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Research Article

Tempered fractional differential equations on hyperbolic space

Roberto Garra[1*], Enzo Orsingher[2,1]

[1]Universitá Telematica Internazionale, Section of Mathematics, Italy
[2]Department of Statistical Sciences Sapienza University of Rome, Italy

[*]Email address for correspondence: roberto.garra@uninettunouniversity.net

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Abstract

In this paper we study linear fractional differential equations involving tempered Caputo-type derivatives in the hyperbolic space. We consider in detail the three-dimensional case for its simple and useful structure. We also discuss the probabilistic meaning of our results in relation to the distribution of an hyperbolic Brownian motion time-changed with the inverse of a tempered stable subordinator. The generalization to an arbitrary dimension n can be easily obtained. We also show that it is possible to construct a particular solution for the non-linear porous-medium type tempered equation by using elementary functions.

Keywords: Tempered fractional derivatives, Hyperbolic geometry, Exact solutions

AMS subject classification: 35R11, 34A08

1. Introduction

In this paper we study fractional differential equations involving tempered time-fractional derivatives. We consider fractional diffusion equations on the hyperbolic Poincaré half-space. It is well-known that the probabilistic motivation for the study of tempered fractional equations is given by the connection between tempered fractional calculus and the inverse of the tempered stable subordinators as pointed out for example in [1]. Our aim is to obtain analytical and probabilistic results for the tempered fractional equations on the hyperbolic Poincaré half-space. There are few studies about fractional equations in hyperbolic spaces, we refer in particular to [2] and [3]. In these papers the authors studied fractional equations involving the Caputo fractional derivative, obtaining the representation of the solution with a probabilistic interpretation. In this paper we are able to find the Laplace transform of the solution for the tempered fractional diffusion on the 3-dimensional hyperbolic Poincaré half-space $\mathbb{H}^3$ and we prove the connection with the density of an hyperbolic Brownian motion time-changed with the inverse of the tempered stable subordinator.

In the second part of the paper we also consider the nonlinear diffusive problem, i.e. the counterpart of the porous medium equation. We show that it is possible to construct a particular solution for this non-trivial problem by separation of variables.

We recall that the structure of the Poincaré half-plane has a physical meaning and motivation in optics, in relation to the model of light passing through a non-homogeneous medium with a space-varying refraction coefficient. We refer for example to [4]. This paper provides an explorative generalization that can be useful for methodological reasons and potential future applications in this context.

2. Preliminaries on tempered fractional calculus

In this section we briefly recall the preliminary notions about tempered fractional derivatives and their probabilistic applications. We refer in particular to the paper [1] and the references therein. We will use the notation adopted in the paper [1] for the sake of simplicity.

Definition 2.1. The Riemann-Liouville fractional tempered derivative of order $\beta > 0$ is defined by

$$\left(\mathbb{D}_t^{\beta,\lambda} g\right)(t) = e^{-\lambda t}\mathbb{D}_t^{\beta}(e^{\lambda t} g) - \lambda^\beta g, \tag{1}$$

where we denoted by $\mathbb{D}_t^{\beta}(\cdot)$ the Riemann-Liouville fractional derivative of order β. The parameter $\lambda > 0$ is the so-called tempering parameter.

Recalling that the Riemann-Liouville fractional derivative of order $\beta > 0$ is defined as follows

$$\left(\mathbb{D}_t^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\beta-1} f(\tau) d\tau, \quad n-1 < \beta < n, \ n \in \mathbb{N}, \tag{2}$$

we can write the tempered fractional derivative in the explicit form

$$\left(\mathbb{D}_t^{\beta,\lambda} g\right)(t) = \frac{e^{-\lambda t}}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\beta-1} e^{\lambda\tau} g(\tau) d\tau - \lambda^\beta g(t). \tag{3}$$

By definition, it is clear that we recover the Riemann-Liouville fractional derivative if the tempering parameter $\lambda = 0$.
There are different approaches to the Caputo-like definition of the tempered fractional derivative. Here we follow the approach developed in [1] in view of the connection between Caputo tempered derivatives and inverse tempered stable subordinators. Moreover, this approach is mathematically based on the analogy with the classical Caputo derivative as a regularization in the time origin of the Riemann-Liouville derivative (see e.g. [5]).

Definition 2.2. The Caputo tempered fractional derivative of order $\beta > 0$ is defined by

$$\left(\partial_t^{\beta,\lambda} g\right)(t) = \mathbb{D}_t^{\beta,\lambda}\left[g - \sum_{j=0}^{n-1}\frac{t^j}{j!} g^{(j)}(0)\right], \tag{4}$$

where $n = \lceil \beta \rceil$ and we denoted by $g^{(j)}$ the ordinary derivatives of order j of the function $g(t)$.

Here we recall two relevant results that we will use in this paper (see [1] for the detailed proofs and discussion).

