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On controlled Stampacchia-type vector variational inequalities*

by

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Abstract: This paper establishes connections between new classes of generalized Stampacchia (weak) vector variational control inequalities and the corresponding multiple-objective extremization problems. In this regard, we introduce the concepts of (strictly) strong convexity and preconvexity, associated with controlled multiple integral type functionals, and a mean value type theorem.

Keywords: extremization problem; preconvexity; variational inequality; (strictly) strong convexity; control

1. Introduction

In 1964, by variational techniques, Hanson (1964) conceived the relationship between mathematical and classical programming. The applications of variational (control) problems have been well stated in Kim (2004). In 1980, Giannessi (1980) studied the notion of a vector variational inequality in a finite-dimensional Euclidian space. Later, Giannessi (1998) studied the relationship between efficient solutions of a differentiable convex vector optimization problem and solutions of a Minty-type vector variational inequality. Oveisiha and Zafarani (2013) investigated generalized Minty vector variational-like inequalities and vector optimization problems in Asplund spaces. Within the same research stream, Yu

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and Yao (1996) extended the existence results with respect to vector variational inequalities for monotone operators in Banach space. Then, Lee (2000) studied the relations between vector variational inequalities and the associated convex vector optimization problems. Continuing the research on Minty's variational principle, Yang and Teo (2004) presented some results, linking the vector variational inequality and the vector optimization problem under the pseudoconvexity and pseudomonotonicity hypotheses. Using the same ideas as their predecessors, Santos, Rojas-Medar and Rufián-Lizana (2006) established a connection between the nonconvex vector optimization problem and the variational-like inequality problem. The Minty vector variational inequality and the Stampacchia vector variational inequality were used by Ansari and Lee (2010) to establish a necessary and sufficient optimality condition for a vector minimal point of a vector optimization problem with pseudoconvex functions involving Dini derivatives. Similar techniques have been used by Al-Homidan and Ansari (2010) to determine the relationship between Minty and Stampacchia vector variational-like inequality problems and vector optimization problems for nondifferentiable and nonconvex functions. Arana-Jiménez et al. (2010) considered efficiency and duality results in multiobjective variational problems, proving that the introduction of new classes of functions may provide a sufficient and necessary condition to establish the duality results of a given problem. Miholca (2014) introduced several types of generalized invexity and established some relations between a multivalued optimization problem and the corresponding vector variational inequality. Jayswal, Singh and Kurdi (2016) established a connection between the solutions of some vector variational-type inequalities with weak formulation and the efficient solutions of the considered multi-objective problem. Later, Jayswal and Singh (2016) established several relationships between Minty and Stampacchia generalized variational inequalities and the associated multiobjective variational problems. For other advancements in this research direction, interested readers can consult the papers by Crespi, Ginchev and Rocca (2004, 2008), Ruiz-Garzon et al. (2010), Yu and Lu (2011).

In this paper, motivated by the previously mentioned developments, we extend the Stampacchia type (weak) vector variational inequalities to the Stampacchia type (weak) vector variational control inequalities. Using the new concepts of (strictly) strong convexity and preconvexity, associated with controlled multiple integral type functionals, and a mean value type theorem, we establish the relationships of these (weak) vector variational control inequalities with the associated multiple-objective optimization problems. Concretely, the new elements included in this study are represented by the presence of the control variable and the associated concepts of (strictly) strong convexity and preconvexity for multiple integral functionals.

This work consists of four sections. In the second section, we present the preliminary notions used to prove the main results of the current paper. In

Section 3, we establish relations between Stampacchia (weak) vector variational control inequalities and the associated multiobjective optimization problems. In addition, the theoretical notions are highlighted by considering some suitable examples. The last section formulates the conclusions of this paper.

