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CAIM
ISSN 2038-0909

Commun. Appl. Ind. Math. 15 (2), 2024, 36–43 DOI: 10.2478/caim-2024-0011

Research Article

A degenerate version of hypergeometric Bernoulli polynomials: announcement of results

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Communicated by Donatella Occorsio

Received on 06 28, 2024. Accepted on 09 04, 2024.

Abstract

This article explores some properties of degenerate hypergeometric Bernoulli polynomials, which are defined through the following generating function

$$\frac{t^m\, e_\lambda^x(t)}{e_\lambda(t) - \sum_{l=0}^{m-1} (1)_{l,\lambda} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \quad |t| < \min\left\{2\pi, \frac{1}{|\lambda|}\right\}, \lambda \in \mathbb{R}\setminus\{0\}.$$

We deduce their associated summation formulas and their corresponding determinant form. Also we focus our attention on the zero distribution of such polynomials and perform some numerical illustrative examples, which allow us to compare the behavior of the zeros of degenerate hypergeometric Bernoulli polynomials with the zeros of their hypergeometric counterpart. Finally, using a monomiality principle approach we present a differential equation satisfied by these polynomials.

Keywords: Degenerate Bernoulli polynomials, hypergeometric Bernoulli polynomials, degenerate hypergeometric Bernoulli Euler polynomials.

AMS subject classification: 33E20; 32A05; 11B83.

1. Introduction

Due to the ease with which polynomials can be used for computation and mathematical operations, special functions and polynomials hold significant value not only in mathematics but also in physics and engineering. Recently, many researchers have reignited their interest in studying Δ_λ-special polynomials and their hybrid forms. The importance of Δ_λ-special polynomials extends beyond their combinatorial and mathematical features to their applications in differential equations, probability theory, and symmetric identities. Using the classical finite difference operator Δ_λ, Costabile and Longo (as referenced in [1] and its citations) introduced a new analogue of the Appell polynomials, known as the Δ_λ-Appell polynomials. These Δ_λ-Appell polynomials have been extensively studied due to their remarkable applications not only in various branches of mathematics but also in physics and statistics. To recap, the following is the definition of Δ_λ-Appell polynomials:

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}^+$, then the finite difference operator Δ_λ is given as (see [2]):

$$\Delta_\lambda[f](x) = f(x+\lambda) - f(x).$$

For $n \in \mathbb{N}_0$, the Δ_λ-Appell polynomials $\mathcal{A}_n^\lambda(x)$ are given by the following generating equation (see [1]):

$$a(t)(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{A}_n^\lambda(x)\frac{t^n}{n!}, \tag{1}$$

where

$$a(t)=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!},\quad \alpha_0\neq 0.$$

For $\lambda, x, t\in\mathbb{R}$, the degenerate exponentials are defined as follows (cf., [3,4]):

$$e_\lambda^x(t)=\begin{cases}(1+\lambda t)^{\frac{x}{\lambda}}, & \text{if } \lambda\in\mathbb{R}\setminus\{0\},\\ e^{xt}, & \text{if } \lambda=0,\end{cases}=\begin{cases}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!}, \quad |\lambda t|<1, & \text{if } \lambda\in\mathbb{R}\setminus\{0\},\\ \sum_{n=0}^{\infty}x^n\frac{t^n}{n!}, & \text{if } \lambda=0.\end{cases}$$

As usual, for $x=1$ we use the notation $e_\lambda(t)=e_\lambda^1(t)$, and the generalized falling factorials $(x)_{n,\lambda}$, are given by (cf., [4–6]):

$$(x)_{n,\lambda}=\begin{cases}1, & \text{if } n=0,\\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n\geq 1,\\ 0, & \text{if } n<0,\end{cases}$$

where $x,\lambda\in\mathbb{R}$ and $n\in\mathbb{Z}$.

For instance, for each $\lambda\in\mathbb{R}$ the generalized $\{(x)_{n,\lambda}\}_{n\geq 0}$ falling factorials are Δ_λ-Appell sequences and they satisfy the following relation [1]:

$$\Delta_\lambda^k[(x)_{n,\lambda}]=\frac{n!}{(n-k)!}(x)_{n-k,\lambda},\quad 0\leq k\leq n.$$

