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Logarithmic E-convex multi-objective programming*

by

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Abstract: A significant generalization of convexity is constituted by the concept of E-convexity, and our aim in this paper is to define a new class of functions depending on E-convexity, called logarithmic E-convex functions and to discuss their properties. Logarithmic E-convex functions are extended to lo-garithmic quasi E-convex and logarithmic pseudo E-convex functions and some results concerning them are presented. Logarithmic E-convex multi-objective programming problem is formulated. Finally, the sufficient and necessary conditions for a feasible solution to be an efficient solution for multi-objective programming problem involving logarithmic E-convex functions are derived.

Keywords: multi-objective programming, optimization, logarithmic E-convexity, efficient, weak efficient

1. Introduction

The extension of convexity is an area of active current research in the field of vector optimization. Many relaxations of convexity were introduced and were called generalized convex functions, see Mishra, Wang and Lai (2009), Noor and Noor (2019a,b) and Emam (2023). The concept of E-convexity was presented by Youness (1999) as a generalization of convexity. E-convexity notation considered only the convex combination of the images of points under an operator E. This kind of convexity can be observed as appearing in many fields, such as differential geometry, when a manifold is deformed by an operator E. In the field of physical chemistry, an E-convexity can take place when the binding force f between elements constructs a crystal effect by a solution E. In mathematical programming, the notion of E-convexity of functions plays an important role in

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solving the problem of the type of composite model problem, Jejakumar and Yong (1993), such as the problem: min $\| F(x) \|$ s.t. $x \in M$, where $f = \| \|$, $E(x) = F(x)$ and $(f \circ E)(x) = \| F(x) \|$. We generalized this notion to the so called strong E-convexity in Youness and Emam (2005). This generalization took into account the convex combination of points on the segment $[Ex, x+Ex]$ and the points on the segment $[Ey, y+Ey]$. Strong E-convexity was extended to quasi and pseudo strong E-convexity, see Youness and Emam (2008). In Emam (2012), we captured the efficient points on non-convex Pareto frontier, which are encountered in *E-[0,1]* convex multi-objective programming using approximation algorithm. Also, we characterized the efficient solutions for *E-[0,1]* convex multi-objective programming in Emam (2017). Recently, we studied nonsmooth semi-infinite multi-objective programming involving E-convex functions and derived sufficient efficiency conditions for this kind of problems, see Emam (2020). In this paper, we introduce a new class of functions as a generalization of convex functions, called logarithmic E-convex functions and present some results concerning this class in Section 2. In Section 3, we define logarithmic quasi E-convex and logarithmic pseudo E-convex functions and discuss their properties. We formulate a multi-objective programming problem, which involves logarithmic E-convex functions in Section 4. Sufficient and necessary conditions for a feasible solution to be an efficient solution for this kind of problems are derived. Finally, Section 5 draws a number of conclusions from the work presented here and provides several suggestions for future research. Let us survey, briefly, the definitions of E-convex sets and E-convex functions.

Definition 1.1 (Youness, 1999) *A set $M \subseteq R^n$ is said to be an E-convex set with respect to $E : R^n \to R^n$ if $\beta E(x)+(1-\beta)E(y) \in M$ for each $x, y \in M$, and $0 \leq \beta \leq 1$.*

Every E-convex set is a convex set if E is the identity map. If $M \subseteq R^n$ is an E-convex set, then $E(M) \subseteq M$. If M_1 and M_2 are E-convex sets, then $M_1 \bigcap M_2$ is an E-convex set, but $M_1 \bigcup M_2$ is not necessarily an E-convex set. If $E : R^n \to R^n$ is a linear mapping and M_1 , $M_2 \subseteq R^n$ are E-convex sets, then $M_1 + M_2$ is an E-convex set.

Definition 1.2 (Youness, 1999) *A function $f : M \subseteq R^n \to R$ is said to be an E-convex function, with respect to $E : R^n \to R^n$ on M, if M is an E-convex set and, for each $x, y \in M$, and $0 \leq \beta \leq 1$,*

$$f(\beta E(x) + (1-\beta)E(y)) \leq \beta f(Ex) + (1-\beta)f(Ey).$$

2. Logarithmic E-convex functions

In this section, we introduce a new class of functions, called logarithmic E-convex functions and discuss some of their properties.

Definition 2.1 *A function $f : M \subseteq R^n \to R^+$ is said to be a logarithmic E-convex function on M with respect to $E : R^n \to R^n$, if M is an E-convex set and*

$$\log f(\beta Ex + (1-\beta)Ey) \leq \beta \log f(Ex) + (1-\beta)\log f(Ey), \tag{2.1}$$

for each $x, y \in M$ and $0 \leq \beta \leq 1$, or

$$f(\beta Ex + (1-\beta)Ey) \leq [f(Ex)]^{\beta}[f(Ey)]^{(1-\beta)}. \tag{2.2}$$

If

$$\log f(\beta Ex + (1-\beta)Ey) \geq \beta \log f(Ex) + (1-\beta)\log f(Ey),$$

then f is called logarithmic E-concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called strictly logarithmic E-convex and strictly logarithmic E-concave, respectively.

