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# Filtering theory for an Ornstein-Uhlenbeck process driven power system dynamics*

by

**Ravish H. Hirpara**

Electrical Engineering Department, S.N. Patel Institute of Technology and Research Centre, Umrakh, Bardoli, Surat-394345, Gujarat, India
ravishhirpara@gmail.com, ravish.hirpara@snpitrc.ac.in

**Abstract:** This paper revisits the non-Markovian state vector of an Ornstein-Uhlenbeck process-driven power system dynamics in non-linear filtering framework. The Fokker-Planck setting accounts for the process noise correction term and ignores the observation noise correction term in the analysis of stochastic systems. This paper introduces the notion of the 'Kushner-Stratonovich setting', which accounts for the process noise as well as observation noise correction terms in the conditional moment evolution equation. The Kushner-Stratonovich setting is the cornerstone formalism of non-linear filtering problems of stochastic control systems. We wish to estimate the rotor angle from given observations using two non-linear filters: (i) a Gaussian non-linear filter, and (ii) the extended Kalman filter. This paper develops two non-linear filters for a filtering model for the machine rotor angle, in which the Ornstein-Uhlenbeck process is the process noise and the Brownian noise process is the observation noise. The filter efficacy is examined by utilizing quite extensive numerical experimentations with two sets of data.

**Keywords:** nonlinear filtering; stochastic differential equation (SDE); Ornstein-Uhlenbeck (OU) process; stochastic systems; power system dynamics

## 1. Introduction

Filtering for the stochastic differential system in Itô sense using Kushner's equation has been studied extensively in literature. The Kushner's equation, a filtering density evolution equation, assumes the structure of a partial-integro stochastic differential equation. Liptser and Shiryaev (1977) developed the

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stochastic evolution of filtering density function using a different method and different notations as well. In their proof of the filtering density function, the stochastic evolution of conditional moment was derived. The stochastic evolution of conditional characteristic function becomes a special case of the conditional moment evolution. Interestingly, the filtering density can be defined as the inverse Fourier transform of the conditional characteristic function, which leads to the filtering density evolution equation. More details can be found in the celebrated book, authored by Liptser and Shiryaev (1977). Stratonovich (1963) developed the filtering density evolution equation for the stochastic differential equation with $\frac{1}{2}$-differential. Fujita and Fukao (1970) developed error analysis for the time-varying linear SDE, driven by the coloured noise process, see also Pugachev and Sinitsyn (1987), as well as Särkkä and Solin (2019). Hirpara and Sharma (2015) developed estimation theory for an OU process driven power system dynamics. Recently, Hirpara (2021) revisited the state of power system dynamics and developed estimation theory in the Stratonovich setting. Sharma (2009) had developed nonlinear filtering theory for Duffing-van der Pol system in the presence of white noise process with the Itô setting. Hirpara (2020) also developed nonlinear filtering theory in the presence of white noise process with Stratonovich setting. Zhu (2022) developed an adaptive neural network based on progressive Gaussian approximate filter with variable size. Nagarsheth et al. (2020) developed filtering theory of OU process noise driven for Duffing system.

This paper is devoted to the development and analysis of filtering for the OU process noise-driven stochastic differential system, a special case of the non-Markovian SDE, in combination with observations that are masked by the Brownian noise. Note that the weakly OU process noise considered in this paper is a zero mean, stationary process with finite, non-zero, smaller correlation time. The filtering model considered here can be regarded as an *extension* of the filtering model for the Itô stochastic differential system, and the classical filtering equations become a special case of the filtering equations, derived in this paper. The theory is grounded on the notion of stochastic equivalence. Subsequently, a Kushner-type equation is derived, which allows to obtain the stochastic evolution of conditional moment. The filtering theory, which is developed in this paper, is applied to a scalar power system dynamics driven by the OU process. Numerical simulations are carried out under a variety of conditions to test the efficacy and usefulness of the developed filtering theory.

This paper is organized as follows: Section 2 provides the mathematical preliminaries of the stochastic evolutions of conditional mean and variance for the weakly OU driven stochastic systems. Section 3 discusses the usefulness of the theory of the present paper by analyzing the scalar power system SDE and numerical simulations based on the filtering equations of this paper. Section 4 includes the concluding remarks.