Furthermore, if we consider the degenerate exponential $e_\lambda^x(t)$ as a function of the variable x, that is $f_{t,\lambda}(x)=e_\lambda^x(t)$, then we have

$$\Delta_\lambda[e_\lambda^x](t)=\lambda t e_\lambda^x(t).$$

Δ_λ-Appell polynomials was considered and studied from an algebraic point of view by Costabile and Longo in [1]. More precisely, given the power series

$$a(t)=\alpha_0+\frac{t}{1!}\alpha_1+\frac{t^2}{2!}\alpha_2+\cdots+\frac{t^n}{n!}\alpha_n+\cdots,\quad \alpha_0\neq 0$$

with $\alpha_i\in\mathbb{R}$, $i\in\mathbb{N}_0$, and $\left\{\mathcal{A}_n^\lambda(x)\right\}_{n\in\mathbb{N}_0}$ the sequence of Δ_λ-Appell polynomials determined by (1), Costabile and Longo [1] obtained the following determinantal form for the polynomials $\mathcal{A}_n^\lambda(x)$ when it is assumed that $\{\beta j\}_{j\in\mathbb{N}_0}$ ($\beta_0\neq 0$) is the sequence of Taylor coefficients for the series expansion of $\frac{1}{a(t)}$:

$$\text{(2)}\qquad\begin{cases}\mathcal{A}_0^\lambda(x)=\frac{1}{\beta_0},\\ \mathcal{A}_n^\lambda(x)=\frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix}1 & (x)_1 & \cdots & (x)_{n-1} & (x)_n\\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n\\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1}\\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1\end{vmatrix}, n=1,2,\ldots.\end{cases}$$

Consequently, the polynomial characterization (2) allows to prove some known general properties, by using elementary linear algebra tools as in the setting of Appell polynomials (see [7] for more details).

1.1. *Hypergeometric Bernoulli polynomials*

For a fixed $m \in \mathbb{N}$, the hypergeometric Bernoulli polynomials are defined by means of the following generating function [8–11].

$$\frac{t^m e^{xt}}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}, \quad |t| < 2\pi, \tag{3}$$

and the hypergeometric Bernoulli numbers are defined by $B_n^{[m-1]} := B_n^{[m-1]}(0)$ for all $n \geq 0$. The hypergeometric Bernoulli polynomials also are called generalized Bernoulli polynomials of level m [11,12]. It is clear that if $m = 1$ in (3), then we recover the definition of the classical Bernoulli polynomials $B_n(x)$ and classical Bernoulli numbers, respectively, i.e., $B_n(x) = B_n^{[0]}(x)$ and $B_n = B_n^{[0]}$, respectively, for all $n \geq 0$.

The first four hypergeometric Bernoulli polynomials are as follows:

$$\begin{aligned}
B_0^{[m-1]}(x) &= m!, \\
B_1^{[m-1]}(x) &= m! \left(x - \frac{1}{m+1} \right), \\
B_2^{[m-1]}(x) &= m! \left(x^2 - \frac{2}{m+1} x + \frac{2}{(m+1)^2(m+2)} \right), \\
B_3^{[m-1]}(x) &= m! \left(x^3 - \frac{3}{m+1} x^2 + \frac{6}{(m+1)^2(m+2)} x + \frac{6(m-1)}{(m+1)^3(m+2)(m+3)} \right).
\end{aligned}$$

The following results summarize some properties of the hypergeometric Bernoulli polynomials (cf. [8,9, 11]).

Proposition 1.1. *For a fixed $m \in \mathbb{N}$, let $\left\{ B_n^{[m-1]}(x) \right\}_{n \geq 0}$ be the hypergeometric Bernoulli polynomials. Then the following statements hold:*

a) Summation formula. For every $n \geq 0$,

$$B_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{[m-1]} x^{n-k}.$$

b) Differential relations (Appell polynomial sequences). For $n, j \geq 0$ with $0 \leq j \leq n$, we have

$$[B_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} B_{n-j}^{[m-1]}(x).$$

c) Inversion formula. [9, Equation (2.6)] For every $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(m+k)!} B_{n-k}^{[m-1]}(x).$$

Consequently, the set $\left\{ B_0^{[m-1]}(x), B_1^{[m-1]}(x), \dots, B_n^{[m-1]}(x) \right\}$ is a basis for $\mathbb{P}_n$ (cf., [11]).

d) Recurrence relation. [9, Lemma 3.2] For every $n \geq 1$,

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1} \right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x).$$

e) Integral formulas.

$$\int_{x_0}^{x_1} B_n^{[m-1]}(x)dx = \frac{1}{n+1}\left[B_{n+1}^{[m-1]}(x_1) - B_{n+1}^{[m-1]}(x_0)\right]$$

$$= \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k} B_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}).$$

$$B_n^{[m-1]}(x) = n\int_0^x B_{n-1}^{[m-1]}(t)dt + B_n^{[m-1]}.$$

f) [9, Theorem 3.1] Differential equation. For every $n \geq 1$, the polynomial $B_n^{[m-1]}(x)$ satisfies the following differential equation

$$\frac{B_n^{[m-1]}}{n!}y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!}y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!}y'' + (m-1)!\left(\frac{1}{m+1} - x\right)y' + n(m-1)!y = 0.$$

2. Main results

This section is devoted to show some properties of degenerate hypergeometric Bernoulli polynomials. We just give some sketch proofs of our results, since this an abridged version of the paper [13].

