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Research Article

Mathematical Insights into Hydrostatic Modeling of Stratified Fluids

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Abstract

We review recent mathematical results concerning the analysis of hydrostatic equations in the context of stably stratified fluids. Beginning with the simpler and better understood setting of homogeneous fluids, we emphasize the additional mathematical challenges posed by non-homogeneous framework. We present both positive and negative results, including well-posedness and proof of the hydrostatic limit with a suitable regularization, alongside ill-posedness in the fully inviscid setting and the breakdown of the hydrostatic limit in specific scenarios.

Keywords: PDE, oceanography, stratified fluids, hydrostatic limit, well-posedness, instabilities

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1. The equations for non-homogeneous density fluids

The following system describes the evolution of non-homogeneous incompressible flows under the influence of gravity,

$$\partial_t \rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} \rho + w \partial_z \rho = 0, \tag{1.1}$$

$$\rho\big(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + w \partial_z \mathbf{u}\big) = -\nabla_{\mathbf{x}} P, \tag{1.2}$$

$$\rho\big(\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w \partial_z w\big) = -\partial_z P - g\,\rho, \tag{1.3}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \tag{1.4}$$

$$P|_{z=\zeta} = P_{\mathrm{atm}}, \quad w|_{z=-H} = 0. \tag{1.5}$$

Here, t and $(\mathbf{x}, z)$ are time, and horizontal-vertical space variables, and we denote by $\nabla_{\mathbf{x}}, \nabla_{\mathbf{x}} \cdot, \Delta_{\mathbf{x}}$ the gradient, divergence and Laplacian with respect to $\mathbf{x}$. The unknowns are the velocity field $(\mathbf{u}, w) \in \mathbb{R}^2 \times \mathbb{R}$ (horizontal and vertical), the density $\rho > 0$ and the pressure $P \in \mathbb{R}$, all depending on $(t, \mathbf{x}, z)$. The equation (1.1) (resp. (1.2) or (1.3)) is the conservation of mass (resp. of momentum in the horizontal or vertical direction), while (1.4) is the divergence-free condition related to the incompressibility of the fluid.

In the most general case, the spatial domain Ω_t, which is moving, is delineated by a free surface:

$$\Omega_t = \{(\mathbf{x}, z) \ : \ \mathbf{x} \in \mathbb{R}^2, \ -H < z < \zeta(t, \mathbf{x})\}, \tag{1.6}$$

where H is the depth of the layer at rest and the location of the free surface $\zeta(t, \mathbf{x})$ evolves through the kinematic boundary condition

$$\partial_t \zeta + \mathbf{u}|_{z=\zeta} \cdot \nabla_{\mathbf{x}} \zeta - w|_{z=\zeta} = 0, \tag{1.7}$$

ensuring that fluid particles do not cross the free surface. The gravity field is assumed to be constant and vertical, and $g > 0$ is the gravity acceleration constant. Finally, a atmospheric pressure is prescribed at the surface, with $P|_{z=\zeta} = P_{\mathrm{atm}} \in \mathbb{R}$.

In the modeling of shallow water flows, which are relevant for applications to oceanography (more specifically, to coastal oceanography, see for instance [1]) and where the depth is much smaller than the typical horizontal scale, vertical accelerations in (1.3) can be neglected so that the pressure P approximately satisfies the hydrostatic balance law, that is

$$\partial_z P + g\,\rho = 0. \tag{1.8}$$

Specifically, replacing the vertical momentum equation (1.3) by the identity in (1.8) yields the so-called *hydrostatic equations*:

$$\begin{aligned} \partial_t\rho + \mathbf{u}\cdot\nabla_{\mathbf{x}}\rho + w\partial_z\rho &= 0,\\ \rho\big(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + w\partial_z\mathbf{u}\big) &= -\nabla_{\mathbf{x}}P,\\ \partial_t\zeta + \mathbf{u}|_{z=\zeta}\cdot\nabla_{\mathbf{x}}\zeta - w|_{z=\zeta} &= 0,\\ \nabla_{\mathbf{x}}\cdot\mathbf{u} + \partial_z w &= 0,\\ P = P_{\mathrm{atm}} + g\int_z^\zeta \rho(z',\cdot)\,\mathrm{d}z',\quad w = -\int_{-H}^z &\nabla_{\mathbf{x}}\cdot\mathbf{u}(z',\cdot)\,\mathrm{d}z'. \end{aligned} \tag{1.9}$$