Remark 2.1 *Every logarithmic E-convex function with respect to $E : R^n \to R^n$ is an E-convex function if $\beta f(Ex) + (1-\beta)f(Ey) = [f(Ex)]^{\beta}[f(Ey)]^{(1-\beta)}$.*

Example 2.1 (Youness, 2004) *Let $E : R^2 \to R^2$ be a mapping defined as $E(x,y) = (\frac{x}{2}, y)$ and $M = \{(x,y) \in R^2 : x \leq 2y,\ x+y \leq 3,\ 0 \leq y \leq 3,\ x \geq 0\}$ be an E-convex set. Suppose that $f : R^2 \to R^+$ be defined as $f(x,y) = (y-x)^3$ and since*

$$\log f(\beta Ex + (1-\beta)Ey) \leq \beta \log f(Ex) + (1-\beta)\log f(Ey),$$

then, f is logarithmic E-convex function on M.

Proposition 2.1 *If the functions $f_i : R^n \to R^+$, $i = 1, 2, ..., k$ are logarithmic E-convex on an E-convex set $M \subseteq R^n$, with respect to the mapping $E : R^n \to R^n$, then, for $a_i \geq 0$, $i = 1, 2, ..., k$, the function $h(x) = \prod_{i=1}^{k} a_i f_i(x)$ is logarithmic E-convex on M.*

Proof Since $f_i, i = 1, 2, \ldots, k$, is logarithmic E-convex on an E-convex set M, then $\forall x, y \in M,\ 0 \leq \beta \leq 1$, we have $\beta(Ex) + (1-\beta)(Ey) \in M$ and

$$\begin{aligned} h(\beta(Ex) + (1-\beta)(Ey)) &= \prod_{i=1}^{k} a_i f_i(\beta(Ex) + (1-\beta)(Ey)) \\ &\leq (\prod_{i=1}^{k} a_i f_i(Ex))^{\beta}(\prod_{i=1}^{k} a_i f_i(Ey))^{(1-\beta)} \\ &= (h(E(x))^{\beta}(h(Ey))^{(1-\beta)}. \end{aligned}$$

Thus, $h(x)$ is logarithmic E-convex on M. ■

Theorem 2.1 *Let $E : R^n \to R^n$ be a mapping, $M \subseteq R^n$ be an E-convex set with respect to E. Then $f : R^n \to R^+$ is logarithmic E-convex on M if the level set $L_E(\gamma) = \{Ex \in M : \log f(Ex) \leq \gamma,\ \forall \gamma \in R\}$ is an E-convex set.*

Proof Let f be a logarithmic E-convex function and $Ex, Ey \in L_E(\gamma)$. Then, $\log f(Ex) \leq \gamma$, $\log f(Ey) \leq \gamma$, and

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq \beta \log f(Ex) + (1-\beta)\log f(Ey) \leq \gamma$$

for $0 \leq \beta \leq 1$. Thus

$$\beta(Ex, \gamma) + (1-\beta)(Ey, \gamma) \in L_E(\gamma),$$

which implies that $L_E(\gamma)$ is E-convex.

Conversely, let

$$\bar{\gamma} = \beta \log f(Ex) + (1-\beta)\log f(Ey)$$

and

$$Ex, Ey \in L_E(\bar{\gamma}),$$

then from E-convexity of $L_E(\bar{\gamma})$, we have

$$\beta(Ex, \bar{\gamma}) + (1-\beta)(Ey, \bar{\gamma}) \in L_E(\bar{\gamma}),$$

or

$$(\beta Ex + (1-\beta)Ey, \bar{\gamma}) \in L_E(\bar{\gamma}),$$

for $0 \leq \beta \leq 1$, which implies that

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq \bar{\gamma} = \beta \log f(Ex) + (1-\beta)\log f(Ey).$$

This shows that f is a logarithmic E-convex function on M. ■

Definition 2.2 *Let $E : R^n \to R^n$, we define an E-epigraph of logarithmic E-convex function $f : R^n \to R^+$ as follows:*

$$epi_E(f) = \{(Ex, \gamma) : x \in M,\ \gamma \in R,\ \log f(Ex) \leq \gamma\}. \tag{2.3}$$

The following theorem gives a characterization of a logarithmic function in terms of its $epi_E(f)$.

Theorem 2.2 *Let $E : R^n \to R^n$ be a map, $M \subseteq R^n$ be an E-convex set with respect to E. Then $f : R^n \to R^+$ is a logarithmic E-convex function on M iff its $epi_E(f)$ is an E-convex set.*

PROOF Let f be logarithmic E-convex function and $(Ex,\gamma),(Ey,\alpha)\in epi_E(f)$. Then, $\log f(Ex)\le\gamma$, $\log f(Ey)\le\alpha$ and

$$\log f(\beta(Ex)+(1-\beta)(Ey))\le\beta\log f(Ex)+(1-\beta)\log f(Ey)\le\beta\gamma+(1-\beta)\alpha$$

for $0\le\beta\le1$. Thus,

$$\beta(Ex,\gamma)+(1-\beta)(Ey,\alpha)\in epi_E(f),$$

which implies that $epi_E(f)$ is E-convex.

