{t_1}\eta(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\left\{\zeta^T h_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})-D\left(\zeta^T h_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v})\right)+H\bar{b}\right\}dt$$

$$+\int_{t_0}^{t_1}\xi(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\zeta^T h_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v})dt\le 0.$$

By using the above inequality in (15) and (16), we obtain

$$\int_{t_0}^{t_1}\eta(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\Big\{\iota^L\mathfrak{q}^L_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\iota^U\mathfrak{q}^U_{\bar{\gamma}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\nu^T f_{\bar{\gamma}}(t,\bar{z},\bar{v})$$

$$-D\left(\iota^L\mathfrak{q}^L_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\iota^U\mathfrak{q}^U_{\dot{\bar{\gamma}}}(t,\bar{z},\dot{\bar{z}},\bar{v})-\nu^T\right)$$

$$+(G+F)\bar{b}\Big\}dt\ge 0,$$

$$\int_{t_0}^{t_1}\xi(t,\bar{\gamma},\bar{u},\bar{z},\bar{v})\Big\{\iota^L\mathfrak{q}^L_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\iota^U\mathfrak{q}^U_{\bar{u}}(t,\bar{z},\dot{\bar{z}},\bar{v})+\nu^T f_{\bar{u}}(t,\bar{z},\bar{v})\Big\}dt\ge 0.$$

Using the above relations and the hypotheses that

$$\int_{t_0}^{t_1}\left(\iota^L\mathfrak{q}^L+\iota^U\mathfrak{q}^U\right)dt$$

and

$$\int_{t_0}^{t_1}\nu^T[f(t,\bar{\gamma},\bar{u})-\dot{\bar{\gamma}}]dt$$

are (strictly) (η,ξ)-bonvex at $(\bar{z},\bar{v})$, we get

$$\int_{t_0}^{t_1}\left(\iota^L\mathfrak{q}^L+\iota^U\mathfrak{q}^U\right)(t,\bar{\gamma},\dot{\bar{\gamma}},\bar{u})dt$$

$$>\int_{t_0}^{t_1}\left(\left(\iota^L\mathfrak{q}^L+\iota^U\mathfrak{q}^U\right)(t,\bar{z},\dot{\bar{z}},\bar{v})-\frac{1}{2}\bar{b}^T G\bar{b}\right)dt,$$

a contradiction to (14). Hence, the result $(\bar{\gamma},\bar{u})=(\bar{z},\bar{v})$. ■

4. An illustrative example with physical significance

Now, in order to highlight the theoretical elements derived in the paper, an application (to which the here proposed approach applies and for which the previous methods do not work) is presented.

Let us extremize the mechanical work provided by the interval-valued variable force $\bar{V}(\gamma+1, \gamma^2+3)$, to move its application point along the interval $\mathcal{B} = [0.6, 1]$, so that the following controlled behavior

$$\begin{aligned}
&\gamma(0.6) = 0 = \gamma(1)\\
&\gamma^3 - \gamma \leq 0, \quad t \in \mathcal{B}\\
&\dot{\gamma} = u, \quad t \in \mathcal{B}\\
&-\gamma \leq 0, \quad t \in \mathcal{B}
\end{aligned}$$

is satisfied.

To solve the above-given problem, for $n = l = 1$, we will consider $\mathcal{A}$ as the space of piecewise smooth state functions $\gamma : \mathcal{B} \to \mathbb{R}$. Also, we denote $\mathcal{U}$ as the space of piecewise continuous control functions $u : \mathcal{B} \to \mathbb{R}$. Consider the following interval-valued variational control problem, formulated as:

$$\text{(P1)} \quad \min_{(\gamma,u)} C(\gamma, u) = \left[\int_{0.6}^{1} (\gamma+1)dt, \int_{0.6}^{1} \left(\gamma^2+3\right) dt\right]$$

subject to

$$\begin{aligned}
&\gamma(0.6) = 0 = \gamma(1)\\
&\gamma^3 - \gamma \leq 0, \quad t \in \mathcal{B}\\
&\dot{\gamma} = u, \quad t \in \mathcal{B}\\
&-\gamma \leq 0, \quad t \in \mathcal{B}.
\end{aligned}$$