Definition 2.1. For $\lambda \in \mathbb{R} \setminus \{0\}$ and a fixed $m \in \mathbb{N}$, the degenerate hypergeometric of Bernoulli polynomials are defined by means of the following generating function and series;

$$\frac{t^m\, e_\lambda^x(t)}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!}, \qquad |t| < \min\left\{2\pi, \frac{1}{|\lambda|}\right\}. \tag{4}$$

For $x = 0$ in (4), we have the so-called degenerate hypergeometric of Bernoulli numbers $B_{n,\lambda}^{[m-1]}$ generated by

$$\frac{t^m}{e_\lambda(t) - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}\frac{t^n}{n!}, \qquad |t| < \min\left\{2\pi, \frac{1}{|\lambda|}\right\}.$$

A simple MATLAB code developed to determine the series expansion on the LHS of (4) allows us the first degenerate hypergeometric Bernoulli polynomials:

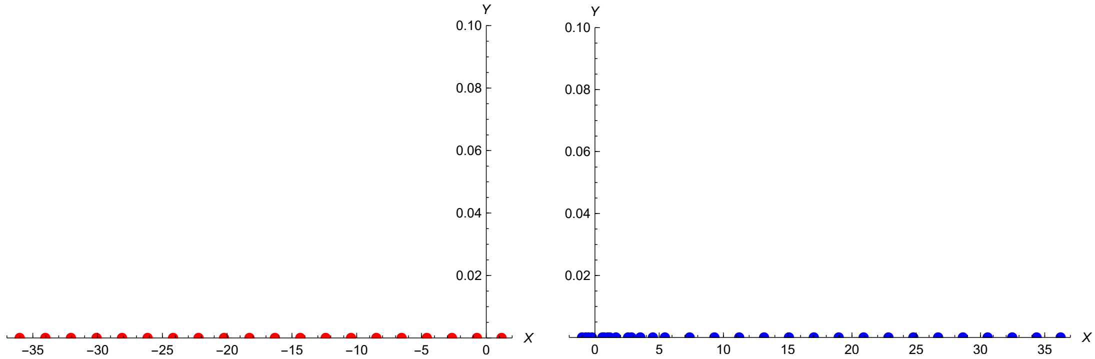
$$\begin{aligned}
B_{0,\lambda}^{[m-1]}(x) &= \frac{m!}{(1)_{m,\lambda}},\\
B_{1,\lambda}^{[m-1]}(x) &= \frac{m!}{(1)_{m,\lambda}}\left[x - \frac{(1-m\lambda)}{(m+1)}\right],\\
B_{2,\lambda}^{[m-1]}(x) &= \frac{m!}{(1)_{m,\lambda}}\left[(x)_{2,\lambda} - \frac{2(1)_{m+1,\lambda}}{(m+1)(1)_{m,\lambda}}x + \frac{2(1)_{m+1,\lambda}(1-m\lambda)}{(m+1)^2(1)_{m,\lambda}} - \frac{2(1)_{m+2,\lambda}}{(m+2)(m+1)(1)_{m,\lambda}}\right],\\
B_{3,\lambda}^{[m-1]}(x) &= \frac{m!}{(1)_{m,\lambda}}\left[(x)_{3,\lambda} - \frac{3(1)_{m+1}}{(m+1)(1)_{m,\lambda}}(x)_{2,\lambda} + \left(\frac{6(1)_{2+m,\lambda}}{(m+2)(m+1)(1)_{m,\lambda}} - \frac{6(1)_{m+1}^2}{(m+1)^2(1)_{m,\lambda}}\right)x\right.\\
&\quad + \left(\frac{6(1)_{m+1,\lambda}^2}{(m+1)^3(1)_{m,\lambda}^2} - \frac{6(1)_{m+2,\lambda}}{(m+2)(m+1)^2(1)_{m,\lambda}}\right)(1-m\lambda)\\
&\quad \left. + \left(\frac{6(1)_{m+3,\lambda}}{(m+3)(m+2)(m+1)(1)_{m,\lambda}} - \frac{6(1)_{m+1,\lambda}(1)_{m+2,\lambda}}{(m+1)^2(m+2)(1)_{m,\lambda}^2}\right)\right].
\end{aligned}$$

Regarding the zero distribution of these polynomials it is remarkable that numerical evidence indicates that this distribution does not align with the behavior of the Bernoulli hypergeometric zeros. For instance, in Table 1 we show the zeros of the hypergeometic polynomial $B_{20}^{[1]}(x)$ and the zeros of the degenerate hypergeometric Bernoully polynomial $B_{20,\lambda}^{[1]}(x)$ for $\lambda = \pm 2$.