Conversely, let $epi_E(f)$ be an E-convex set and

$$(Ex,\log f(Ex)),(Ey,\log f(Ey))\in epi_E(f),$$

then

$$\beta(Ex,\log f(Ex))+(1-\beta)(Ey,\log f(Ey))\in epi_E(f), \tag{2.4}$$

$\forall x,y\in M$ and $0\le\beta\le1$, which implies that

$$(\beta(Ex+(1-\beta)(Ey),\beta(\log f(Ex))+(1-\beta)(\log f(Ey)))\in epi_E(f), \tag{2.5}$$

and hence

$$\log f(\beta(Ex)+(1-\beta)(Ey))\le\beta\log f(Ex)+(1-\beta)\log f(Ey).$$

This shows that f is a logarithmic E-convex function on M. ∎

THEOREM 2.3 *Let $E:R^n\to R^n$ and $J=\{1,2,...\}$. If $(f_j)_{j\in J}$ be a family of functions, which are logarithmic E-convex and bounded from above on an E-convex set $M\subseteq R^n$, then the function $f(x)=\sup_{j\in J}f_j(x)$ is a logarithmic function E-convex on M.*

PROOF Since each f_j is a logarithmic E-convex function on an E-convex set M, then an $epi_E(f_j)$ of f_j,

$$epi_E(f_j)=\{(Ex,\gamma):x\in M,\ \gamma\in R,\ \log f_j(Ex)\le\gamma\},$$

is an E-convex set. Thus

$$\begin{aligned}\bigcap_{j\in J}epi_E(f_j) &= \{(Ex,\gamma):x\in M,\ \gamma\in R,\ \log f_j(Ex)\le\gamma\}\\ &= \{(Ex,\gamma):x\in M,\ \gamma\in R,\ \log f(Ex)\le\gamma\},\end{aligned}$$

is an E-convex set, where $f(x)=\sup_{j\in J}f_j(x)$. Hence, by Theorem (2.2), f is a logarithmic E-convex function on M. ∎

Proposition 2.2 *Let* $g_i : R^n \to R^+$, $i = 1, 2, ..., m$ *be logarithmic* E*-convex functions with respect to* $E : R^n \to R^n$*, then the set*

$$M = \{x \in R^n : \log g_i(x) \leq 0 \ , i = 1, 2, ..., m\} \tag{2.6}$$

is an E*-convex set if* $E(M) \subseteq M$.

Proof Let $x, y \in M \subseteq R^n$, then $Ex, Ey \in M$ and $\log g_i(Ex) \leq 0$, $\log g_i(Ey) \leq 0$, $i = 1, 2, ..., m$. Since $g_i(x), i = 1, 2, ..., m$ are logarithmic E-convex functions, then

$$\begin{aligned} \log g_i(\beta(Ex) + (1-\beta)(Ey)) &\leq \beta \log(g_i(Ex) + (1-\beta) \log(g_i(Ey) \\ &\leq 0, \qquad\qquad\qquad\qquad i = 1, 2, ..., m, \end{aligned}$$

where $0 \leq \beta \leq 1$ and hence $\beta(Ex) + (1-\beta)(Ey) \in M$, which implies that M is an E-convex set. ∎

Theorem 2.4 *Let* $f : R^n \to R^+$ *be a differentiable logarithmic* E*-convex function on an open* E*-convex set* $M \subseteq R^n$ *with respect to* $E : R^n \to R^n$*, then*

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \leq \log f(Ex) - \log f(Ey), \tag{2.7}$$

for each $x, y \in M$.

Proof Since M is an open E-convex set, then there is an open neighborhood N with radius r of $Ey \in E(M)$ such that $N(Ey, r) \subseteq M$.

Let $Ex \in E(M)$ and $Ex \neq Ey$, then

$$\beta(Ex) + (1-\beta)(Ey) \in N(Ey, r) \subseteq M, \text{ for } 0 < \beta < min\{\tfrac{r}{2||Ex-Ey||}, 1\},$$

and since f is a logarithmic E-convex function, then

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq \beta \log f(Ex) + (1-\beta) \log f(Ey),$$

or

$$\log f(Ey + \beta[Ex - Ey]) - \log f(Ey) \leq \beta[\log f(Ex) - \log f(Ey)], \tag{2.8}$$

for $0 < \beta < min\{\frac{r}{2||Ex-Ey||}, 1\}$. Upon dividing (2.8) by β and taking $\beta \to 0$, we get

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \leq \log f(Ex) - \log f(Ey),$$

for each $x, y \in M$. ∎

THEOREM 2.5 *Let $f : R^n \to R^+$ be a differentiable logarithmic E-convex function on an open E-convex set $M \subseteq R^n$. Then*

$$(Ex - Ey)(\log f(Ex)\nabla f(Ex) - \log f(Ey)\nabla f(Ey)) \geq 0,$$

for each $x, y \in M$.

PROOF Since f is a logarithmic E-convex function on an open E-convex set M, then, from (2.7), we have

$$(Ex - Ey)\log f(Ey)\nabla f(Ey) \leq \log f(Ex) - \log f(Ey),$$

$$(Ey - Ex)\log f(Ex)\nabla f(Ex) \leq \log f(Ey) - \log f(Ex),$$

and, by adding the above two inequalities, we get

$$(Ex - Ey)(\log f(Ex)\nabla f(Ex) - \log f(Ey)\nabla f(Ey)) \geq 0,$$

for each $x, y \in M$. ■

3. Logarithmic generalized E-convex Functions

In this section, we extend logarithmic E-convex functions to the so called logarithmic quasi E-convex and logarithmic pseudo E-convex functions, and discuss their properties.

DEFINITION 3.1 *A function $f : M \subseteq R^n \to R^+$ is said to be a logarithmic quasi E-convex function on M with respect to $E : R^n \to R^n$, if M is an E-convex set and for each $x, y \in M$ and $0 \leq \beta \leq 1$,*

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq max\{\log f(Ex), \log f(Ey)\}. \tag{3.1}$$

If

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \geq \max\{\log f(Ex), \log f(Ey)\},$$

then f is called logarithmic quasi E-concave function on M. If the inequality signs in the previous two inequalities are strict, then f is called strictly logarithmic quasi E-convex and strictly logarithmic quasi E-concave, respectively.