## 2. Mathematical preliminaries

The filtering for the Itô stochastic differential system is accomplished using the filtering model (Jazwinski, 1970), as follows

$$dx_t = m(x_t,\ t)dt + \sigma(x_t,\ t)dv_t, \tag{1}$$

and

$$dz_t = h(x_t,\ t)dt + d\eta_t, \tag{2}$$

where $m(x_t, t)$ is the system nonlinearity term, $\sigma(x_t,\ t)$ is the process noise coefficient and $v_t$ is the Brownian motion process with variance $\mathrm{var}(dv_t) = Idt$. The term $dz_t$ is the observation vector, $h(x_t, t)$ is the function of state vector $x_t$, and $d\eta_t$ is the Brownian noise process with variance $\mathrm{var}(d\eta_t) = \phi_\eta dt$. In this paper we consider a non-Markovian stochastic differential system, meaning the weakly OU process noise process driven stochastic power system dynamics. The problem of analyzing the Itô stochastic differential system and its filtering has received quite some attention. The classical approach to accomplish filtering for the Itô SDE is to use the Kushner equation. In contrast to this classical filtering approach, the present paper introduces filtering for the stochastic differential system affected by weakly OU process noise, where $\xi_t$ is the weakly OU process noise. As a special case, we consider the stochastic differential system

$$\dot{x}_t = m(x_t) + \sigma(x_t)\xi_t, \tag{3}$$

with the observation equation

$$dz_t = h(x_t)dt + d\eta_t, \tag{4}$$

where variance $\mathrm{var}(d\eta_t) = \phi_\eta dt$. The evolution of conditional probability density for given initial states for a non-Markovian process assumes the structure of an infinite series, popularly known as stochastic equation (kinetic equation). Here, we explain why the weakly OU process noise is more appealing in contrast to the OU process noise. Most notably, the stochastic equation for the weakly OU process noise-driven stochastic differential equation involves only two spatial derivatives and the resulting equation is the Fokker-Planck equation. The Fokker-Planck equation describes the evolution of conditional probability density for given initial states for a Markov process, which satisfies a stationary Gaussian white noise-driven stochastic differential equation. Thus, the weakly OU process noise-driven and the white noise-driven stochastic differential systems would be associated with the same Fokker-Planck equation and the two can be considered 'stochastically' equivalent. The weakly OU process noise-driven stochastic differential system offers a quite simplified exposition and also confirms an actual physical situation. A simple calculation shows that the white noise-driven stochastic differential equation, associated with the Fokker-Planck equation (Stratonovich, 1963, p. 97) assumes the structure

$$\dot{x}_t = (s_1(x_t) - \frac{\frac{\partial}{\partial x_t} s_2(x_t)}{4}) + \sqrt{s_2(x_t)} w_0(t), \tag{5}$$

where the input process $w_0(t)$ is zero mean, stationary, Gaussian white noise process. Owing to equation (4.180) from Stratonovich (1963), a special case of equation (4.180), i.e. the coefficients $s_1(x)$ and $s_2(x)$ for the stochastic differential system of equation (3) can be re-cast as

$$s_1(x_t) = m(x_t) + \frac{\lambda_1}{2}\sigma(x_t)\frac{\partial}{\partial x_t}\sigma(x_t) + \lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}), \tag{6}$$

$$s_2(x_t) = \lambda_1\sigma^2(x_t) + 2\lambda_2\sigma^3(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}), \tag{7}$$

where

$$\lambda_1 = 2\int_{-\infty}^{0} R_{\xi\xi}(\tau)d\tau,\ \lambda_2 = \int_{-\infty}^{0} |\tau| R_{\xi\xi}(\tau)d\tau,$$

and the autocorrelation $R_{\xi\xi}(\tau) = E(\xi_{t+\tau}\xi_t)$. Equations (6)-(7) are valid for zero mean, stationary with finite non-zero smaller correlation time weakly OU process noise input process $\xi_t$. Equation (5), in conjunction with equations (6)-(7), assumes the structure

$$\dot{x}_t = m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}) - \frac{\lambda_2}{2}\sigma^3(x_t)\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)})$$
$$+\sqrt{\lambda_1}\sigma(x_t)\sqrt{1 + \frac{2\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})} w_0(t). \tag{8}$$

Note that equation (8) is *stochastically* equivalent to the SDE equation (3), since both are associated with the same Fokker-Planck equation. Equation (8) can be 're-formulated' in the Itô sense, i.e.

$$\begin{aligned} dx_t = {} & (m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}) \\ & -\frac{\lambda_2}{2}\sigma^3(x_t)\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)}))dt+ \\ & +\sqrt{\lambda_1}\sigma(x_t)\sqrt{1 + \frac{2\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})} dv_t, \end{aligned} \tag{9}$$

where

$$a(x_t) = m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}) - \frac{\lambda_2}{2}\sigma^3(x_t)\,\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)}), \tag{10}$$

$$b(x_t) = \sqrt{\lambda_1}\sigma(x_t)\sqrt{1 + \frac{2\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})}. \tag{11}$$

Equation (2), the Itô stochastic differential equation, describes a rigorous interpretation of the SDE in contrast to equation (8).