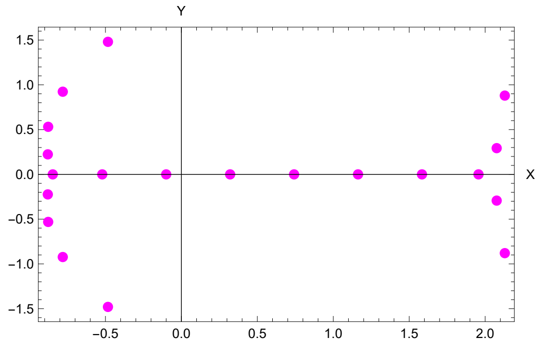
Polynomial	Real zeros	Complex zeros
$B_{20}^{[1]}(x)$	-0.846947, -0.521141, -0.100094,0.320946 0.741987,1.16303,1.58408,1.95623	-0.87952 - 0.22361i,-0.87952 + 0.22361 i -0.877128 - 0.531072i,-0.877128 + 0.531072i -0.780997 - 0.923175i,-0.780997 + 0.923175 i -0.483561 - 1.48024 i,0.483561 + 1.48024 2.07599 - 0.292932 i,2.07599 + 0.292932i 2.1295 - 0.87976i,2.1295 + 0.87976 i
$B_{20,-2}^{[1]}(x)$	-36.014385604730116, -34.03462061653576, -32.057093041815556,-30.084233697177908, -28.114706874568874, -26.142560318296454, -24.176515777209143, -22.20878854882131, -20.241514006453894,-18.278718989438424, -16.316085443481715, -14.355027759517451, -12.395901053934262, -10.43908269608266, -8.48510191126173,-6.534760439158918, -4.589320326980511, -2.650961604085424, -0.7240544243679391, 1.179851643411796	
$B_{20,2}^{[1]}(x)$	-0.2369742575673881, 1.6323795170176767, 3.5314424969544493, 5.444908212413409, 7.36690893492561,9.294459747499188, 11.225768972545103, 13.159634467825722, 15.095134452408075, 17.031833262528618, 18.96807703290493, 20.905126749860337, 22.83767978195378, 24.777519729851992, 26.705159537842224, 28.6317749160933, 30.55657554578635, 32.46828298150592, 34.36760676184257, 36.23696419872402	

Table 1. Zeros of $B_{n,\lambda}^{[m-1]}(x)$, for $n = 20$, $m = 2$ and $\lambda = 1, \pm 2$.

While, Figure 1 shows the distribution of each one zeros given in Table 1.



(a) Zeros of $B^{[1]}_{20,-2}(x)$. (b) Zeros of $B^{[1]}_{20,2}(x)$.



(c) Zeros of $B^{[1]}_{20}(x)$.

Figure 1. Zero distribution of $B^{[m-1]}_{n,\lambda}(x)$, for $n = 20$, $m = 2$ and $\lambda = 1, \pm 2$.

Finally, Figure 2 shows the induced meshes of $B^{[1]}_{j,\lambda}(x)$ for $\lambda = \pm 2$, $j = 1, 2, \ldots, 20$. Notice that the superposition of meshes points near Y-axis yields an "eclipse effect".

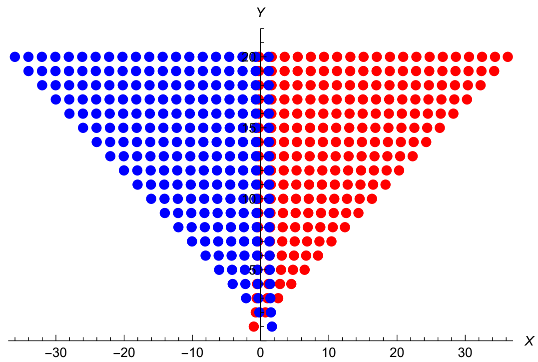


Figure 2. Induced meshes of $B^{[m-1]}_{j,\lambda}(x)$, for $m = 2$ with $\lambda = -2$ (red points) and $\lambda = 2$ (blue points), $j = 1, 2, \ldots, 20$.

In what follows we present some properties satisfied by the polynomial sequence $\left\{B^{[m-1]}_{n,\lambda}(x)\right\}_{n\geq 0}$.