2. Some preliminary results

Let $D \subset \mathbb{R}^q$ be a compact set (for instance, a hyper-parallelepiped having the diagonally located points a and b in $\mathbb{R}^q$). Also, let $\theta : D \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^r$ be a C^1-class function. For notation simplicity, we will write x and x_τ instead of $x(t)$ and $x_\tau(t)$, respectively, where $x : D \mapsto \mathbb{R}^n$ is a piecewise smooth function with partial derivative $x_\tau := \frac{\partial x}{\partial t^\tau}$, $\tau = \overline{1,q}$. Also, we will write u instead of $u(t)$, where $u : D \mapsto \mathbb{R}^m$ is a piecewise continuous function. We denote the partial derivatives of a scalar function $\phi : D \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ with respect to x and x_τ, respectively, by ϕ_x, ϕ_{x_τ} and, in a similar manner, we consider the partial derivative of ϕ with respect to u.

Further, let us denote by X the space of all piecewise smooth state functions $x : D \mapsto \mathbb{R}^n$ such that $x(a) = \alpha$, $x(b) = \beta$, with the norm $\|x\| = \|x\|_\infty + \sum_{\tau=1}^{q} \|x_\tau\|_\infty$, and consider U to be the space of piecewise continuous control functions $u : D \mapsto \mathbb{R}^m$, having the uniform norm $\|\cdot\|_\infty$.

Now, we consider the following multiple-objective optimization problem:

$$(P) \quad \min_{(x,u)} \int_D \theta(t,x,u) d\omega = \left(\int_D \theta^1(t,x,u) d\omega, \cdots, \int_D \theta^r(t,x,u) d\omega \right),$$

subject to

$$x(a) = \alpha, \quad x(b) = \beta,$$

$$g(t,x,u) \leq 0, \quad t \in D,$$

$$h^i_\tau(t,x,u) = x^i_\tau, \quad t \in D,\ i = \overline{1,n},\ \tau = \overline{1,q},$$

where $d\omega := dt^1, \cdots dt^q$, $g : D \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ and $h : D \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{nq}$ are assumed to be continuously differentiable functions. We denote by $\mathbb{K} \subset X \times U$ the convex set of all feasible pairs of P.

Definition 2.1 *A point $(y,v) \in \mathbb{K}$ is named the efficient pair of P if for all $(x,u) \in \mathbb{K}$ the following inequality cannot hold*

$$\int_D \theta^p(t,x,u) d\omega \leq \int_D \theta^p(t,y,v) d\omega,$$

with $<$ for at least one $p \in \mathbb{P}$, where $\mathbb{P} = \{1, \cdots, r\}$.

Definition 2.2 *A point $(y,v)\in\mathbb{K}$ is named the weak efficient pair of P if for all $(x,u)\in\mathbb{K}$ the following inequality cannot hold*

$$\int_D \theta^p(t,x,u)d\omega < \int_D \theta^p(t,y,v)d\omega, \quad \forall p\in\mathbb{P}.$$

In order to formulate and prove the main results, which are derived in this paper, we present the notions of *strong preconvexity* and *strong convexity* for the involved multiple integral functionals.

Definition 2.3 *A scalar functional $\Phi:\mathbb{K}\to\mathbb{R}$, $\Phi(x,u)=\int_D \phi(t,x,u)d\omega$ is named strongly preconvex on $\mathbb{K}$ if there exists $\delta>0$ such that, for all (x,u), $(y,v)\in\mathbb{K}$ and $\sigma\in[0,1]$, the inequality*

$$\int_D \phi\left(t,y+\sigma(x-y),v+\sigma(u-v)\right)d\omega \leq \sigma\int_D \phi(t,x,u)d\omega+(1-\sigma)\int_D \phi(t,y,v)d\omega \\ -\delta\sigma(1-\sigma)\|(x,u)-(y,v)\|^2,$$

is satisfied.

Definition 2.4 *A scalar functional $\Phi:\mathbb{K}\to\mathbb{R}$, $\Phi(x,u)=\int_D \phi(t,x,u)d\omega$ is named strongly convex on $\mathbb{K}$ if there exists $\delta>0$ such that, for all (x,u), $(y,v)\in\mathbb{K}$, the inequality*

$$\int_D [\phi_x(t,y,v)(x-y)+\phi_u(t,y,v)(u-v)]d\omega+\delta\|(x,u)-(y,v)\|^2 \\ \leq \int_D \phi(t,x,u)d\omega-\int_D \phi(t,y,v)d\omega,$$

is satisfied.