In particular, there is no evolution equation for the vertical velocity, which can only be derived through the divergence-free condition (1.4). This significant aspect will be central to the mathematical challenges encountered in the hydrostatic limit. Additionally, there is a strong connection with the inviscid primitive equations, as discussed in [2].

The *hydrostatic approximation* procedure can actually be clarified by introducing a non-dimensional quantity known as the *shallowness parameter*

$$\varepsilon^2 := \frac{H^2}{L^2}, \tag{1.10}$$

representing the ratio between the depth H and the typical horizontal scale L, in the regime $L \gg H$ or $0 < \varepsilon \ll 1$. Dimensionless quantities are then defined as follows,

(1.11)
$$\tilde{x} := \frac{x}{L},\quad \tilde{z} := \frac{z}{H},\quad \tilde{t} := \frac{\sqrt{gH}}{L}t,\qquad \tilde{\mathbf{u}} := \frac{\mathbf{u}}{\sqrt{gH}},\quad \tilde{w} := \frac{L}{H}\frac{w}{\sqrt{gH}},\quad \tilde{P} := \frac{P}{gH},\quad \tilde{P}_{\mathrm{atm}} := \frac{P_{\mathrm{atm}}}{gH},\quad \tilde{\zeta} := \frac{\zeta}{H},$$

so that, dropping the tilde, equations (1.1)-(1.4) read

$$\begin{aligned} \partial_t\rho + \mathbf{u}\cdot\nabla_{\mathbf{x}}\rho + w\partial_z\rho &= 0,\\ \rho\big(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + w\partial_z\mathbf{u}\big) &= -\nabla_{\mathbf{x}}P,\\ \varepsilon^2\rho\big(\partial_t w + \mathbf{u}\cdot\nabla_{\mathbf{x}}w + w\partial_z w\big) &= -\partial_z P - \rho,\\ \partial_t\zeta + \mathbf{u}|_{z=\zeta}\cdot\nabla_{\mathbf{x}}\zeta - w|_{z=\zeta} &= 0,\\ \nabla_{\mathbf{x}}\cdot\mathbf{u} + \partial_z w &= 0, \end{aligned} \tag{1.12}$$

with (1.7) and the boundary conditions

$$P|_{z=\zeta} = P_{\mathrm{atm}},\quad w|_{z=-1} = 0. \tag{1.13}$$

Heuristically, the limit $\varepsilon \to 0$ leads to (1.8) and therefore to the *hydrostatic equations* (1.9). As we shall see, this limit can be understood as a long-wave asymptotic, and it is also related to the long-time dynamics of the original equations (1.1)-(1.5).

Mathematically, the two main challenges in the hydrostatic limit procedure are the study of the well-posedness of the limiting system (1.9) and its rigorous derivation in the limit $\varepsilon \to 0$.

In this note, we review some recent mathematical results on the well-posedness of the hydrostatic equations, as well as the rigorous justification Vs failure of the hydrostatic limit.

2. The case of 2d homogeneous fluids

In this section, we first simplify the situation by considering a homogeneous fluid, that is with constant density. If satisfied at the initial time, this property is then formally preserved by the evolution. We also restrict ourselves to a 2d setting, where the horizontal variable is now $x \in \mathbb{R}$.