Theorem 2.1. *For the n-th degenerate hypergeometric Bernoulli polynomial $B^{[m-1]}_{n,\lambda}(x)$ the following implicit summation formula holds.*

$$B^{[m-1]}_{n,\lambda}(x) = \sum_{k=0}^{n} \binom{n}{k} (x)_{k,\lambda} B^{[m-1]}_{n-k,\lambda}.$$

Sketch of Proof. By (4), using the Cauchy product rule, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}(x)\frac{t^n}{n!} &= \frac{t^m}{e_\lambda(t)-\sum_{l=0}^{m-1}(1)_{l,\lambda}} e_\lambda^x(t) \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}^{[m-1]}\frac{t^n}{n!}\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}B_{n-k,\lambda}^{[m-1]}\frac{t^n}{n!}.\end{aligned}$$

This finishes the proof. □

From (1) (cf. [1, Theorem 2]), we can deduce a determinantal form for degenerate Bernoulli hypergeometric polynomials.

Proposition 2.1. *The degenerate hypergeometric Bernoulli polynomials has the following determinantal representation:*

$$\begin{cases} B_{0,\lambda}^{[m-1]}(x) = \frac{1}{\beta_0}, \\ B_{n,\lambda}^{[m-1]}(x) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & (x)_{1,\lambda} & \cdots & (x)_{n-1,\lambda} & (x)_{n,\lambda} \\ \beta_0 & \beta_1 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots, \end{cases}$$

where the numerical sequence $\{\beta_n\}_{n\geq 0}$

$$\left[\frac{t^m}{e_\lambda(t)-\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}\right]^{-1} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}.$$

On the other hand, it is well known that the monomiality principle is based on an abstract definition of the concept of derivative and multiplicative operators which allows us to treat different families of special polynomials as ordinary monomials. Many relevant properties of a broad classes of special polynomials can be conveniently framed within the context of the monomiality principle [14–16]. In the particular case of degenerate hypergeometric Bernoulli polynomials, it is possible to show that they are quasi-monomials with respect to the following derivative and multiplicative operators (see [13]):

$$\hat{\mathcal{D}}_\lambda^{[m-1]} = \frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda} \tag{5}$$

and

$$\hat{M}_\lambda^{[m-1]} = \frac{\lambda m}{e^{\lambda\frac{\partial}{\partial x}}-1} + \frac{x}{e^{\lambda\frac{\partial}{\partial x}}} - \frac{(e^{\frac{\partial}{\partial x}})^{1-\lambda} - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}{e^{\frac{\partial}{\partial x}} - \sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{(e^{\lambda\frac{\partial}{\partial x}}-1)^l}{\lambda^l l!}}. \tag{6}$$

The following result is one of aforementioned relevant properties in the setting of degenerate hypergeometric Bernoulli polynomials.

Proposition 2.2. *For* $n \in \mathbb{N}$ *and a fixed* $m \geq 2$*, the* n*-th degenerate hypergeometric Bernoulli polynomial* $B_{n,\lambda}^{[m-1]}(x)$ *satisfy the following differential equation:*

$$\left[m + \frac{x}{e_\lambda^\lambda(t)}\frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda} - \frac{e_\lambda(t)e_\lambda^{-1}(t) - \sum_{l=0}^{m-2}(1)_{l+1,\lambda}\frac{t^l}{l!}}{e_\lambda(t)-\sum_{l=0}^{m-1}(1)_{l,\lambda}\frac{t^l}{l!}}\frac{e^{\lambda\frac{\partial}{\partial x}}-1}{\lambda} - n\right]B_{n,\lambda}^{[m-1]}(x) = 0.$$

Sketch of Proof. It is suffices to compose the operators (5) and (6) and use the identity

$$\hat{M}_\lambda^{[m-1]} \circ \hat{\mathcal{D}}_\lambda^{[m-1]}\left[B_{n,\lambda}^{[m-1]}(x)\right] - nB_{n,\lambda}^{[m-1]}(x) = 0.$$

□

3. Conclusions

In this paper, we have presented some properties of the degenerate hypergeometric Bernoulli polynomials [13], such that their associated implicit addition formula (Theorem 2.1) and their determinant form (Proposition 2.1). Furthermore, a brief discussion about the zero distribution of these polynomials is done using illustrative numerical examples. Finally, using an approach based on the principle of monomiality, we have presented the corresponding differential equation satisfied by the degenerate hypergeometric Bernoulli polynomials.

Acknowledgements

The research of Y. Quintana has been partially supported by Grant CEX2019-000904-S, funded by: MCIN/AEI/10.13039/501100011033.

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