The feasible solution set for (P1) is given by

$$Z = \left\{(\gamma, u) \in \mathcal{A} \times \mathcal{U} : \gamma\,(0.6) = \gamma\,(1) = 0,\ \gamma^3 - \gamma \leq 0,\ -\gamma \leq 0,\ \dot{\gamma} - u = 0\right\}.$$

In the following, we consider

$$\iota(t) = \left(t^2+2, t^2+2\right),\ \zeta(t) = \left(\frac{2t^3+t^2+4t+2}{2}, \frac{2t^3+t^2+4t+2}{2}\right),$$

$$b(t) = \nu(t) = t.$$

The dual problem, associated with the above primal problem (P1), is given as:

$$
\begin{aligned}
&\text{(D1)} \quad \max J(z,v,\iota,\zeta,\nu,b) = \\
&\left[\int_{0.6}^{1}\left(z+1-\frac{1}{2}b^T(0)b\right)dt, \int_{0.6}^{1}\left(z^2+3-\frac{1}{2}b^T(2)b\right)dt\right] \\
&= \left[\int_{0.6}^{1}(z+1)dt, \int_{0.6}^{1}\left(z^2+3-b^Tb\right)dt\right]
\end{aligned}
$$

subject to

$$
\begin{aligned}
&z(0.6)=0=z(1), \\
&\left(t^2+2\right)(1)+\left(t^2+2\right)(2z)+\left[\frac{2t^3+t^2+4t+2}{2}\ \frac{2t^3+t^2+4t+2}{2}\right]\left[\begin{array}{c} 3z^2-1 \\ -1 \end{array}\right] \\
&+1+\left(2\left(t^2+2\right)+6z\left(\frac{2t^3+t^2+4t+2}{2}\right)\right)t=0, \quad t\in\mathcal{B},
\end{aligned}
$$

$$
\begin{aligned}
&\left(t^2+2\right)(0)+\left(t^2+2\right)(0)+t=0, \quad t\in\mathcal{B}, \\
&\int_{0.6}^{1}\left[\frac{2t^3+t^2+4t+2}{2}\ \frac{2t^3+t^2+4t+2}{2}\right]\left[\begin{array}{c} z^3-z \\ -z \end{array}\right] \\
&-\frac{1}{2}t\left(6z\frac{2t^3+t^2+4t+2}{2}\right)tdt \geq 0, \quad t\in\mathcal{B}, \\
&\int_{0.6}^{1} t(u-\dot{z})dt \geq 0, \quad t\in\mathcal{B}, \\
&\iota:\mathcal{B}\to\mathbb{R}^2,\ \iota=\left(t^2+2,t^2+2\right)\geq 0,\ \iota\neq 0, \\
&\zeta:\mathcal{B}\to\mathbb{R}^2,\ \zeta=\left(\frac{2t^3+t^2+4t+2}{2},\frac{2t^3+t^2+4t+2}{2}\right)\geq 0,
\end{aligned}
$$

where

$$
\begin{aligned}
G = {}& \iota^L\mathfrak{q}^L_{\gamma\gamma}(t,z,\dot{z},v)+\iota^L\mathfrak{q}^L_{u\gamma}(t,z,\dot{z},v)+\iota^U\mathfrak{q}^U_{\gamma\gamma}(t,z,\dot{z},v)+\iota^U\mathfrak{q}^U_{u\gamma}(t,z,\dot{z},v) \\
& -D\left(\iota^L\mathfrak{q}^L_{\gamma\dot{\gamma}}+\iota^L\mathfrak{q}^L_{u\dot{\gamma}}+\iota^U\mathfrak{q}^U_{\gamma\dot{\gamma}}+\iota^U\mathfrak{q}^U_{u\dot{\gamma}}\right)(t,z,\dot{z},v),
\end{aligned}
$$