2.1. *The Benney system*

In the 2d setting with constant density $\rho = 1$, the system (1.12) reads as follows

$$\begin{aligned} \partial_t u + u\partial_x u + w\partial_z u &= -\partial_x P, \\ \varepsilon^2 \big(\partial_t w + u\partial_x w + w\partial_z w\big) &= -\partial_z P - 1, \\ \partial_x u + \partial_z w &= 0, \end{aligned}$$

with

$$\begin{aligned} \partial_t \zeta + u|_{z=\zeta}\partial_x \zeta - w|_{z=\zeta} &= 0, \\ P|_{z=\zeta} - P_{\mathrm{atm}} = 0, \quad w|_{z=-1} &= 0. \end{aligned}$$

Setting $\varepsilon = 0$ yields

$$\partial_z P + 1 = 0 \quad \Longrightarrow \quad P = P_{\mathrm{atm}} - (z - \zeta).$$

Next, using the divergence-free and the boundary conditions gives

$$\begin{aligned} \partial_x \left(\int_{-1}^{\zeta(t,x)} u(t,x,z)\,\mathrm{d}z \right) &= \partial_x \zeta(t,x) \cdot u(t,x,\zeta(t,x)) + \int_{-1}^{\zeta(t,x)} \partial_x u(t,x,z)\,\mathrm{d}z \\ &= \partial_x \zeta(t,x) u(t,x,\zeta(t,x)) - w(t,x,\zeta(t,x)), \end{aligned}$$

and substituting them into the equations, we obtain the so-called **Benney system** (see [3]),

$$\begin{aligned} \partial_t u + u\partial_x u + w\partial_z u + \partial_x \zeta &= 0, \\ \partial_t \zeta + \partial_x \left(\int_{-1}^{\zeta(t,x)} u(t,x,z)\,\mathrm{d}z \right) &= 0, \\ \partial_x u + \partial_z w &= 0, \\ w|_{z=-1} &= 0. \end{aligned} \tag{2.1}$$

We can also refer to it as the hydrostatic homogeneous Euler equations with free-surface, in *Eulerian coordinates*. The system (2.1) has very interesting mathematical properties, including for instance an infinite number of conservation laws (at least for the 2d case, that is $x \in \mathbb{R}$). For example, direct computations lead to the identities

$$\begin{aligned} \partial_t \left[\int_{-1}^{\zeta} u\mathrm{d}z \right] + \partial_x \left[\int_{-1}^{\zeta} u^2 \mathrm{d}z + \frac{\zeta^2}{2} + \zeta \right] &= 0, \\ \partial_t \left[\int_{-1}^{\zeta} \frac{u^2}{2}\mathrm{d}z + \frac{\zeta^2}{2} \right] + \partial_x \left[\int_{-1}^{\zeta} \frac{u^3}{2}\mathrm{d}z + \zeta \int_{-1}^{\zeta} u\mathrm{d}z \right] &= 0, \end{aligned}$$

and in particular to the conservation of the total energy

$$\mathcal{E}_B(t) := \frac{1}{2}\|u(t)\|^2_{L^2(\Omega_t)} + \frac{1}{2}\|\zeta(t)\|^2_{L^2(\mathbb{R})}. \tag{2.2}$$

However, notice that it is not clear if such energy can be extended to control higher norms of the solution and, this way, does not provide straightforward well-posedness result. This is due to the loss of derivative encoded in the inertial terms $w\partial_z u$ (see below in Section (2.3) for more details). In particular, taking $k \geq 1$ horizontal derivatives of the equation for u and multiplying by $\partial_x^k u$ provides a term of the form

$$\int_{\Omega_t} (\partial_x^k w \partial_z u)\partial_x^k u,$$

which cannot be directly controlled by (powers of) $\|u(t)\|_{H^k(\Omega_t)}$. To our knowledge, it seems to be an open problem to prove a direct local in time well-posedness result for (2.1). In fact, some nontrivial hyperbolicity conditions can be obtained through a reformulation of the Benney equation.