The OU process is a Gauss-Markov process and satisfies the stochastic differential equation

$$d\xi_t = -\frac{1}{\tau_{cor}}\xi_t dt \;+ \frac{\sqrt{2N}}{\tau_{cor}}dv(t),$$

where the term $\frac{\sqrt{2N}}{\tau_{cor}}$ has an interpretation as the process noise coefficient. For the OU process, a weakly OU process noise process, the terms $\lambda_1$, $\lambda_2$ of equations (6)-(7) become

$$\lambda_1 = 2N, \lambda_2 = N\tau_{cor}. \tag{12}$$

The definition of the terms $\lambda_1$, $\lambda_2$ is associated with equations (6)-(7). From equations (10)-(12), we have the following system non-linearity $a(x_t)$ and process noise coefficient $b(x_t)$ for the OU process-driven stochastic differential system:

$$\begin{aligned} a(x_t) &= m(x_t) - \frac{N\tau_{cor}}{2}\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}) \\ &-\frac{N\tau_{cor}}{2}\sigma^3(x_t)\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)}), \end{aligned} \tag{13}$$

$$b(x_t) = \sqrt{2N}\sigma(x_t)\sqrt{1 + \tau_{cor}\sigma(x_t)\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})}. \tag{14}$$

Here, we intend to develop the filtering equations of the weakly OU process noise-driven SDE, in which the observation equation is masked by the Brownian noise. This structure of the observation equation allows us to write the filtering density $p(x,\ t\,|z_\tau\ , t_0\ \le\ \tau\ \le\ t)$ in the Kushner setting (Jazwinski, 1970, p. 178). Thanks to Theorem (6.5) of Jazwinski (1970), p. 178, the Kushner-type equation for the system of equations (3)-(4), where equation (3) is 'stochastically equivalent' to equation (2), can be stated as

$$dp = L(p)dt + (h(x_t) - \langle h(x_t)\rangle)\phi_\eta^{-1} \times (dz_t - \langle h(x_t)\rangle\, dt)p, \tag{15}$$

where $p = p(x,\ t\,|z_\tau\ , t_0 \le \tau \le t)$. From equation (15), the Fokker-Planck operator

$$L(.) = -\frac{\partial m(x_t)(.)}{\partial x_t} + \frac{\lambda_2}{2}\frac{\partial \sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\,\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})(.)}{\partial x_t}$$

$$+\frac{\lambda_2}{2}\frac{\partial \sigma^3(x_t)\,\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)})(.)}{\partial x_t}$$

$$+\frac{\lambda_1}{2}\frac{\partial^2 \sigma^2(x_t)(1+2\frac{\lambda_2}{\lambda_1}\sigma(x_t)\,\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})\,)\,(.)}{\partial x_t^2}. \tag{16}$$

Consider now the scalar function $\varphi(x_t)$, system non-linearity $m(x_t)$, and the process noise coefficient $\sigma(x_t)$, which are twice continuously differentiable with respect to $x_t$. More precisely, $\varphi : R \to R, R$ is the solution space of the equation (3). The stochastic evolution $d\,\langle\varphi(x_t)\rangle$ of conditional moments is defined as

$$d\,\langle\varphi(x_t)\rangle = \int \varphi(x_t)dp(x,\ t\,|z_\tau\,, t_0 \le \tau \le t)dx, \tag{17}$$

where $\langle\varphi(x_t)\rangle = E(\varphi(x_t)\,|z_\tau\ ,\ t_0 \le \tau \le t)$. From equations (15)-(16), we have