REMARK 3.1 *Every logarithmic E-convex function with respect to $E : R^n \to R^n$ is a logarithmic quasi E-convex function. However, the converse is not true.*

EXAMPLE 3.1 *Let $E : R \to R$ be defined as $E(x) = 4x + 2$ and $f : [2, \infty) \to R^+$ be defined as $f(x) = (\frac{x-2}{4})^3$. Suppose that $x \leq y$, then*

$$\log f(Ex) \leq \log f(Ey) \Rightarrow \log f(\beta(Ex)+(1-\beta)(Ey)) = \log f(4(\beta x+(1-\beta)y)+2)$$

$$= \log(\beta x + (1-\beta)y)^3 \le \log y^3 = \log f(Ey) = \max\{\log f(Ex), \log f(Ey)\}.$$

Hence, f is a logarithmic quasi E-convex function, but it is not a logarithmic E-convex function, because

$$\log f(\beta Ex + (1-\beta)Ey) > \beta \log f(Ex) + (1-\beta)\log f(Ey),$$

by taking $x = 2, y = 4, \beta = \frac{1}{2}$, for example.

THEOREM 3.1 *Let $E : R^n \to R^n$ be a map, $M \subseteq R^n$ be an E-convex set with respect to E. Then $f : R^n \to R^+$ is logarithmic quasi E-convex on M iff the level set $L_E(\gamma) = \{Ex \in M : \log f(Ex) \le \gamma,\ \forall \gamma \in R\}$ is an E-convex set.*

PROOF Let f be logarithmic quasi E-convex function and $Ex, Ey \in L_E(\gamma)$. Then $\log f(Ex) \le \gamma$, $\log f(Ey) \le \gamma$, and

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \le max\{\log f(Ex), \log f(Ey)\} \le \gamma \tag{3.2}$$

for $0 \le \beta \le 1$. Thus,

$$\beta(Ex, \gamma) + (1-\beta)(Ey, \gamma) \in L_E(\gamma),$$

which implies that $L_E(\gamma)$ is E-convex.

Conversely, let $\bar{\gamma} = max\{\log f(Ex), \log f(Ey)\}$ and $Ex, Ey \in L_E(\bar{\gamma})$, then from E-convexity of $L_E(\bar{\gamma})$, we have

$$\beta(Ex, \bar{\gamma}) + (1-\beta)(Ey, \bar{\gamma}) \in L_E(\bar{\gamma}),$$

or

$$(\beta Ex + (1-\beta)Ey, \bar{\gamma}) \in L_E(\bar{\gamma}),$$

for $0 \le \beta \le 1$, which implies that

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \le \bar{\gamma} = max\{\log f(Ex), \log f(Ey)\}.$$

This shows that f is a logarithmic quasi E-convex function on M. ■

THEOREM 3.2 *Let $E : R^n \to R^n$ be a map, and $M \subseteq R^n$ be an E-convex set with respect to E. Then $f : R^n \to R^+$ is a logarithmic quasi E-convex function on M if its $epi_E(f)$ is an E-convex set.*

PROOF Let $epi_E(f)$ be an E-convex set and $(Ex, \log f(Ex)), (Ey, \log f(Ey)) \in epi_E(f)$, then

$$\beta(Ex, \log f(Ex)) + (1-\beta)(Ey, \log f(Ey)) \in epi_E(f),$$

$\forall x, y \in M$ and $0 \leq \beta \leq 1$, which implies that

$$(\beta(Ex + (1-\beta)(Ey), \beta(\log f(Ex)) + (1-\beta)(\log f(Ey))) \in epi_E(f),$$

and hence

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq \log f(Ey) + \beta[\log f(Ex) - \log f(Ey)]. \quad (3.3)$$

So, if $\log f(Ex) \leq \log f(Ey)$, (3.3) becomes

$$log f(\beta(Ex) + (1-\beta)(Ey)) \leq \log f(Ey) = max\{\log f(Ex), \log f(Ey)\}.$$

Thus, f is a logarithmic quasi E-convex function on M. ∎

THEOREM 3.3 *Let $f : R^n \to R^+$ be a differentiable logarithmic quasi E-convex function on an open E-convex set $M \subseteq R^n$. Then*

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \leq 0, \quad (3.4)$$

for each $x, y \in M$.

PROOF Since M is an open E-convex set, then there is an open neighborhood of $Ey \in E(M)$, such that $N(Ey, r) \subseteq M$.

Let $Ex \in E(M)$ and $Ex \neq Ey$, then

$$\beta(Ex) + (1-\beta)(Ey) \in N(Ey, r) \subseteq M,$$

for $0 < \beta < min\{\frac{r}{2||Ex-Ey||}, 1\}$.

Suppose that $\log f(Ex) \leq \log f(Ey)$ and since f is a logarithmic quasi E-convex function, then

$$\log f(\beta(Ex) + (1-\beta)(Ey)) \leq \log f(Ey),$$

or

$$\log f(Ey + \beta[Ex - Ey]) - \log f(Ey) \leq 0, \quad (3.5)$$

for $0 < \beta < min\{r/2||Ex - Ey||, 1\}$. By dividing (3.5) by β and taking $\beta \to 0$, we get

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \leq 0,$$

for each $x, y \in M$. ∎

PROPOSITION 3.1 *Let $g_i : R^n \to R^+$, $i = 1, 2, ..., m$ be logarithmic quasi E-convex functions with respect to $E : R^n \to R^n$, then the set*

$$M = \{x \in R^n : \log g_i(x) \leq 0\ , i = 1, 2, ..., m\}$$

is an E-convex set if $E(M) \subseteq M$.