$$
H=\zeta^T h_{\gamma\gamma}(t,z,\dot{z},v)+\zeta^T h_{u\gamma}(t,z,\dot{z},v)-D\left(\zeta^T h_{\gamma\dot{\gamma}}(t,z,\dot{z},v)+\zeta^T h_{u\dot{\gamma}}(t,z,\dot{z},v)\right),
$$

$$
F=\nu^T f_{\gamma\gamma}(t,z,\dot{z},v)+\nu^T f_{u\gamma}(t,z,\dot{z},v)-D\left(\nu^T f_{\gamma\dot{\gamma}}(t,z,\dot{z},v)+\nu^T f_{u\dot{\gamma}}(t,z,\dot{z},v)\right),
$$

imply

$$
\begin{aligned}
G &= \left(t^2+2\right)\mathfrak{q}^L_{\gamma\gamma}(t,z,\dot{z},v) + \left(t^2+2\right)\mathfrak{q}^L_{u\gamma}(t,z,\dot{z},v) + \left(t^2+2\right)\mathfrak{q}^U_{\gamma\gamma}(t,z,\dot{z},v) \\
&+ \left(t^2+2\right)\mathfrak{q}^U_{u\gamma}(t,z,\dot{z},v) \\
&- D\left[\left(t^2+2\right)\mathfrak{q}^L_{\gamma\dot{\gamma}} + \left(t^2+2\right)\mathfrak{q}^L_{u\dot{\gamma}} + \left(t^2+2\right)\mathfrak{q}^U_{\gamma\dot{\gamma}} + \left(t^2+2\right)\mathfrak{q}^U_{u\dot{\gamma}}\right](t,z,\dot{z},v) \\
&= \left(t^2+2\right)(0) + \left(t^2+2\right)(0) + \left(t^2+2\right)(2) + \left(t^2+2\right)(0)(t,z,\dot{z},v) \\
&= 2\left(t^2+2\right), \\
H &= \left(\left[\frac{2t^3+t^2+4t+2}{2}\;\frac{2t^3+t^2+4t+2}{2}\right] left[\begin{array}{c} \gamma^3-\gamma \\ -\gamma \end{array}\right]_{\gamma\gamma}\right. \\
&+ \left[\frac{2t^3+t^2+4t+2}{2}\;\frac{2t^3+t^2+4t+2}{2}\right]\left[\begin{array}{c} \gamma^3-\gamma \\ -\gamma \end{array}\right]_{u\gamma}(t,z,\dot{z},v) \\
&- D\left(\left[\frac{2t^3+t^2+4t+2}{2}\;\frac{2t^3+t^2+4t+2}{2}\right]\left[\begin{array}{c} \gamma^3-\gamma \\ -\gamma \end{array}\right]_{\gamma\dot{\gamma}}\right. \\
&+ \left[\frac{2t^3+t^2+4t+2}{2}\;\frac{2t^3+t^2+4t+2}{2}\right]\left[\begin{array}{c} \gamma^3-\gamma \\ -\gamma \end{array}\right]_{u\dot{\gamma}}(t,z,\dot{z},v),
\end{aligned}
$$

that is,

$$
H = \frac{2t^3+t^2+4t+2}{2}(6z) - 0 = 3z\left(2t^3+t^2+4t+2\right)
$$

and $F = 0$.