$$\begin{aligned} d\,\langle\varphi(x_t)\rangle &= \int \varphi(x)\,(L(p)dt + (h(x_t) - \langle h(x_t)\rangle)\phi_\eta^{-1}(dz_t - \langle h(x_t)\rangle\,dt)p)dx \\ &= \langle\varphi,\ Lp\rangle\,dt + (\langle\varphi\,h\rangle - \langle\varphi\rangle\ \langle h\rangle)\phi_\eta^{-1}(dz_t - \langle h\rangle\,dt), \\ &= \langle L'\varphi,\ p\rangle\,dt + (\langle\varphi\,h\rangle - \langle\varphi\rangle\ \langle h\rangle)\phi_\eta^{-1}(dz_t - \langle h\rangle\,dt), \end{aligned} \tag{18}$$

where the notation $\langle\rangle$ denotes conditional expectation. The backward operator (Karatzas and Shreve, 1991, p. 281) for the stochastic differential system of equation (3), which is stochastically equivalent to equation (8) and further recast as equation (2), can be stated as

$$L'(.) = m(x_t)\frac{\partial(.)}{\partial x_t} - \frac{\lambda_2}{2}\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\,\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})\frac{\partial(.)}{\partial x_t}$$

$$-\frac{\lambda_2}{2}\sigma^3(x_t)\,\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)})\frac{\partial(.)}{\partial x_t}$$

$$+\frac{\lambda_1}{2}\sigma^2(x_t)\,(1+2\frac{\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)}))\frac{\partial^2(.)}{\partial x_t^2}. \tag{19}$$

From equations (2)-(19), we have

$$d\,\langle\varphi(x_t)\rangle = (\Big\langle m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)})$$

$$
\begin{aligned}
&- \frac{\lambda_2}{2}\sigma^3(x_t)\,\frac{\partial^2}{\partial x_t \partial x_t}(\frac{m(x_t)}{\sigma(x_t)})\,)\frac{\partial \varphi(x_t)}{\partial x_t}\Big\rangle \\
&+\frac{1}{2}\left\langle \lambda_1\sigma^2(x_t)\,\times\,(1+2\frac{\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)})\,)\frac{\partial^2\varphi(x_t)}{\partial x_t^2}\right\rangle)dt \\
&+(\langle \varphi(x_t)\,h(x_t)\rangle - \langle \varphi(x_t)\rangle\ \langle h(x_t)\rangle)\phi_\eta^{-1}\times(dz_t - \langle h(x_t)\rangle\,dt),
\end{aligned}
\tag{20}
$$

where $\langle \varphi h\rangle = E(\varphi(x_t)h(x_t)\,|z_\tau\,, t_0 \le \tau \le t)$. Interestingly, equation (20) is a consequence of the notion of conditional expectation, integration by parts formula, and the adjoint property of the Fokker-Planck operator. For $\varphi(x_t) = x_t$ and $\varphi(x_t) = \tilde{x}_t^2$, we have the stochastic evolutions of conditional mean and variance, where $\tilde{x}_t = x_t - \langle x_t\rangle$,

$$
\begin{aligned}
&d\,\langle x_t\rangle = \\
&\Big\langle m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\,\frac{\partial}{\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}) \\
&- \frac{\lambda_2}{2}\sigma^3(x_t)\frac{\partial^2}{\partial x_t\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)})\Big\rangle\,dt \\
&+(\langle x_t\ h(x_t)\rangle - \langle x_t\rangle\ \langle h(x_t)\rangle)\phi_\eta^{-1}(dz_t - \langle h(x_t)\rangle\,dt),
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&dP_t = (\Big\langle 2x_t(m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)}) \\
&- \frac{\lambda_2}{2}\sigma^3(x_t)\,\frac{\partial^2}{\partial x_t\partial x_t}(\frac{m(x_t)}{\sigma(x_t)}))\Big\rangle \\
&-2\,\langle x_t\rangle\,\Big\langle m(x_t) - \frac{1}{2}\lambda_2\sigma^2(x_t)\frac{\partial}{\partial x_t}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)}) \\
&- \frac{\lambda_2}{2}\sigma^3(x_t)\,\frac{\partial^2}{\partial x_t\partial x_t}(\frac{m(x_t)}{\sigma(x_t)})\Big\rangle \\
&+\Big\langle \lambda_1\sigma^2(x_t)(1+2\frac{\lambda_2}{\lambda_1}\sigma(x_t)\frac{\partial}{\partial x_t}\,(\frac{m(x_t)}{\sigma(x_t)}))\Big\rangle - (\langle x_t h(x_t)\rangle \\
&- \langle x_t\rangle\,\langle h(x_t)\rangle)^2\phi_\eta^{-1})dt \\
&+(\left\langle x_t^2\ h(x_t)\right\rangle - \left\langle x_t^2\right\rangle\langle h(x_t)\rangle - 2\,\langle x_t\rangle\,\langle x_t h(x_t)\rangle \\
&+2\left\langle x_t^2\right\rangle\langle h(x_t)\rangle)\phi_\eta^{-1}(dz_t - \langle h(x_t)\rangle\,dt).
\end{aligned}
\tag{22}
$$