PROOF Let $x, y \in M \subseteq R^n$, then $Ex, Ey \in M$ and $\log g_i(Ex) \leq 0$, $\log g_i(Ey) \leq 0$, $i = 1, 2, ..., m$. Since $g_i(x), i = 1, 2, ..., m$ are logarithmic quasi E-convex functions, then

$$\begin{aligned} \log g_i(\beta(Ex) + (1-\beta)(Ey)) &\leq max\{\log(g_i(Ex), \log(g_i(Ey)\} \\ &\leq 0, \qquad\qquad i = 1, 2, ..., m, \end{aligned}$$

where $0 \leq \beta \leq 1$, and hence $\beta(Ex) + (1-\beta)(Ey) \in M$, which implies that M is an E-convex set. ■

DEFINITION 3.2 *A differentiable function $f : M \subseteq R^n \to R^+$ is said to be a logarithmic pseudo E-convex function on M with respect to $E : R^n \to R^n$, if M is an E-convex set and*

$$\log f(Ex) < \log f(Ey) \Rightarrow (Ex - Ey) \log f(Ey) \nabla f(Ey) < 0, \tag{3.6}$$

for each $x, y \in M$.

If $\log f(Ex) > \log f(Ey) \Rightarrow (Ex - Ey) \log f(Ey) \nabla f(Ey) > 0$, then f is called a logarithmic pseudo E-concave function on M.

REMARK 3.2 *If $f : M \subseteq R^n \to R^+$ is a differentiable logarithmic pseudo E-convex function on an E-convex set M with respect to $E : R^n \to R^n$, then*

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \geq 0 \Rightarrow \log f(Ex) \geq \log f(Ey), \tag{3.7}$$

for each $x, y \in M$.

EXAMPLE 3.2 *Let $E : R \to R$ be defined as $E(x) = \frac{5x}{2} + 2$ and $f : [2, \infty) \to R^+$ be defined as $f(x) = (\frac{2x-4}{5})^3$. Suppose that $x < y$, then*

$$\log x^3 < \log y^3 \Rightarrow (3y^2)\frac{5}{2}(x - y) \log y^3 < 0, y \in (1, \infty)$$

i.e.

$$\log f(Ex) < \log f(Ey) \Rightarrow (Ex - Ey) \log f(Ey) \nabla f(Ey).$$

Hence, f is a logarithmic pseudo E-convex function at y.

DEFINITION 3.3 *A differentiable function $f : M \subseteq R^n \to R^+$ is said to be a strictly logarithmic pseudo E-convex function on M with respect to $E : R^n \to R^n$, if*

$$\log f(Ex) \leq \log f(Ey) \Rightarrow (Ex - Ey) \log f(Ey) \nabla f(Ey) < 0, \tag{3.8}$$

or

$$(Ex - Ey) \log f(Ey) \nabla f(Ey) \geq 0 \Rightarrow \log f(Ex) > \log f(Ey), \tag{3.9}$$

for each $x, y \in M$, where M is an E-convex set.

Definition 3.4 *Let $E : R^n \to R^n$, then $\bar{x} \in R^n$ is said to be a global E-minimizer of $f : M \to R^+$ if $f(E\bar{x}) \leq f(Ex)$ holds for each $x \in M$.*

Proposition 3.2 *Let $f : M \to R^+$ be a differentiable logarithmic E-convex function on M with respect to $E : R^n \to R^n$ and $\nabla f(E\bar{x}) = 0$, then $\bar{x}$ is an E-minimizer of f.*

Proof Let $f : M \to R^+$ be a differentiable logarithmic E-convex function on M with respect to $E : R^n \to R^n$, then

$$(Ex - E\bar{x}) \log f(E\bar{x}) \nabla f(E\bar{x}) \leq \log f(Ex) - \log f(E\bar{x}),$$

for each $x \in M$. Since $\nabla f(E\bar{x}) = 0$, and by substitution in the above inequality, we get

$$\log f(E\bar{x}) \leq \log f(Ex),$$

which implies that

$$f(E\bar{x}) \leq f(Ex),$$

for each $x \in M$. Hence, $\bar{x}$ is an E-minimizer of f. ■

Proposition 3.3 *Let $f : M \to R^+$ be a differentiable logarithmic pseudo E-convex function on M with respect to $E : R^n \to R^n$ and $\nabla f(E\bar{x}) = 0$, then $\bar{x}$ is an E-minimizer of f.*

Proof The proof proceeds in the same manner as for Proposition 3.2. ■

4. Efficiency criteria

Let $E : R^n \to R^n$ be a one to one and onto mapping, $f_j : R^n \to R^+$, $j = 1, 2, ..., k$, and $g_i : R^n \to R^+$, $i = 1, 2, ..., m$ be differentiable logarithmic E-convex functions on R^n. A logarithmic E-convex multi-objective programming problem can be formulated as follows:

$$(P) \qquad \begin{array}{l} \text{Min } f_j(x),\ j = 1, 2, ..., k, \\ \textit{subject to} \\ x \in M = \{x \in R^n : \log g_i(x) \leq 0,\ i = 1, 2, ..., m\}. \end{array}$$

Let us now formulate the associated problem:

$$(P_E) \qquad \begin{array}{l} Min\ f_j(Ex),\ j = 1, 2, ..., k, \\ \textit{subject to} \\ x \in M_E = \{x \in R^n : \log g_i(Ex) \leq 0,\ i = 1, 2, ..., m\}. \end{array}$$