2.2. *Zakharov formulation*

Following Zakharov [4,5] (still in the 2d case with constant density), we derive a *semi-Lagrangian formulation* of the previous system by introducing:

- A parameter $\lambda \in [0,1]$, that is to be understood as an (artificial) index for the continuous fluid layer.
- A re-parameterization $z = \Phi = \Phi(t,x,\lambda) \in (-1, \zeta(t,x))$ solution in $\mathbb{R}_x \times (0,1)$ to the quasilinear system

$$\begin{aligned} \partial_t \Phi + u(t,x,\Phi)\partial_x \Phi &= w(t,x,\Phi), \\ \Phi(0,x,\lambda) &= -(1-\lambda) + \lambda\zeta_0(x). \end{aligned}$$

  Observe that initially, we have $\partial_\lambda \Phi(0,x,\lambda) = \zeta_0(x) + 1 > 0$, as well as $\Phi(0,x,0) = -1$ and $\Phi(0,x,1) = \zeta_0(x)$. Formally, we also have $\Phi(t,x,1) = \zeta(t,x)$.

The effect of such change of variable, if suitably defined, is to straighten the moving domain Ω_t into the flat domain $(x,\lambda) \in \mathbb{R} \times (0,1)$. Setting

$$h(t,x,\lambda) := -\partial_\lambda \Phi(t,x,\lambda), \quad v(t,x,\lambda) := u(t,x,\Phi(t,x,\lambda)),$$

the Benney system (2.1) now reads

$$\begin{aligned} \partial_t h + \partial_x(vh) &= 0, \\ \partial_t v + v\partial_x v - \partial_x \int_0^1 h(t,x,\lambda)\,\mathrm{d}\lambda &= 0, \end{aligned} \qquad (x,\lambda) \in \mathbb{R} \times (0,1). \tag{2.3}$$

As will become clear later on, we note that the above formulation is entirely analogous to the non-homogeneous hydrostatic equations in *isopycnal coordinates*, as seen in (5.4). Notice that in both (2.3) and (5.4), vertical derivatives have been completely eliminated. System (2.1) can be viewed as an infinite superposition of pressureless Euler equations, where the Cauchy theory is nontrivial due to the apparent loss of derivatives.

Remark 2.1. From (2.3), one can obtain the so-called Vlasov-Benney system. By setting (in the sense of distributions)

$$f(t,x,\xi) := \int_0^1 h(t,x,\lambda)\delta_{\xi = v(t,x,\lambda)}\,\mathrm{d}\lambda,$$

the distribution function f solves the following kinetic equation

$$\partial_t f + \xi\partial_x f - \partial_x \left(\int_{\mathbb{R}} f\,\mathrm{d}\xi \right) \partial_\xi f = 0. \tag{2.4}$$

Surprisingly, this equation also arises as the quasineutral limit of the Vlasov-Poisson equations for ions. Compared to the fluid systems (2.1) and (2.3), its Cauchy theory in finite regularity seems to be better understood: we refer to [6] and to the literature therein.

The invertibility and regularity property of the mapping $(\mathrm{Id}, \Phi(t)) : \mathbb{R}^d \times (0,1) \to \mathbb{R}^d \times \Omega_t$ and the equivalence between (2.1) and (2.3), assuming one can define local strong solutions, has been rigorously set in [7]. In particular, one has the formula

$$\Phi(t,x,\lambda) = -1 - \int_0^\lambda h(t,x,\lambda')\,\mathrm{d}\lambda'.$$

Based on the formulation (2.3) of Zakharov, a suitable generalized notion of hyperbolicity is due to Teshukov [8,9], where Riemann invariants were found (see also [7]). It has been understood later that such hyperbolicity condition is linked to the stability of shear flows (see [10,11]), which is a very standard feature in hydrodynamics. These stability criteria will be clarified in the following.

2.3. *The case of 2d homogeneous fluids with rigid lid*

The rigid-lid assumption simply consists in replacing the free surface $\zeta(t,x)$ by a straight (fixed) line. For instance, we can take z belonging to the interval $[0,1]$, with no-slip boundary condition at the top and bottom layers

$$w|_{z=0} = w|_{z=1} = 0. \tag{2.5}$$

Further assuming periodic boundary conditions in the horizontal variable x, the H^s theory of the homogeneous hydrostatic equations in $(x,z) \in \mathbb{T} \times [0,1]$ is due to Brenier [12], Grenier [13], and Masmoudi & Wong [14]. The existence and uniqueness of local in time solutions is achieved under the *Rayleigh condition* $\partial_{zz} u > 0$ (see (R) and Theorem (2.1) below), which echoes the famous stability criteria for shear flows in hydrodynamics (see [15]).