After a few steps of simplifications up to the second order Taylor series expansions, we obtain

$$
d\,\langle x_t\rangle = (m(\langle x_t\rangle) - \frac{\lambda_2}{2}\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial\,\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\,\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})
$$

$$
\begin{aligned}
& -\frac{\lambda_2}{2}\sigma^3(\langle x_t\rangle)\,\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) + \frac{1}{2}P_t(\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}m(\langle x_t\rangle) \\
& -\frac{\lambda_2}{2}(\sigma^3(\langle x_t\rangle)\frac{\partial^4}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +7\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial^3}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +5\sigma^2(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +10\sigma(\langle x_t\rangle)\,(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^2\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +\sigma^2(\langle x_t\rangle)\frac{\partial^3}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +2(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^3\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) + 6\sigma(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle) \\
& \times\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}))))dt \\
& +P_t\frac{\partial}{\partial\langle x_t\rangle}h(\langle x_t\rangle)\phi_\eta^{-1}\times(dz_t-(h(\langle x_t\rangle)+\frac{1}{2}P_t\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}h(\langle x_t\rangle))dt), \quad (23)
\end{aligned}
$$

$$
\begin{aligned}
dP_t = {} & (2P_t(\frac{\partial}{\partial x_t}m(\langle x_t\rangle)-\frac{1}{2}\lambda_2(4\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +\sigma^2(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +2\sigma(\langle x_t\rangle)\,(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^2\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +\sigma^3(\langle x_t\rangle)\frac{\partial^3}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}))) \\
& +\lambda_1\sigma^2(\langle x_t\rangle)\,(1+2\frac{\lambda_2}{\lambda_1}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})) \\
& +P_t(\lambda_2\sigma^3(\langle x_t\rangle)\,\frac{\partial^3}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +6\lambda_2\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial x_t}\sigma(\langle x_t\rangle)\,\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}) \\
& +3\lambda_2\sigma^2(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})
\end{aligned}
$$

$$+6\lambda_2\sigma(\langle x_t\rangle)\,(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^2\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})$$

$$+\lambda_1\sigma(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)+\lambda_1(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^2)$$

$$-P_t^2\phi_\eta^{-1}(\frac{\partial}{\partial\langle x_t\rangle}h(\langle x_t\rangle))^2)dt+P_t^2\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}h(\langle x_t\rangle)\phi_\eta^{-1}$$

$$\times(dz_t-(h(\langle x_t\rangle)+\frac{1}{2}P_t\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}h(\langle x_t\rangle))dt). \tag{24}$$

Equations (23)-(24) are the proposed Gaussian filtering equations.

**Extended Kalman Filtering (EKF) equations**

After a few steps of simplifications up to the first order Taylor series expansions in equations (2)-(22), we obtain

$$d\langle x_t\rangle=(m(\langle x_t\rangle)-\frac{\lambda_2}{2}\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}$$

$$-\frac{\lambda_2}{2}\sigma^3(\langle x_t\rangle)\,\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}))dt$$

$$+P_t\frac{\partial}{\partial\langle x_t\rangle}h(\langle x_t\rangle)\phi_\eta^{-1}\times(dz_t-h(\langle x_t\rangle)dt), \tag{25}$$

$$dP_t=(2P_t(\frac{\partial}{\partial\langle x_t\rangle}m(\langle x_t\rangle)-\frac{1}{2}\lambda_2(4\sigma^2(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\,\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})$$

$$+\sigma^2(\langle x_t\rangle)\frac{\partial^2}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\sigma(\langle x_t\rangle)\frac{\partial}{\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})$$

$$+2\sigma(\langle x_t\rangle)\,(\frac{\partial}{\partial\langle x_t\rangle}\sigma(\langle x_t\rangle))^2\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})$$

$$+\sigma^3(\langle x_t\rangle)\frac{\partial^3}{\partial\langle x_t\rangle\,\partial\langle x_t\rangle\,\partial\langle x_t\rangle}\,(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)})))$$

$$+\lambda_1\sigma^2(\langle x_t\rangle)\,(1+2\frac{\lambda_2}{\lambda_1}\sigma(\langle x_t\rangle)\,\frac{\partial}{\partial\langle x_t\rangle}(\frac{m(\langle x_t\rangle)}{\sigma(\langle x_t\rangle)}))-P_t^2\phi_\eta^{-1}(\frac{\partial}{\partial\langle x_t\rangle}h(\langle x_t\rangle))^2)dt. \tag{26}$$