DEFINITION 4.1 (ABDULALEEM, 2024) *A feasible solution $E(x^*)$ for (P) is called a weak E-efficient solution for (P) iff there is no other feasible E(x) such that*

$$f_j(Ex) < f_j(Ex^*), \quad \forall\, j.$$

DEFINITION 4.2 (ABDULALEEM, 2024) *A feasible solution $E(x^*)$ for (P) is called an E-efficient solution for (P) iff there is no other feasible E(x) such that for some $i \in \{1, 2, ..., k\}$,*

$$f_j(Ex) < f_j(Ex^*), \quad f_j(Ex) \leq f_j(Ex^*), \quad \forall\, j \neq i.$$

REMARK 4.1 *Let $E(x^*) \in M$ be an E-efficient solution for (P), then $x^* \in M_E$ is an efficient solution for (P_E).*

For a feasible point $x^* \in M_E$, we denote by $I(x^*)$ the index set for binding constraints at x^*, i.e.

$$I(x^*) = \{i : \log g_i(Ex^*) = 0\}.$$

Now, let us derive the sufficient and necessary conditions for a feasible solution to be an E-efficient solution for problem (P) as in the following theorems:

THEOREM 4.1 *(Sufficient Conditions) Let $E(x^*)$ be a feasible solution for (P). If there exist scalars $w_j \geq 0,\ j = 1, 2, ..., k, \sum_{j=1}^{k} w_j = 1, u_i \geq 0,\ i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies*

$$\sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(E(x^*) + \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) \nabla g_i(Ex^*) = 0; \quad (4.1)$$

if $f_j, j = 1, 2, ..., k$ and $g_i, i \in I(x^*)$ are logarithmic E-convex at x^* with respect to E; then $E(x^*)$ is a weak E-efficient solution for (P).

PROOF Suppose that $E(x^*)$ is not a weak E-efficient solution for (P). Then, there exists a feasible $E(x) \in M$ such that

$$f_j(Ex) < f_j(Ex^*),\ \forall\, j,$$

or

$$\sum_{j=1}^{k} w_j \log f_j(Ex) - \sum_{j=1}^{k} w_j \log f_j(Ex^*) < 0. \quad (4.2)$$

Since f_j are differentiable logarithmic E-convex functions at x^*, then

$$(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(E(x^*) \leq \log f_j(Ex) - \log f_j(Ex^*),$$

which leads to

$$(Ex - Ex^*)\sum_{j=1}^{k} w_j \log f_j(Ex^*)\nabla f_j(E(x^*)$$
$$\leq \sum_{j=1}^{k} w_j \log f_j(Ex) - \sum_{j=1}^{k} w_j \log f_j(Ex^*). \tag{4.3}$$

From (4.2) and (4.3), we get

$$(Ex - Ex^*)\sum_{j=1}^{k} w_j \log f_j(Ex^*)\nabla f_j(E(x^*) < 0. \tag{4.4}$$

Also, logarithmic E-convexity of $g_i,\ i \in I(x^*)$ at x^* implies

$$(Ex - Ex^*)\sum_{i\in I(x^*)} u_i \log g_i(Ex^*)\nabla g_i(E(x^*)$$
$$\leq \sum_{i\in I(x^*)} u_i \log g_i(Ex) - \sum_{i\in I(x^*)} u_i \log g_i(Ex^*),$$
$$\Rightarrow (Ex - Ex^*)\sum_{i\in I(x^*)} u_i \log g_i(Ex^*)\nabla g_i(Ex^*) \leq 0. \tag{4.5}$$

Upon adding (4.4) and (4.5), we arrive at a contradiction to (4.1). Hence, $E(x^*)$ is a weak E-efficient solution for (P). ■

Theorem 4.2 *Let $E(x^*)$ be a feasible solution for (P). If there exist scalars $w_j \geq 0,\ j = 1, 2, ..., k, \sum_{j=1}^{k} w_j = 1, u_i \geq 0,\ i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies (4.1), $f_j, j = 1, 2, ..., k$ are strictly logarithmic E-convex and $g_i, i \in I(x^*)$ are logarithmic E-convex at x^* with respect to E, then $E(x^*)$ is an E-efficient solution for (P).*

Proof Suppose that $E(x^*)$ is not an E-efficient solution for (P). Then, there exists a feasible $E(x) \in M$ and index r such that

$$f_r(Ex) < f_r(Ex^*),$$
$$f_j(Ex) \leq f_j(Ex^*),\ \forall\, j \neq r.$$

These two inequalities lead to

$$\sum_{j=1}^{k} w_j \log f_j(Ex) - \sum_{j=1}^{k} w_j \log f_j(Ex^*) \leq 0. \tag{4.6}$$

Since f_j are differentiable strictly logarithmic E-convex functions at x^*, then

$$(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(E(x^*) < \log f_j(Ex) - \log f_j(Ex^*),$$

which leads to

$$\begin{aligned}&(Ex - Ex^*) \sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(E(x^*) \\ &< \sum_{j=1}^{k} w_j \log f_j(Ex) - \sum_{j=1}^{k} w_j \log f_j(Ex^*),\end{aligned} \tag{4.7}$$

next, from (4.6) and (4.7), we get

$$(Ex - Ex^*) \sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(E(x^*) < 0. \tag{4.8}$$

Now, we can proceed as in the proof of Theorem (4.1) by adding (4.7) and (4.8), which leads to a contradiction with (4.1), in order to prove that $E(x^*)$ is an E-efficient solution for (P). ■