Obviously, a strong convex functional is also a convex functional, but the converse is not true.

Exemple 2.1 *Consider $\phi:[0,1]^2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, with $\phi(t,x,u)=x+u$ and $x(t)=t^1+t^2$, $u(t)=c\in\mathbb{R}$. The functional $\Phi(x,u)=\int_{[0,1]^2}\phi(t,x,u)d\omega$ is convex at $(y,v)=(0,0)$ since, for all $(x,u)\in\mathbb{R}^2$, we have*

$$\int_{[0,1]^2}\phi(t,x,u)d\omega-\int_{[0,1]^2}\phi(t,y,v)d\omega \geq \\ \int_{[0,1]^2}\left[\phi_x(t,y,v)(x-y)+\phi_u(t,y,v)(u-v)\right]d\omega.$$

Indeed, we get

$$\int_{[0,1]^2} \phi(t,x,u)d\omega - \int_{[0,1]^2} \phi(t,y,v)d\omega = \int_{[0,1]^2} (x+u)d\omega - \int_{[0,1]^2} (y+v)d\omega$$
$$= 1+c,$$

and

$$\int_{[0,1]^2} \left[\phi_x(t,y,v)(x-y) + \phi_u(t,y,v)(u-v)\right] d\omega = \int_{[0,1]^2} \left[(x-y)+(u-v)\right] d\omega$$
$$= 1+c.$$

The functional Φ *is not strongly convex at* $(y,v)=(0,0)$*, since we get*

$$\int_{[0,1]^2} [\phi_x(t,y,v)(x-y) + \phi_u(t,y,v)(u-v)]d\omega + \delta\|(x,u)-(y,v)\|^2$$
$$- \left[\int_{[0,1]^2} \phi(t,x,u)d\omega - \int_{[0,1]^2} \phi(t,y,v)d\omega\right]$$
$$= \delta\|(x,u)-(y,v)\|^2 \nleq 0,$$

for any $\delta > 0$.

Definition 2.5 *A scalar functional* $\Phi : \mathbb{K} \to \mathbb{R}, \Phi(x,u) = \int_D \phi(t,x,u)d\omega$ *is called strictly strongly convex on* $\mathbb{K}$ *if there exists* $\delta > 0$ *such that, for all* $(x,u),(y,v) \in \mathbb{K}$ *and* $(x,u) \neq (y,v)$*, we have*

$$\int_D [\phi_x(t,y,v)(x-y) + \phi_u(t,y,v)(u-v)]d\omega + \delta\|(x,u)-(y,v)\|^2$$
$$< \int_D \phi(t,x,u)d\omega - \int_D \phi(t,y,v)d\omega.$$

Based on the *convex set* concept, we introduce the following definition for the *path* notion.

Definition 2.6 *Let* (x,u) *and* (y,v) *be arbitrary in* $\mathbb{K}$*. A set* $P_{(x,u),(y,v)}$ *is named closed path linking* (y,v) *and* (x,u)*, if*

$$P_{(x,u),(y,v)} = \{(z,w) = (x,u) + \sigma((y,v)-(x,u)) : \sigma \in [0,1]\}.$$

Similarly, $P^0_{(x,u),(y,v)}$ is called *an open path linking* (x,u) *and* (y,v), if

$$P^0_{(x,u),(y,v)} = \{(z,w) = (x,u) + \sigma((y,v)-(x,u)) : \sigma \in (0,1)\}.$$

Next, we will formulate a mean value type theorem for differentiable functionals defined on a convex set.