Let us write down the homogeneous hydrostatic equations in $\mathbb{T} \times [0,1]$:

$$\begin{aligned} \partial_t u + (u\partial_x + w\partial_z)u + \partial_x p &= 0, \\ \partial_x u + \partial_z w &= 0, \\ w|_{z=0} = w|_{z=1} &= 0, \end{aligned} \tag{HH}$$

where the pressure $p = p(t,x)$ is *function only of* x.

The homogeneous hydrostatic equations with rigid lid (HH) were rigorously derived by Grenier [13] from the homogeneous incompressible Euler equation,

$$\begin{aligned} \partial_t u + (u\partial_x + w\partial_z)u + \partial_x p &= 0, \\ \partial_t w + (u\partial_x + w\partial_z)w + \partial_z p &= 0, \\ \partial_x u + \partial_z w &= 0, \end{aligned} \tag{E}$$

under the hydrostatic approximation, already introduced before. In particular, notice that in the framework of homogeneous fluids, the approximating vertical momentum equation reads

$$\varepsilon^2(\partial_t w + (u\partial_x + w\partial_z)w) + \partial_z p = 0, \tag{2.8}$$

which, under the hydrostatic limit $\varepsilon \to 0$, yields the equation $\partial_z p = 0$, namely

$$p = p(t,x) \quad \textbf{is independent of } z. \tag{1.Key-HH}$$

This is a key feature of the *homogeneous* hydrostatic equations (HH).

Moreover, integrating the divergence-free condition in $z \in [0,1]$ and using the boundary conditions in (2.5), one has that $\partial_x \int_0^1 u(\cdot,z)\,\mathrm{d}z = w|_{z=0} - w|_{z=1} = 0$, namely

$$\int_0^1 u\,\mathrm{d}z \quad \textbf{is independent of } x. \tag{2.Key-HH}$$

Now, given an initial datum $u_{\rm in}(x,z)$ such that $\int_0^1 u_{\rm in}(x,z)\,{\rm d}z = 0$, integrating the first equation of (HH) in z in $[0,1]$ gives an explicit expression for the pressure,

$$\partial_x p = -\int_0^1 u\partial_x u + w\partial_z u \,{\rm d}z, \text{ where } \int_0^1 w\partial_z u \,{\rm d}z = -\int_0^1 \partial_z w u \,{\rm d}z = \int_0^1 \partial_x u u \,{\rm d}z, \tag{2.11}$$

and therefore

$$\partial_x p = -\partial_x \int_0^1 u^2 \,{\rm d}z, \quad \text{i.e.} \quad p(t,x) = -\int_0^1 u^2 \,{\rm d}z \quad (\text{up to a function of } t \text{ only}). \tag{2.12}$$

This crucial observation leads to **conservation of the energy** $\mathcal{E}(t) := \frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{T}\times[0,1])}$ for the homogeneous hydrostatic equations with rigid-lid assumption, in fact

$$\begin{aligned}(\partial_x p, u)_{L^2(\mathbb{T}\times[0,1])} &= -\int_{\mathbb{T}} \left(\int_0^1 \partial_x(u^2)\,{\rm d}z\right)\left(\int_0^1 u\,{\rm d}z\right){\rm d}x = \int_{\mathbb{T}} \left(\int_0^1 u^2\,{\rm d}z\right)\left(\int_0^1 \partial_x u\,{\rm d}z\right){\rm d}x \\ &= -\int_{\mathbb{T}} \left(\int_0^1 u^2\,{\rm d}z\right)\left(\int_0^1 \partial_z w\,{\rm d}z\right){\rm d}x = 0.\end{aligned}$$

However, energy conservation is obviously not enough to overcome the loss of derivative due to the term $w\partial_z u$. In particular, system (HH) shows two main difficulties:

A) the pressure and the horizontal velocity are linked by an integral expression (2.12);
B) as said before, there is no equation for the vertical velocity w, which is recovered by the divergence-free condition $w = -\partial_z^{-1}\partial_x u$. This generates a **loss of derivative in x** as in the case of the Prandtl equation for boundary layer theory (see for instance [16] and references therein).