Equations (25)-(26) are the extended Kalman filtering equations.

## 3. Filtering theory for an Ornstein-Uhlenbeck process driven power system dynamics and numerical simulations

In deterministic setting, the swing equation of a single machine-infinite bus (SMIB) system is given by the following second-order non-linear differential equation, see Kundur (1994),

$$M\ddot{\delta} + D\dot{\delta} + \frac{VE'_a}{X}\sin\delta = P_m. \tag{27}$$

Note that the terms $V, M, D, E'_a, \delta, X, P_m$ denote, respectively, the voltage magnitude of the infinite bus, combined inertia constant and the damping coefficient of the generator and turbine, the transient EMF (electromotive force) of the generator, the rotor angle of the generator, the total reactance, and the input mechanical power. After accounting for random power fluctuations in the swing equation of the single machine-infinite bus system, we can get

$$M\ddot{\delta} + D\dot{\delta} + \frac{VE'_a}{X}\sin\delta = P_m + \xi_t, \tag{28}$$

where $\xi_t$ is the OU process. After accomplishing the phase space formulation, we have

$$x_1 = \delta_t, x_2 = \dot{\delta}_t,$$

$$\dot{x}_1 = x_2 \tag{29}$$

$$\dot{x}_2 = -\frac{D}{M}x_2 - \frac{VE'_a}{M\ X}\sin x_1 + \frac{P_m}{M} + \frac{\xi_t}{M}. \tag{30}$$

Now, consider that the contribution to the evolution of the phase variable $\delta_t = x_1$, coming from the damping term, is significantly greater than the inertial term (Hänggi and Jung, 1995, p. 241), then the term

$$-\frac{D}{M}x_2 - \frac{VE'_a}{M\ X}\sin x_1 + \frac{P_m}{M} + \frac{\xi_t}{M}$$

vanishes, i.e. if we model a system without physical or virtual inertia, power system is best described as a Kuramoto oscillator (see Dörfler and Bullo, 2010; Tönjes, 2010; Simpson-Porco, 2012; Schäfer et al., 2017*) with the equation of motion,

$$\dot{\delta}_t = -\frac{VE'_a}{D\ X}\sin\delta_t + \frac{P_m}{D} + \frac{1}{D}\xi_t. \tag{31}$$

*See Supplemental Material to this paper at http://link.aps.org/supplemental /10.1103/PhysRevE.95.060203 for discussion of robustness of the phenomena reported and a comparison of the escape time across different power grid models.

Thus, the above equation can be recast with the use of a more convenient notation by choosing the state variable notation $x_t$ for the rotor angle $\delta_t$. Thus,

$$\dot{x}_t = m(x_t,\, t) \,+ \sigma(x_t, t)\xi_t, \tag{32}$$

where $m(x_t, t) = -\frac{VE'_a}{D\,X}\sin x_t + \frac{P_m}{D}, \sigma(x_t, t) = \sigma(t) = \frac{1}{D}$,
and observation equation,

$$dz_t = x_t + d\eta_t, \tag{33}$$

where variance $\mathrm{var}(d\eta_t) = r^2 dt$.

After combining equations (23)-(24) with equations (32)-(33), as a result, we have the following system of filtering equations for the power system driven by the OU process,

$$d\langle x_t\rangle = (\frac{VE'_a}{D\,X}\sin\langle x_t\rangle\,(1+\frac{N\tau_{cor}}{2D^2})(\frac{P_t}{2}-1)+\frac{P_m}{D})dt+r^{-2}P_t(dz_t-\langle x_t\rangle\,dt), \tag{34}$$

$$dP_t = (-2P_t\frac{VE'_a}{D\,X}\cos\langle x_t\rangle - 2N\tau_{cor}\frac{VE'_a}{D^3\,X}\cos\langle x_t\rangle + \frac{2N}{D^2} - P_t^2 r^{-2})dt. \tag{35}$$