Theorem 2.1 *Let* $\Phi : \mathbb{K} \to \mathbb{R}, \Phi(x,u) = \int_D \phi(t,x,u) d\omega$ *be a differentiable functional and* $P_{(x,u),(y,v)}$ *be an arbitrary closed path contained in* $\mathbb{K}$. *Then, there exists* $(x_0,u_0) \in P^0_{(x,u),(y,v)}$ *such that the following relation holds*

$$\int_D \phi(t,y,v) d\omega - \int_D \phi(t,x,u) d\omega = \int_D \left[\phi_x(t,x_0,u_0)(y-x) + \phi_u(t,x_0,u_0)(v-u)\right] d\omega.$$

Proof Consider $f : [0,1] \to \mathbb{R}$, a real-valued function, given by

$$\begin{aligned} f(\sigma) = & \int_D \phi(t, x+\sigma(y-x), u+\sigma(v-u)) d\omega - \int_D \phi(t,x,u) \\ & - \sigma \left[\int_D g(t,y,v) d\omega - \int_D \phi(t,x,u) d\omega \right]. \end{aligned} \tag{1}$$

Since $f(0) = f(1) = 0$, we use Rolle's theorem that involves the fact that there exists $\epsilon \in (0,1)$ with $f'(\epsilon) = 0$. By considering relation (1), we get

$$\begin{aligned} 0 = f'(\epsilon) = & \int_D \phi_x(t, x+\epsilon(y-x), u+\epsilon(v-u))(y-x) d\omega \\ & + \int_D \phi_u(t, x+\epsilon(y-x), u+\epsilon(v-u))(v-u) d\omega \\ & - \int_D \phi(t,y,v) d\omega + \int_D \phi(t,x,u) d\omega, \end{aligned}$$

that is,

$$\begin{aligned} \int_D \phi(t,y,v) d\omega - \int_D \phi(t,x,u) d\omega = & \int_D [\phi_x(t, x+\epsilon(y-x), u+\epsilon(v-u))(y-x) \\ & + \phi_u(t, x+\epsilon(y-x), u+\epsilon(v-u))(v-u)] d\omega. \end{aligned}$$

By taking $(x_0,u_0) := (x+\epsilon(y-x), u+\epsilon(v-u))$, the above relation completes the proof, namely

$$\int_D \phi(t,y,v) d\omega - \int_D \phi(t,x,u) = \int_D \left[\phi_x(t,x_0,u_0)(y-x) + \phi_u(t,x_0,u_0)(v-u)\right] d\omega. \square$$

The following lemma will help us to prove our main results, presented in the next section of the paper.

Lemma 2.1 *Let* $\Phi : \mathbb{K} \to \mathbb{R}$, $\Phi(x,u) = \int_D \phi(t,x,u) d\omega$ *be a differentiable functional. If the functional* Φ *is strongly convex on* $\mathbb{K}$, *then it is strongly preconvex on* $\mathbb{K}$.

Proof Since $\mathbb{K}$ is a convex set, we have

$$(x_1,u_1) = (x,u) + \sigma((y,v) - (x,u)) \in \mathbb{K}, \quad \forall (x,u),\ (y,v) \in \mathbb{K},\ \sigma \in [0,1].$$

Using strong convexity of $\Phi(x,u) = \int_D \phi(t,x,u)d\omega$, we obtain that there exists $\delta > 0$ such that, for all (x_1,u_1), $(y,v) \in \mathbb{K}$, we have

$$\begin{aligned}\int_D [\phi_x(t,x_1,u_1)(y-x_1) + \phi_u(t,x_1,u_1)(v-u_1)]\, d\omega + \delta\|(y,v)-(x_1,u_1)\|^2 \leq \\ \int_D \phi(t,y,v)d\omega - \int_D \phi(t,x_1,u_1)d\omega.\end{aligned} \tag{2}$$