Notice that we can rewrite the equation for u as

$$\partial_t u + u\partial_x u - \left(\int_0^1 \partial_x u\,{\rm d}z\right)\partial_z u = \partial_x \int_0^1 u^2\,{\rm d}z,$$

which is a nonlinear and nonlocal transport equation, without a skew-symmetry structure providing good energy estimates.

To overcome such difficulties, a key idea due to Masmoudi & Wong [14] is to use the *hydrostatic vorticity* $\omega = \partial_z u$. In [14], the authors prove the following crucial result.

Lemma 2.2. *Let $\int_0^1 u_{in}\,dz = 0$. Then:*

- *If $(u(t,x,z), w(t,x,z), p(t,x))$ is a classical solution to (HH) with initial data u_{in} and with $\partial_z u(t,x,z) \in C^1([0,T];\mathbb{T}\times[0,1])$, then $(\omega(t,x,z), u(t,x,z), w(t,x,z))$ with the* hydrostatic vorticity

$$\omega(t,x,z) := \partial_z u(t,x,z) \tag{2.13}$$

solves the following:

$$\begin{aligned}&\partial_t \omega + u\partial_x\omega + w\partial_z\omega = 0,\\ &u = -\partial_z\mathcal{A}(\omega), \qquad (x,z)\in\mathbb{T}\times(0,1),\\ &w = \partial_x\mathcal{A}(\omega),\end{aligned} \tag{2.14}$$

with $\omega|_{t=0} = \omega_{in} := \partial_z u_{in}$, and where

$$A(f) := -\int_0^1 G(z,\xi)f(t,x,\xi)\,{\rm d}\xi, \quad G(z,\xi) := \frac{1}{2}(|z-\xi| - z - \xi + 2z\xi)$$

is the unique solution to

$$\begin{aligned}-\partial_{zz}\mathcal{A}(f) &= f,\\ \mathcal{A}(f)|_{z=0,z=1} &= 0.\end{aligned} \tag{2.15}$$

- *If* $(\omega(t,x,z), u(t,x,z), w(t,x,z))$ *is a classical solution to* (2.14)*, then* $(u(t,x,z), w(t,x,z), p(t,x))$ *solves* (HH) *and* $\int_0^1 u \, \mathrm{d}z = 0$.

In [14], the authors are able to overcome difficulties A) and B) and prove the well-posedness of (HH) in finite (Sobolev) regularity (previous results were in the context of analytic/infinite regularity, see the references in [14]) for solutions satisfying the

$$\text{(R)} \qquad \textbf{Rayleigh condition}: \quad \partial_{zz}u(x,z) = \partial_z\omega(x,z) > \sigma > 0, \qquad (x,z) \in \mathbb{T} \times (0,1).$$

More precisely, for any $0 < \sigma < 1$, their solutions live in the functional space

$$H^s_\sigma(\mathbb{T} \times (0,1)) := \Big\{\omega : \mathbb{T} \times (0,1) \to \mathbb{R}\,;\, \omega \in H^s(\mathbb{T} \times (0,1)),\, \sigma \le \partial_z\omega \le \sigma^{-1}\Big\}. \tag{2.17}$$

In this space, the H^s norm is equivalent to the weighted norm

$$\|\omega\|_{\widetilde{H}^s} := \sqrt{\left\|\frac{\partial_x^s\omega}{\sqrt{\partial_z\omega}}\right\|^2_{L^2} + \sum_{|\alpha|\le s, D^\alpha \ne \partial_x^s} \|D^\alpha\omega\|^2_{L^2}}. \tag{2.18}$$