Then, after combining equations (25)-(26) with equations (32)-(33), as a result, we obtain the following system of extended Kalman filtering equations for the power system driven by the OU process,

$$d\langle x_t\rangle = (-\frac{VE'_a}{D\,X}\sin\langle x_t\rangle\,(1+\frac{N\tau_{cor}}{2D^2}) + \frac{P_m}{D})dt + r^{-2}P_t(dz_t - \langle x_t\rangle\,dt), \tag{36}$$

$$dP_t = (-P_t\frac{VE'_a}{D\,X}\cos\langle x_t\rangle\,(2+\frac{N\tau_{cor}}{D^2}) - 2N\tau_{cor}\frac{VE'_a}{D^3\,X}\cos\langle x_t\rangle + \frac{2N}{D^2} - P_t^2 r^{-2})dt. \tag{37}$$

The first set of initial conditions and system parameters is the following, see Hirpara (2021):

$V = 1.0pu, E'_a = 1.2pu, D = 5pu/rad/\sec$,
$P_m = 1.0pu, X'_d = 0.15pu, X_l = 0.1pu, X = 0.25pu, \tau_{cor} = 0.1, \langle x(0)\rangle = 1$ rad, $r = 5, P(0) = 0$ rad$^2$, $N = 0.05$.

Note that the initial conditions and system parameters are associated with equation (32). This set of data corresponds to lower intensity of noise, introduced into the observations as well as relatively less of uncertainties in initial conditions. In Figs. 1 through 4 the solid line (-) denotes the trajectory resulting

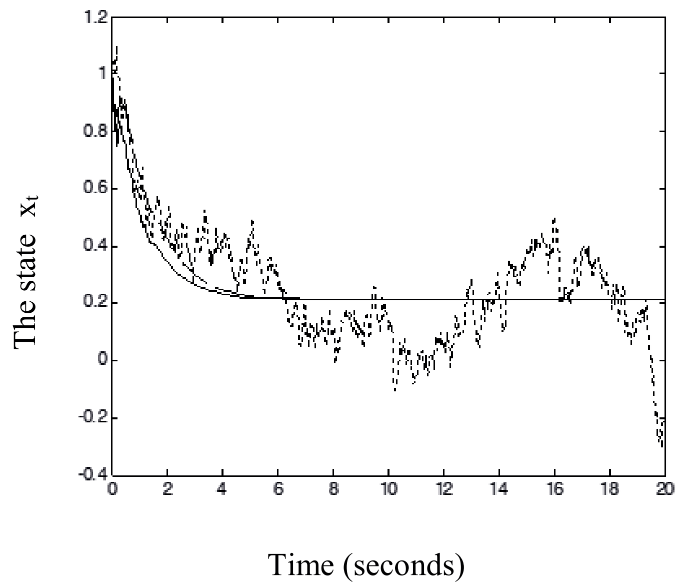


Figure 1. A comparison between the true and estimated states

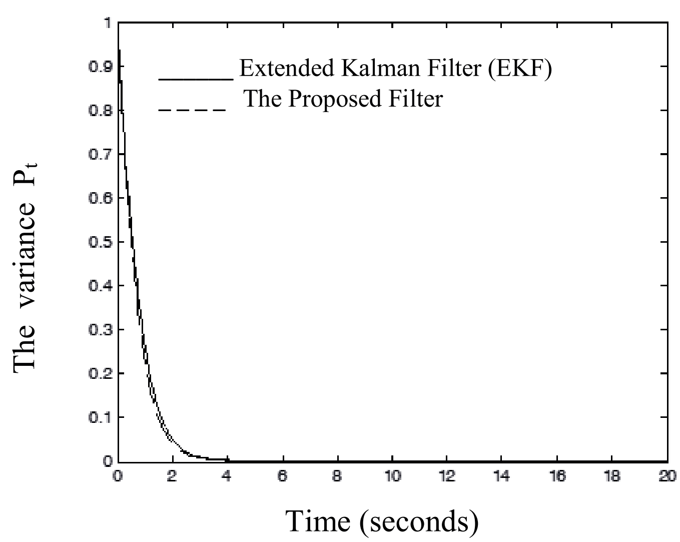


Figure 2. A comparison between the variance trajectories

from the extended Kalman filtering equations. The dashed line (–) corresponds to the trajectory resulting from the proposed Gaussian filtering equations and the dotted line (...) denotes the true state.