Example 4.1 (Youness, 2004) *Let* $E : R^2 \to R^2$ *be defined as* $E(x, y) = (\frac{x}{2}, y)$. *Consider the bi-objective programming problem*

$$min f_1(x, y) = x^3$$

$$min f_2(x, y) = (y - x)^3$$

$$s.t.\ \ (x, y) \in M = \{(x, y) \in R^2 : x - 2y \le 0,\ x + y - 3 \le 0,\ 3 \ge y \ge 0,\ x \ge 0\}$$

where f_1 *and* f_2 *are logarithmic* E*-convex functions on the* E*-convex set* M*, and now formulate the problem* (P_E) *as follows:*

$$min f_1(E(x, y)) = \left(\frac{x}{2}\right)^3$$

$$min f_2(E(x, y)) = \left(y - \frac{x}{2}\right)^3$$

$$s.t.\ \ (x, y) \in M_E = \{(x, y) \in R^2 : x - 4y \le 0,\ x + 2y - 6 \le 0,\ 3 \ge y \ge 0,\ x \ge 0\}.$$

By applying conditions (4.1)*, we conclude that the set of efficient solutions for* (P_E) *is* $X_E = \{(x^*, y^*) \in M_E : x^* = 4y^*\}$*, and the images of these efficient solutions, by mapping E, are the efficient solutions for (P), i.e.* $X = \{(x^*, y^*) \in M : x^* = 2y^*\}$.

Theorem 4.3 *Let $E(x^*)$ be a feasible solution for (P). If there exist scalars $w_j \geq 0,\ j = 1, 2, ..., k, \sum_{j=1}^{k} w_j = 1, u_i \geq 0,\ i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies (4.1), $f_j, j = 1, 2, ..., k$ are logarithmic pseudo E-convex and $g_i, i \in I(x^*)$ are logarithmic quasi E-convex at x^* with respect to E, then $E(x^*)$ is a weak E-efficient solution for (P).*

Proof Suppose that $E(x^*)$ is not a weak E-efficient solution for (P). Then, there exists a feasible $E(x) \in M$, such that

$$f_j(Ex) < f_j(Ex^*)$$

$$\Rightarrow \log f_j(Ex) < \log f_j(Ex^*)\ \forall j,$$

and since f_j are differentiable logarithmic pseudo E-convex at x^*, then

$$(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(E(x^*) < 0,$$

which leads to

$$(Ex - Ex^*) \sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(E(x^*) < 0. \tag{4.9}$$

Since $g_i,\ i \in I(x^*)$, is differentiable logarithmic quasi E-convex at x^*, then

$$\sum_{i \in I(x^*)} u_i \log g_i(Ex) \leq \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) = 0,$$

$$\Rightarrow (Ex - Ex^*) \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) \nabla g_i(Ex^*) \leq 0. \tag{4.10}$$

Upon adding (4.9) and (4.10), we arrive at a contradiction to (4.1). Hence, $E(x^*)$ is a weak E-efficient solution for (P). ■

Theorem 4.4 *Let $E(x^*)$ be a feasible solution for (P). If there exist scalars $w_j \geq 0,\ j = 1, 2, ..., k, \sum_{j=1}^{k} w_j = 1, u_i \geq 0,\ i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies (4.1), $f_j, j = 1, 2, ..., k$ are strictly logarithmic pseudo E-convex and $g_i, i \in I(x^*)$ are logarithmic quasi E-convex at x^* with respect to E, then $E(x^*)$ is an E-efficient solution for (P).*

Proof Suppose that $E(x^*)$ is not an E-efficient solution for (P). Then, there exists a feasible $E(x) \in M$ and an index r, such that

$$f_r(Ex) < f_r(Ex^*),$$
$$f_j(Ex) \leq f_j(Ex^*),\ \forall j \neq r,$$

or

$$\log f_r(Ex) < \log f_r(Ex^*),$$
$$\log f_j(Ex) \le \log f_j(Ex^*),\ \forall j \neq r,$$

and, since f_j are differentiable strictly logarithmic pseudo E-convex at x^*, then

$$(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(Ex^*) < 0,$$

which leads to

$$(Ex - Ex^*) \sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(Ex^*) < 0. \tag{4.11}$$

Now, we can proceed as in the proof of Theorem (4.3) by adding (4.10) and (4.11), which leads to a contradiction with (4.1), in order to prove that $E(x^*)$ is an E-efficient solution for (P). ■

Theorem 4.5 *Let $E(x^*)$ be a feasible solution for (P). If there exist scalars $w_j \ge 0,\ j = 1, 2, ..., k, \sum_{j=1}^{k} w_j = 1, u_i \ge 0,\ i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies (4.1), $f_j, j = 1, 2, ..., k$ are logarithmic quasi E-convex and $g_i, i \in I(x^*)$ are strictly logarithmic pseudo E-convex at x^* with respect to E, then $E(x^*)$ is an E-efficient solution for (P).*

Proof Suppose that $E(x^*)$ is not an E-efficient solution for (P). There exists a feasible $E(x) \in M$ and an index r, such that

$$f_r(Ex) < f_r(Ex^*),$$
$$f_j(Ex) \le f_j(Ex^*),\ \forall j \neq r,$$

or

$$\log f_r(Ex) < \log f_r(Ex^*),$$
$$\log f_j(Ex) \le \log f_j(Ex^*),\ \forall j \neq r,$$

and, since f_j is differentiable logarithmic quasi E-convex at x^*, then

$$(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(Ex^*) \le 0,$$

which leads to

$$(Ex - Ex^*) \sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(Ex^*) \le 0. \tag{4.12}$$