Similarly, we use the strong convexity of $\Phi(x,u) = \int_D \phi(t,x,u)d\omega$ and we obtain that there exists $\delta > 0$ such that, for all (x_1,u_1), $(x,u) \in \mathbb{K}$, we have

$$\begin{aligned}\int_D [\phi_x(t,x_1,u_1)(x-x_1) + \phi_u(t,x_1,u_1)(u-u_1)]\, d\omega + \delta\|(x,u)-(x_1,u_1)\|^2 \leq \\ \int_D \phi(t,x,u)d\omega - \int_D \phi(t,x_1,u_1)d\omega.\end{aligned} \tag{3}$$

Now, we multiply the inequalities (2) and (3) by σ and $1-\sigma$, respectively. Next, upon adding both inequalities, it follows that there exists $\delta > 0$ such that for all (x,u), $(y,v) \in \mathbb{K}$ and $\sigma \in [0,1]$, we obtain

$$\begin{aligned}\int_D \phi(t,x+\sigma(y-x),u+\sigma(v-u))d\omega \leq \sigma \int_D \phi(t,y,v)d\omega + (1-\sigma)\int_D \phi(t,x,u) \\ -\delta\sigma(1-\sigma)\|(y,v)-(x,u)\|^2,\end{aligned}$$

and this completes the proof. □

Now, we use the idea of a generalization for the Stampacchia variational inequality, provided by Oveisiha and Zafarani (2013), and state generalized Stampacchia vector variational control inequalities, which help us in the investigation of the efficient pairs of the above-mentioned multiobjective variational control problem P:

GSVVI$_\gamma$: For a given γ, let us find $(y,v) \in \mathbb{K}$ such that there exists no $(x,u) \in \mathbb{K}$, fulfilling

$$\int_D [\theta_x^p(t,y,v)\,(x-y) + \theta_u^p(t,y,v)(u-v)]\, d\omega + \gamma\|(x.u)-(y,v)\|^2 \leq 0,$$

with $<$ for at least one $p \in \mathbb{P}$;

GWSVVI$_\gamma$: For a given γ, let us find $(y,v)\in\mathbb{K}$ such that there exists no $(x,u)\in\mathbb{K}$, fulfilling

$$\int_D [\theta^p_x(t,y,v)\,(x-y)+\theta^p_u(t,y,v)(u-v)]\,d\omega+\gamma\|(x.u)-(y,v)\|^2<0,\quad \forall p\in\mathbb{P};$$

GWMVVI$_\gamma$: For a given γ, let us find $(y,v)\in\mathbb{K}$ such that there exists no $(x,u)\in\mathbb{K}$, fulfilling

$$\int_D [\theta^p_x(t,x,u)\,(y-x)+\theta^p_u(t,x,u)(v-u)]\,d\omega+\gamma\|(y,v)-(x,u)\|^2>0,\quad \forall p\in\mathbb{P}.$$

Particular cases

If $\gamma=0$ in **GSVVI**$_\gamma$ (**GWSVVI**$_\gamma$), we obtain (weak) vector variational control inequality, studied by Treanţă, Antczak and Saeed (2023), or Treanţă and Saeed (2023). If we remove the control variable, consider $\gamma=0$ and $D=I=[a,b]\subset\mathbb{R}$, then we obtain the class of variational inequalities, formulated in Kim (2004).

In the next example, we show that there exists a solution of **GSVVI**$_\gamma$, but it is not a solution for vector variational control inequality, studied by Treanţă and Guo in (2023). Also, a practical application (mechanical work) of the theoretical developments, derived in the current paper, can be found in Treanţă, Antczak and Saeed (2023).

Exemple 2.2 *Consider* $\theta:[0,1]^2\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^2$, *given by*

$$\theta(t,x,u)=(\theta^1(t,x,u),\theta^2(t,x,u))=(-x-u,-x+u^2),$$

that generates the following vector functional

$$\int_{[0,1]^2}\theta(t,x,u)d\omega=\left(\int_{[0,1]^2}\theta^1(t,x,u)d\omega,\int_{[0,1]^2}\theta^2(t,x,u)d\omega\right),$$

with $x:[0,1]\to\mathbb{R}$, $x(t)=t^1+t^2$, *and* $u:[0,1]\to\mathbb{R}$, $u(t)=1$.