Proposition 2.1. *The Laplace transform of the Caputo tempered derivative of order $\beta > 0$ is given by*

$$\mathcal{L}\left[\partial_t^{\beta,\lambda} g\right](s) = \psi^\lambda(s)\tilde{g}(s) - \sum_{j=0}^{n-1} s^{-j-1}\psi^\lambda(s) g^{(j)}(0), \tag{5}$$

where s is the Laplace parameter, $\tilde{g}(s)$ is the Laplace transform of the function $g(t)$ and

$$\psi^\lambda(s) = (s+\lambda)^\beta - \lambda^\beta. \tag{6}$$

We finally recall the relevant connection between fractional tempered equations and their probabilistic meaning.
Let us denote with E_t^λ, $t > 0$, the inverse tempered stable subordinator, i.e.

$$E_t^\lambda = \inf\{x > 0 : D_x^\lambda > t\} \tag{7}$$

where D_x^λ is a tempered stable subordinator. According to [1], we have that the density of the inverse tempered stable subordinator E_t^λ, namely $h(x,t)$ solves the tempered fractional equation

$$\partial_t^{\beta,\lambda} h(x,t) = -\frac{\partial}{\partial x} h(x,t), \quad \beta \in (0,1) \tag{8}$$

with $h(x,0)=0$. This result can be simply obtained by using the Laplace transform of the tempered derivatives and recalling that

$$\tilde{h}(x,s) = \frac{\psi^\lambda(s)}{s} e^{-x\psi^\lambda(s)}, \tag{9}$$

where $\psi^\lambda(s)$ is defined in (6).
We observe that, if the tempering parameter $\lambda = 0$, we recover the well-known connection between the Caputo-fractional equation

$$\frac{\partial^\beta}{\partial t^\beta} h(x,t) = -\frac{\partial}{\partial x} h(x,t), \quad \beta \in (0,1) \tag{10}$$

and the density of the inverse of the stable subordinator (according to this notation E_t).

3. Tempered fractional differential equations on hyperbolic spaces

The diffusion equation on the hyperbolic Poincaré half-space $\mathbb{H}^n = \{\mathbf{x}, y : \mathbf{x} \in \mathbb{R}^{n-1}, y > 0\}$ is given by

$$\frac{\partial u}{\partial t} = \frac{1}{\sinh^{n-1}\eta} \frac{\partial}{\partial \eta}\left(\sinh^{n-1}\eta \frac{\partial}{\partial \eta}\right) u. \tag{11}$$

The case $\mathbb{H}^3$ is particularly interesting and simple. In this case the solution for the Cauchy problem with initial datum $u(\eta,0) = \delta(\eta)$ is given by (see e.g. [3])

$$u(\eta,t) = \frac{e^{-t}}{2\sqrt{\pi t^3}} \frac{\eta e^{-\frac{\eta^2}{4t}}}{\sinh \eta}. \tag{12}$$

On the other hand, we underline that the solutions for $n > 3$ can be obtained from the two-dimensional and three-dimensional cases by means of the Millson recursive formula (see equation (2.24) in [3]).
There are few studies in the literature about the fractional differential equations on hyperbolic spaces. As far as we know, the first paper on this topic was the paper by Lao and Orsingher [2]. Then, in the paper by D'Ovidio et al. [3], the authors studied fractional telegraph equations on hyperbolic spaces. In the paper [6], some results were obtained for the linear and nonlinear equation on Poincaré half plane for equations involving time-fractional derivatives w.r.t. another function.
Here we start our analysis from the following tempered fractional equation

$$\partial_t^{\beta,\lambda} u(\eta,t) = \frac{1}{\sinh^2\eta} \frac{\partial}{\partial \eta}\left(\sinh^2\eta \frac{\partial}{\partial \eta}\right) u(\eta,t), \quad \beta \in (0,1) \tag{13}$$

that is the tempered fractional generalization of the diffusive governing equation for the density of the 3-dimensional hyperbolic Brownian motion in the Poincaré hyperbolic half-space $\mathbb{H}^3$. This case is, in our view, particularly interesting for the simple structure of the Laplace transform of the probability density (and the probability density itself).

Theorem 3.1. *The Laplace transform of the solution for the tempered fractional equation* (13) *is given by*

$$\tilde{u}(\eta,s) = \frac{(s+\lambda)^\beta - \lambda^\beta}{s \sinh \eta} e^{-\eta\sqrt{1+\psi^\lambda(s)}} = \frac{\psi^\lambda(s)}{s \sinh \eta} e^{-\eta\sqrt{1+\psi^\lambda(s)}} \tag{14}$$