Thus, Fig. 1 corresponds to the first set of initial conditions and demonstrates that the second-order filter produces bounded state estimates. Moreover, the variance term is accounted for in the stochastic evolution of a conditional mean. Because of the presence of the variance term in the conditional mean evolution, the second-order filtering accounts for random initial conditions as well as random perturbations. Then, Fig. 2 shows a comparison between the variance trajectories of the extended Kalman filtering equation and the proposed Gaussian filtering equation for power system dynamics. The proposed Gaussian filtering equation has less variance compared to the Extended Kalman filtering equation.

The second set of initial conditions is the same as the first set with $r = 50$, i.e. larger observed noise intensity.

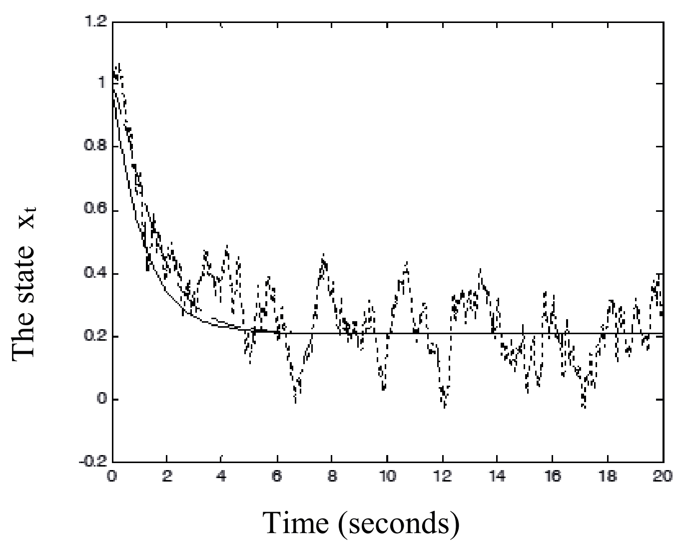


Figure 3. A comparison between the true and estimated states

It is interesting to note that the difference between estimated state trajectory and simulated state becomes definitely larger, which is attributed to large measurement noise (Germani et al., 2007, p. 2171; Sharma, 2009), as well as relatively larger value of the variance of the phase variable at the initial time instant. The classical filtering density evolution equation does not account for

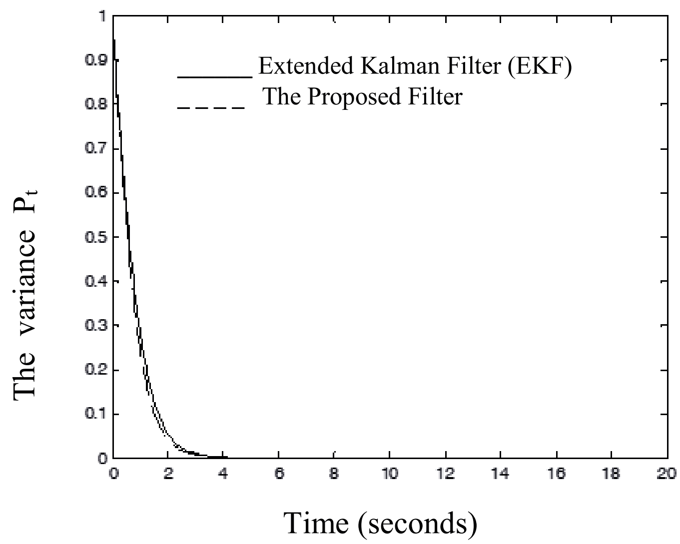


Figure 4. A comparison between the variance trajectories

the ‘process-observation noise’ coupling term, a combined effect of the process noise and the observation noise. This argument suggests that both of the noise processes, the process noise and observation noise process, are stochastically independent. That allows two different noise processes to model the process noise and the observation noise.

## 4. Conclusions

This paper contributes towards the development of a filtering theory for a weakly OU process noise-driven non-Markovian stochastic differential system, where the observations are masked by the Brownian noise. The proposed filtering theory has been successfully applied to a scalar power system driven by weakly OU process noise. Numerical simulations have been carried with different varieties of conditions. The results obtained from the proposed filtering theory confirm the effectiveness and usefulness of the method, introduced in the paper to estimate the states of a non-Markovian stochastic differential system, driven by the OU process.

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