By direct computation, we find that $(y,v)=(0,0)$ *is a solution of* ***GSVVI***$_\gamma$. *Indeed, for* $\gamma=\frac{3}{2}$, *we get*

$$\Big(\int_{[0,1]^2}\left[\theta^1_x(t,y,v)\,(x-y)+\theta^1_u(t,y,v)(u-v)\right]d\omega+\gamma\|(x,u)-(y,v)\|^2,$$

$$\int_{[0,1]^2}\left[\theta^2_x(t,y,v)\,(x-y)+\theta^2_u(t,y,v)(u-v)\right]d\omega+\gamma\|(x,u)-(y,v)\|^2\Big)\nleq(0,0).$$

On the other hand, it can be easily shown that, for $\gamma=0$, *the corresponding class of variational control inequalities, studied by Treanţă and Guo in (2023), is not solvable at* $(0,0)$.

3. Main results

In the following, we investigate the connections, which appear between the solutions of the above-mentioned generalized Stampacchia vector variational control inequalities and the associated multiple-objective optimization problem P.

Next, we establish a connection between the considered multiple objective variational control problem P and the associated class of generalized Stampacchia vector variational control inequalities.

THEOREM 3.1 *For each* $p \in \mathbb{P}$, *consider the functional* $\Theta^p = \int_D \theta^p(t,\cdot,\cdot)d\omega$ *that is strongly convex on* $\mathbb{K}$ *with constant* δ_p. *If* $(y,v) \in \mathbb{K}$ *is a solution of* $\boldsymbol{GSVVI}_\gamma$, *where* $\gamma = \min\{\delta_1,...,\delta_r\}$, *then it is an efficient pair of* P.

PROOF Consider that (y,v) is a solution of $\mathbf{GSVVI}_\gamma$. Then, for a real constant γ, there exists no $(x,u) \in \mathbb{K}$ satisfying

$$\int_D \theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2 \leq 0, \qquad (4)$$

with $<$ for at least one $p \in \mathbb{P}$. We use the assumption of strong convexity of the functionals $\Theta^p = \int_D \theta^p(t,\cdot,\cdot)d\omega$ and we get that there exists a real constant $\delta_p > 0$ such that

$$\int_D [\theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \delta_p\|(x,u)-(y,v)\|^2$$
$$\leq \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u) \in \mathbb{K},\ p \in \mathbb{P}.$$

In particular, for $\gamma = \min\{\delta_1,...,\delta_r\}$, we get

$$\begin{aligned}&\int_D [\theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2 \\ &\quad \leq \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u) \in \mathbb{K},\ p \in \mathbb{P}.\end{aligned} \qquad (5)$$

Using (4) and (5), we obtain that there is no $(x,u) \in \mathbb{K}$ such that

$$\int_D \theta^p(t,x,u)d\omega \leq \int_D \theta^p(t,y,v)d\omega,$$

with $<$ for at least one $p \in \mathbb{P}$. Hence, the pair (y,v) is an efficient pair of P and the proof is complete. □

Now, we establish a connection between the class of generalized Stampacchia weak vector variational control inequalities and the class of generalized Minty weak vector variational control inequalities.

THEOREM 3.2 *Let the functional* $\Theta^p = \int_D \theta^p(t,\cdot,\cdot)d\omega$ *be strongly convex on* $\mathbb{K}$ *with constant* δ_p, $p \in \mathbb{P}$. *If* $(y,v) \in \mathbb{K}$ *solves* $\boldsymbol{GWSVVI}_\gamma$, *then it solves* $\boldsymbol{GWMVVI}_\gamma$, *where* $\gamma = \min\{\delta_1,...,\delta_r\}$.