Proof. By using (5) in (13), we obtain that the Laplace transform of the fundamental solution satisfies the equation

$$\psi^\lambda(s)\tilde{u}(\eta,s) = \frac{1}{\sinh^2\eta} \frac{d}{d\eta} \sinh^2\eta \frac{d}{d\eta} \tilde{u}(\eta,s), \tag{15}$$

whose solution is given by (14) by direct calculations.
Indeed, by substitution we have that

$$\left(\frac{1}{\sinh^2\eta}\frac{d}{d\eta}\sinh^2\eta\frac{d}{d\eta}\right)\left(\frac{\psi^\lambda(s)}{s\sinh\eta}e^{-\eta\sqrt{1+\psi^\lambda(s)}}\right)=\frac{(\psi^\lambda(s))^2}{s\sinh\eta}e^{-\eta\sqrt{1+\psi^\lambda(s)}},$$

as claimed. □

Theorem 3.2. *The fundamental solution for the equation (13) coincides with the density of the hyperbolic Brownian motion time-changed with the inverse of the tempered stable subordinator E_t^λ, namely $B_3^{hp}(E_t^\lambda)$.*

Proof. Let us denote by $g(x,t)$ the density of the process $B_3^{hp}(E_t^\lambda)$. By definition, the Laplace transform of $g(x,t)$ is given by

$$\tilde{g}(s,\eta)=\int_0^\infty k(\eta,\xi)\int_0^\infty e^{-st}h(\xi,t)d\xi dt, \tag{17}$$

where $h(\eta,t)$ is the density of the inverse tempered stable subordinator (see the previous section) and

$$k(\eta,t)=\frac{e^{-t}}{2\sqrt{\pi t^3}}\frac{\eta e^{-\frac{\eta^2}{4t}}}{\sinh\eta} \tag{18}$$

is the fundamental solution of the equation governing the process B_3^{hp}.
By using the fact that

$$\tilde{h}(x,s)=\frac{\psi^\lambda(s)}{s}e^{-x\psi^\lambda(s)} \tag{19}$$

we have that

$$\tilde{g}(\eta,s)=\frac{\psi^\lambda(s)}{2\sqrt{\pi}s\sinh\eta}\int_0^\infty e^{-\xi(1+\psi^\lambda(s))}\frac{e^{-\frac{\eta^2}{4\xi}}}{\sqrt{\xi^3}}d\xi=\frac{\psi^\lambda(s)}{s\sinh\eta}e^{-\eta\sqrt{1+\psi^\lambda(s)}}, \tag{20}$$

that coincides with (14), as claimed. □

Corollary 3.1. *For $\lambda=0$, we obtain that the fundamental solution for the equation*

$$\frac{\partial^\beta}{\partial t^\beta}u(\eta,t)=\frac{1}{\sinh^2\eta}\frac{\partial}{\partial\eta}\left(\sinh^2\eta\frac{\partial}{\partial\eta}\right)u(\eta,t), \tag{21}$$

coincides with the density of the hyperbolic Brownian motion time-changed with the inverse of the stable subordinator E_t, namely $B_3^{hp}(E_t)$. The fractional derivative comparing in (21) is the Caputo fractional derivative of order $\beta\in(0,1)$.

4. Tempered fractional porous medium type equation

In the recent paper [6], the authors have considered some particular results for nonlinear fractional equations in the hyperbolic space.
This topic is object of interest in view of the relevance of nonlinear differential equations for the applications. Here we consider the following tempered fractional porous-medium-type equation in the PoincarÃ¨ half-space $\mathbb{H}^3$

$$\mathbb{D}_t^{\beta,\lambda}u(\eta,t)=\frac{1}{\sinh^2\eta}\left(\frac{\partial}{\eta}\sinh^2\eta\frac{\partial}{\eta}\right)u^n(\eta,t),\quad n\in\mathbb{N},\ n\geq 2,\ \beta\in(0,1). \tag{22}$$

We have the following result

Theorem 4.1. *The tempered fractional porous medium type equation* (22) *admits a solution of the form*

$$u(\eta,t)=t^{\beta-1}e^{-\lambda t}E_{\beta,\beta}(\lambda^{\beta}t^{\beta})\,\sqrt[n]{c_1\,(\coth(\eta))+c_2}, \tag{23}$$

where c_1 and c_2 are positive real constants and $E_{\beta,\beta}(\cdot)$ is the two-parameter Mittag-Leffler function.

Proof. We search a solution by separation of variable

$$u(\eta,t)=f(t)g(\eta), \tag{24}$$

where $f(t)$ is such that

$$\mathbb{D}_t^{\beta,\lambda}f(t)=0, \tag{25}$$

and $g(\eta)$ is such that

$$\frac{1}{\sinh^2\eta}\frac{d}{d\eta}\sinh^2\eta\frac{d}{d\eta}g^n(\eta)=0, \tag{26}$$

that admits a solution of the form

$$g(\eta)=\sqrt[n]{c_1\;\;(\coth(\eta))+c_2}, \tag{27}$$

where c_1 and c_2 are suitable positive real constants.
Observe that the fractional equation (25) is equivalent to

$$e^{-\lambda t}\mathbb{D}_t^{\beta}(e^{\lambda t}f)=\lambda^{\beta}f \tag{28}$$

whose solution is given by

$$f(t)=t^{\beta-1}e^{-\lambda t}E_{\beta,\beta}(\lambda^{\beta}t^{\beta}). \tag{29}$$

Then, by direct calculation we have the claimed result. □

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