For simplicity, we take $(z, v) = (0, 0)$. Then, the feasible solution set for (D1) is given by

$$
Z_1 = \left\{\left((0,0), \left(t^2+2, t^2+2\right), \left(\frac{2t^3+t^2+4t+2}{2}, \frac{2t^3+t^2+4t+2}{2}\right), t, t\right)\right.
$$

$$
\left. \in \mathcal{A}\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2, t\in\mathcal{B}\right\}.
$$

Now, for all feasible solutions,

$$
(z(t), v(t), \iota(t), \zeta(t), \nu(t), b(t))
$$

$$
= \left((0,0), \left(t^2+2, t^2+2\right), \left(\frac{2t^3+t^2+4t+2}{2}, \frac{2t^3+t^2+4t+2}{2}\right), t, t\right)
$$

of (D1), the objective value is $\left[0.4, \frac{2.816}{3}\right]$. Also, for all feasible solutions of (P1), the objective value is between $[0.4, 1.2]$ and $[0.8, 1.6]$.

Let $\eta, \xi : \mathcal{B} \times (\mathcal{A} \times \mathcal{U})^2 \to \mathbb{R}$ be defined by $\eta(t, \gamma, u, z, v) = \xi(t, \gamma, u, z, v) = \frac{-\gamma^3+2\gamma+z}{2}$. Since

$$\begin{aligned}
&\int_{0.6}^{1} \left(\left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right)(t, \gamma, \dot{\gamma}, u) - \left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right)(t, z, \dot{z}, v) + \frac{1}{2} b^T G b \right) dt \\
&- \int_{0.6}^{1} \eta(t, \gamma, u, z, v) \left(\iota^L \mathfrak{q}^L_\gamma(t, z, \dot{z}, v) + \iota^U \mathfrak{q}^U_\gamma(t, z, \dot{z}, v)\right. \\
&\left. - D\left(\iota^L \mathfrak{q}^L_{\dot{\gamma}}(t, z, \dot{z}, v) + \iota^U \mathfrak{q}^U_{\dot{\gamma}}(t, z, \dot{z}, v)\right) + Gb\right) dt \\
&- \int_{0.6}^{1} \xi(t, \gamma, u, z, v) \left(\iota^L \mathfrak{q}^L_u(t, z, \dot{z}, v) + \iota^U \mathfrak{q}^U_u(t, z, \dot{z}, v)\right) dt \\
&= \int_{0.6}^{1} \left(\left(t^2+2\right)\left(\gamma^2+\gamma+4\right) - \left(t^2+2\right)\left(z^2+z+4\right) + \left(t^4+2t^2\right)\right) dt \\
&- \int_{0.6}^{1} \left(\frac{-\gamma^3+2\gamma+z}{2}\right) \left(\left(t^2+2\right)(1) + \left(t^2+2\right)(2z) + \left(2t^3+4t\right)\right) dt \\
&= \int_{0.6}^{1} \left(t^2+2\right)\left(\gamma^2+\gamma\right) + \left(t^4+2t^2\right) - \left(\frac{-\gamma^3+2\gamma}{2}\right)\left(2t^3+t^2+4t+2\right) dt \\
&:= \int_{0.6}^{1} \psi dt \geq 0,
\end{aligned}$$

with

$$\psi := \left(t^2+2\right)\left(\gamma^2+\gamma\right) + \left(t^4+2t^2\right) - \left(\frac{-\gamma^3+2\gamma}{2}\right)\left(2t^3+t^2+4t+2\right),$$

is true for $(z, v) = (0, 0)$, $t \in \mathcal{B}$ and $\forall (\gamma, u) \in Z$ (its graphical representation is shown in Fig. 1), thus $\int_{0.6}^{1} \left(\iota^L \mathfrak{q}^L + \iota^U \mathfrak{q}^U\right) dt$ is (η, ξ)-bonvex at $(z, v) = (0, 0)$.