PROOF We assume that (y,v) is a solution of $\mathbf{GWSVVI}_\gamma$, that is, for a given γ, there is no $(x,u) \in \mathbb{K}$ satistying

$$\int_D [\theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2 < 0, \quad p \in \mathbb{P}. \tag{6}$$

We obtain the following inequality by using the strong convexity of Θ^p, for a real $\delta > 0$,

$$\int_D [\theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \delta_p\|(x,u)-(y,v)\|^2$$
$$\leq \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u) \in \mathbb{K},\, p \in \mathbb{P}.$$

For $\gamma = \min\{\delta_1,...,\delta_r\}$, we get

$$\begin{aligned}&\int_D [\theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2 \\ &\quad\leq \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u) \in \mathbb{K},\, p \in \mathbb{P}.\end{aligned} \tag{7}$$

In inequality (7) we interchange (x,u) with (y,v) and obtain

$$\begin{aligned}&\int_D [\theta_x^p(t,x,u)(y-x) + \theta_u^p(t,x,u)(v-u)]d\omega + \gamma\|(y,v)-(x,u)\|^2 \\ &\quad\leq \int_D \theta^p(t,y,v)d\omega - \int_D \theta^p(t,x,u)d\omega, \quad \forall (x,u) \in \mathbb{K},\, p \in \mathbb{P}.\end{aligned} \tag{8}$$

Using (7) and (8) we arrive at

$$\begin{aligned}&\int_D \theta_x^p(t,x,u)(y-x) + \theta_u^p(t,x,u)(v-u)]d\omega + \gamma\|(y,v)-(x,u)\|^2 \\ &\leq -\left[\int_D \theta_x^p(t,y,v)(x-y) + \theta_u^p(t,y,v)(u-v)]d\omega\right] - \gamma\|(x,u)-(y,v)\|^2.\end{aligned} \tag{9}$$

Now, we combine (6) and (9) and, for $\gamma \in \mathbb{R}$, we can say that there is no $(x,u) \in \mathbb{K}$, satisfying

$$\int_D [\theta_x^p(t,x,u)(y-x) + \theta_u^p(t,x,u)(v-u)]d\omega + \gamma\|(y,v)-(x,u)\|^2 > 0, \quad p \in \mathbb{P}.$$

Hence, the pair (y,v) is a solution of $\mathbf{GWMVVI}_\gamma$. □

Next, we establish a connection between the considered multiple objective variational control problem P and the associated class of generalized Stampacchia weak vector variational control inequalities.

THEOREM 3.3 *Let the functional* $\Theta^p = \int_D \theta^p(t,\cdot,\cdot)d\omega$ *be a strongly convex functional on* $\mathbb{K}$ *with respect to* δ_p*, for each* $p\in\mathbb{P}$*. If* (y,v) *is a solution of* $\boldsymbol{GWSVVI}_\gamma$*, where* $\gamma=\min\{\delta_1,...,\delta_r\}$*, then it is a weak efficient pair of* P*.*

PROOF Assume, by contrary, that (y,v) is not a weak efficient pair of P. Thus, there exists $(x,u)\in\mathbb{K}$ such that

$$\int_D \theta^p(t,x,u)d\omega < \int_D \theta^p(t,y,v)d\omega, \quad \forall p\in\mathbb{P}. \tag{10}$$

Since the functional $\Theta^p = \int_D \theta^p(t,\cdot,\cdot)d\omega$ is strongly convex, it follows that there exists $\delta_p>0$, a real constant, such that

$$\begin{aligned}&\int_D [\theta^p_x(t,y,v)(x-y)+\theta^p_u(t,y,v)(u-v)]d\omega + \delta_p\|(x,u)-(y,v)\|^2\\ &\qquad\le \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u)\in\mathbb{K},\ p\in\mathbb{P}.\end{aligned}$$

If we consider $\gamma=\min\{\delta_1,...,\delta_r\}$, we have

$$\begin{aligned}&\int_D [\theta^p_x(t,y,v)(x-y)+\theta^p_u(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2\\ &\qquad\le \int_D \theta^p(t,x,u)d\omega - \int_D \theta^p(t,y,v)d\omega, \quad \forall (x,u)\in\mathbb{K},\ p\in\mathbb{P}.\end{aligned} \tag{11}$$