Since $g_i,\ i \in I(x^*)$ are differentiable strictly logarithmic pseudo E-convex at x^*, then

$$\sum_{i \in I(x^*)} u_i \log g_i(Ex) \le \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) = 0$$

$$\Rightarrow (Ex - Ex^*) \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) \nabla g_i(Ex^*) < 0. \tag{4.13}$$

Upon adding (4.12) and (4.13), we arrive at a contradiction with (4.1). Hence, $E(x^*)$ is a weak E-efficient solution for (P). ∎

Example 4.2 *Let $E : R \to R$ be defined as $E(x) = \frac{5x}{2} + 2$. Consider the bi-objective problem*

$$min f_1(x) = \left(\frac{2x-4}{5}\right)^3$$

$$min f_2(x) = \left(\frac{x-2}{4}\right)^3$$

$$s.t. \;\; x \in M = \{x \in R : \, x \geq 2\}$$

where f_1, and f_2 are logarithmic pseudo E-convex functions at x^*, $x^* \in (1, \infty)$; now, formulate the problem (P_E) as follows:

$$min f_1(Ex) = x^3$$

$$min f_2(Ex) = \left(\frac{5x}{8}\right)^3$$

$$s.t. \quad (x, y) \in M_E = \{x \in R : \, x \geq 0, \}$$,

and by applying conditions (4.1) *we conclude that the efficient solution for (P_E) is $x^* = 0$ and the efficient solution for (P) is $Ex^* = 2$.*

Theorem 4.6 *(Necessary Conditions) Assume that Ex^* is a weak E-efficient solution for problem (P) and there exists a feasible point y such that $\log g_i(Ey) < 0$, $i = 1, 2, ..., m$, and each g_i, $i \in I(x^*)$ is logarithmic E-convex at x^* with respect to $E : R^n \to R^n$. Then, there exist scalars $w_j \geq 0$, $j = 1, 2, ..., k$ and $u_i \geq 0, i \in I(x^*)$, such that (x^*, w_j, u_i) satisfies* (4.1).

Proof Let the system

$$\begin{aligned} &(Ex - Ex^*) \log f_j(Ex^*) \nabla f_j(Ex^*) < 0, \;\; j = 1, 2, ..., k \\ &(Ex - Ex^*) \log g_i(Ex^*) \nabla g_i(Ex^*) \leq 0, \qquad i \in I(x^*), \end{aligned} \tag{4.14}$$

have a solution for $Ex \neq Ex^*$. Since

$$\log g_i(Ey) - \log g_i(Ex^*) < 0, \;\; i \in I(x^*),$$

and from the logarithmic E-convexity of g_i *at* x^*, we get

$$(Ey - Ex^*) \log g_i(Ex^*) \nabla g_i(Ex^*) < 0, \; i \in I(x^*). \tag{4.15}$$

Therefore, from (4.14) and (4.15)

$$[(Ex - Ex^*) + \rho(Ey - Ex^*)] \log g_i(Ex^*) \nabla g_i(Ex^*) < 0, \; \forall \rho > 0, \; i \in I(x^*).$$

Hence, for some positive λ, small enough:

$$\log g_i(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ey - Ex^*)]) < \log g_i(Ex^*) = 0, \; i \in I(x^*).$$

Similarly, for $i \notin I(x^*)$, $\log g_i(Ex^*) < 0$ and for $\lambda > 0$ small enough:

$$\log g_i(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ey - Ex^*)]) \leq 0, \; i \notin I(x^*).$$

Thus, $Ex^* + \lambda[(Ex - Ex^*) + \rho(Ey - Ex^*)]$ is feasible for (P) and (4.14) gives

$$\log f_j(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ey - Ex^*)]) < \log f_j(Ex^*) \forall j,$$

or

$$f_j(Ex^* + \lambda[(Ex - Ex^*) + \rho(Ey - Ex^*)]) < f_j(Ex^*) \forall j,$$

which contradicts the E-efficiency of Ex^* for (P). Hence, the system (4.14) has no solution and the result follows by applying the Farkas Lemma as in Bazaraa, Sherali and Shetty (2006), namely

$$\sum_{j=1}^{k} w_j \log f_j(Ex^*) \nabla f_j(Ex^*) + \sum_{i \in I(x^*)} u_i \log g_i(Ex^*) \nabla g_i(Ex^*) = 0, \quad u_i \geq 0.$$

■

Remark 4.2 *The above theorem continues to hold when Ex^* is an E-efficient solution for problem (P) and the functions $g_i, \; i \in I(x^*)$, are strictly logarithmic E-convex at x^* with respect to the mapping $E : R^n \to R^n$.*

5. Conclusion and future work

In this paper, we introduced a new class of functions as a generalization of convex functions, called logarithmic E-convex functions, and presented some results concerning this new class in Section 2. In Section 3, we extended the here introduced logarithmic E-convex functions to logarithmic quasi E-convex and logarithmic pseudo E-convex functions and discussed their properties. Logarithmic E-convex multi-objective programming problem formulation was considered in Section 4. Finally, we derived the sufficient and necessary conditions for a feasible solution to be an efficient solution for multi-objective programming problem involving logarithmic E-convex functions.

There are many open points in this area that should be studied in the future, for example

1. Study of duality for multi-objective programming involving logarithmic E-convex functions.

2. Study of efficiency criteria for logarithmic E-convex multi-objective programming without differentiability.

3. Parametric study of the new kind of generalized E-convex functions in multi-objective programming.

4. Developing numerical methods for solving multi-objective programming involving logarithmic E-convex functions.

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