Similarly, the inequality

$$\begin{aligned}
&\int_{0.6}^{1} \left(\zeta^T h(t, \gamma, \dot{\gamma}, u) - \zeta^T h(t, z, \dot{z}, v) + \frac{1}{2} b^T H b \right) dt \\
&\quad - \int_{0.6}^{1} \eta(t, \gamma, u, z, v) \left(\zeta^T h_\gamma(t, z, \dot{z}, v) - D\left(\zeta^T h_{\dot{\gamma}}(t, z, \dot{z}, v)\right) + Hb\right) dt \\
&\quad - \int_{0.6}^{1} \xi(t, \gamma, u, z, v) \zeta^T h_u(t, z, \dot{z}, v) dt \\
&= \int_{0.6}^{1} \left(\left[\frac{2t^3+t^2+4t+2}{2} \; \frac{2t^3+t^2+4t+2}{2} \right] \begin{bmatrix} \gamma^3 - \gamma \\ -\gamma \end{bmatrix} \right. \\
&\quad - \begin{bmatrix} \frac{2t^3+t^2+4t+2}{2} & \frac{2t^3+t^2+4t+2}{2} \end{bmatrix} \begin{bmatrix} z^3 - z \\ -z \end{bmatrix}
\end{aligned}$$

$$+\frac{1}{2}t\left(6z\frac{2t^3+t^2+4t+2}{2}\right)t\right)dt - \int_{0.6}^{1}\left(\frac{-\gamma^3+2\gamma+z}{2}\right)$$
$$\left(\left[\begin{array}{cc} \frac{2t^3+t^2+4t+2}{2} & \frac{2t^3+t^2+4t+2}{2} \end{array}\right]\left[\begin{array}{c} 3z^2-1 \\ -1 \end{array}\right]\right.$$
$$+6z\left(\frac{2t^3+t^2+4t+2}{2}\right)t\right)dt$$
$$= \int_{0.6}^{1}\left(\left(\frac{2t^3+t^2+4t+2}{2}\right)\left(\gamma^3-2\gamma\right)-\right.$$
$$\left.\left(\frac{-\gamma^3+2\gamma}{2}\right)\left(-\left(2t^3+t^2+4t+2\right)\right)\right)dt$$
$$= 0 \geq 0,$$

implies $\int_{0.6}^{1}\zeta^T h dt$ is (η, ξ)-bonvex at $(z, v) = (0, 0)$.

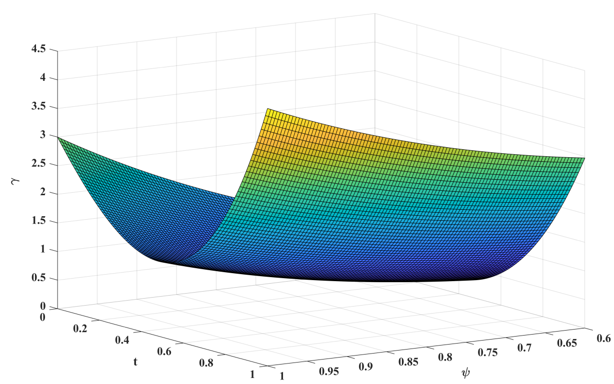

Figure 1. The 3-D plot of ϕ, where $0 \leq \gamma \leq 1$ and $0.6 \leq t \leq 1$

For all feasible solutions of (P1) and (D1), we have

$$\int_{0.6}^{1}\mathfrak{q}(t,\gamma,\dot{\gamma},u)dt \not<_{LU} \int_{0.6}^{1}\left(\mathfrak{q}(t,z,\dot{z},v)-\frac{1}{2}b^T G b\right)dt,$$

and, in consequence,

$$C(\gamma, u) \not<_{LU} J(z, v, \iota, \zeta, \nu, b)$$

and Theorem 2 is verified.

5. Conclusions

A dual-type problem of second order associated with the new class of (η, ξ)-bonvex interval-valued variational control problems has been formulated and studied. For this kind of problem, various duality theorems (weak, strong, strictly converse) have been established and proven under the new (η, ξ)-bonvexity assumptions of the involved functionals. In addition, we have presented an illustrative numerical example of a main result (namely, Theorem 2), obtained in the paper. Due to the applicability of control in various branches of science, the present work could represent a basis study of real importance for the academic community. An immediate further development of the paper is the inclusion of curvilinear or multiple integrals as functionals involved in the considered variational models. This would considerably increase the role of the present study.

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