Taking into account inequalities (10) and (11), we obtain that there exists $(x,u)\in\mathbb{K}$ such that

$$\int_D [\theta^p_x(t,y,v)(x-y)+\theta^p_u(t,y,v)(u-v)]d\omega + \gamma\|(x,u)-(y,v)\|^2 < 0, \quad p\in\mathbb{P},$$

which contradicts that (y,v) is a solution of $\mathbf{GWSVVI}_\gamma$. □

EXEMPLE 3.1 *Consider* $\theta:[0,1]\times\mathbb{R}^2\to\mathbb{R}^2$*, given by*

$$\theta(t,x,u) = (\theta^1(t,x,u),\theta^2(t,x,u)) = (x^2,1+x^2),$$

that generates the following vector functional

$$\int_0^1 \theta(t,x,u)dt = \left(\int_0^1 \theta^1(t,x,u)dt, \int_0^1 \theta^2(t,x,u)dt\right),$$

with $x, y : [0,1] \to \mathbb{R}$, $x(t) = c_1 \cdot t$ *and* $y(t) = c_2 \cdot t$, $\forall c_1, c_2 \in \mathbb{R}$ *and* $u, v : [0,1] \to \mathbb{R}$, $u(t) = c_1$, $v(t) = c_2$. *For* $\delta_1 = \dfrac{1}{24}$, *we obtain*

$$
\begin{aligned}
&\int_0^1 [\theta_x^1(t,y,v)(x-y) + \theta_u^1(t,y,v)(u-v)]dt + \delta_1 \|(x,u) - (y,v)\|^2 \\
&- \int_0^1 \theta^1(t,x,u)dt + \int_0^1 \theta^1(t,y,v)dt \\
&= \int_0^1 [2y \cdot (x-y) + 0 \cdot (u-v)]dt + \delta_1 \|(x,u) - (y,v)\|^2 - \int_0^1 x^2 dt + \int_0^1 y^2 dt \\
&= \int_0^1 [2xy - 2y^2]dt + \delta_1 \|((c_1 - c_2)t, c_1 - c_2)\|^2 - \int_0^1 x^2 dt + \int_0^1 y^2 dt \\
&= \int_0^1 [2xy - 2y^2]dt + 4\delta_1 (c_1 - c_2)^2 - \int_0^1 x^2 dt + \int_0^1 y^2 dt \\
&= -\frac{(c_1 - c_2)^2}{6} \leq 0.
\end{aligned}
$$

Similarly, for $\delta_2 = \dfrac{1}{24}$, *we obtain*

$$
\begin{aligned}
&\int_0^1 [\theta_x^2(t,y,v)(x-y) + \theta_u^2(t,y,v)(u-v)]dt + \delta_2 \|(x,u) - (y,v)\|^2 \\
&- \int_0^1 \theta^2(t,x,u)dt + \int_0^1 \theta^2(t,y,v)dt \\
&= -\frac{(c_1 - c_2)^2}{6} \leq 0,
\end{aligned}
$$

involving the fact that the two functionals are strongly convex with respect to δ_1 *and* δ_2. *For* $\gamma = \frac{1}{24}$ *and* (y, v) *a solution of* $\boldsymbol{GWSVVI}_\gamma$, *by direct computation, it follows that* (y, v) *is a weak efficient pair of the corresponding extremization problem.*

4. Conclusions

In this paper, we have defined the notions of (strictly) strongly convex and preconvex functionals, given by controlled multiple integrals. By considering these concepts, we have studied the relationships between the solutions of generalized Stampacchia (weak) vector variational control inequalities and (weak) efficient pairs of the associated multiobjective optimization problems. More exactly, the new elements included in this study have been represented by the presence of the control variable and the associated concepts of (strictly) strong convexity and preconvexity for multiple integral functionals. As further developments, based on this study, the authors suggest the analysis of well-posedness and of saddle-point criteria for such type of problems.

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