The key observation due to [14] is that the above norm allows to exploit the $L^2(\mathbb{T} \times [0,1])$ orthogonality of w and ω

$$\int_0^1 \int_{\mathbb{T}} w\omega \; \mathrm{d}z \; \mathrm{d}x = \frac{1}{2}\int_0^1 \int_{\mathbb{T}} \partial_x(u^2) \; \mathrm{d}z \; \mathrm{d}x = 0,$$

and further to extend this idea to the derivatives

$$\int_0^1 \int_{\mathbb{T}} \partial_x^k w \partial_x^k \omega \; \mathrm{d}z \; \mathrm{d}x = 0, \quad k \in \mathbb{N}. \tag{3.Key-HH}$$

This cancellation solves the issue of the loss of x derivatives in the energy estimates. With these ingredients at hand, Masmoudi & Wong prove the following.

Theorem 2.1 (Well-posedness of (HH)). *For any* $s \ge 4$ *and* $\sigma > 0$*, let* $\omega_{in} = \partial_z u_{in} \in H^s_{2\sigma}(\mathbb{T} \times (0,1))$ *and* u_{in} *such that* $\int_0^1 u_{in} \, dz = M_0$*. Then there exists* $T = T(\|\omega_{in}\|_{H^s_{2\sigma}}, \sigma, s) > 0$ *such that equations* (HH) *with boundary conditions* (2.5) *admit a unique classical solution* (u, w, p) *with*

$$\omega = \partial_z u \in C^0([0,T]; H^s_\sigma(\mathbb{T} \times (0,1))) \cap C^1([0,T]; H^{s-1}(\mathbb{T} \times (0,1))),$$

and $\int_0^1 u \, dz = M_0$.

Using similar ideas, [14] also provides a rigorous derivation of the homogeneous hydrostatic system (HH) in the hydrostatic limit $\varepsilon \to 0$, for data that satisfy the stability criterion (R). We refer to [14] for more details.

Concerning finite-time singularity formation for solutions to (HH), different blow-up mechanisms have been studied (see [17] and references therein). For possible ill-posedness scenarios (around non-convex data), we refer to Section 4, where the more general case of non-homogeneous fluids will be described.

The above setting of the hydrostatic equations for homogeneous fluids will be compared with the case of **non-homogeneous** and more precisely **stably stratified fluids**.
We anticipate that the stably-stratified hydrostatic equations share the difficulties A) and B) (though difficulty A) has a different integral equation) with the homogeneous case, but, at least in general, they do not share with them any of the Key Property (1.Key-HH)-(2.Key-HH)-(3.Key-HH) ((2.Key-HH) still holds for non-homogeneous fluids with rigid lid). In particular, about (3.Key-HH): the orthogonality still holds but the energy functional does not provide a closed energy estimate.

Remark 2.3. In the case of homogeneous fluids without rigid-lid assumption (the Benney system in (2.1)), the energy functional (3.Key-HH) (as well as any straightforward modification of it) does not work, even in the L^2 case, namely $\|\omega(t)\|_{L^2}^2$. In these terms, the homogeneous hydrostatic Euler equations without rigid-lid (Benney system in (2.1)) seem to encode an additional complexity with respect to the equations with the rigid-lid assumption (HH). The case of non-homogeneous fluids adds further difficulties, which we discuss below.

Remark 2.4. Likewise Remark 2.1 in the case of free-surface flow, the hydrostatic system (HH) is naturally linked to a kinetic equation [18]: the distribution function

$$f(t,x,\xi) := \int_0^1 \delta_{\xi=u(t,x,z)}\,\mathrm{d}z, \quad \big(i.e.\ \ f(t,x,\cdot) = u(t,x,\cdot)\#\mathrm{Leb}_{[0,1]}\big),$$

is a solution to the so-called *kinetic Euler equation*

$$\partial_t f + \xi\partial_x f - \partial_x P \partial_\xi f = 0, \tag{2.20}$$

$$-\partial_x^2 P = \partial_x^2\left(\int \xi^2 f\,\mathrm{d}\xi\right). \tag{2.21}$$

This equation emerges as the quasineutral limit of the Vlasov-Poisson equations for electrons (see [6] and references therein) as well as in the incompressible optimal transport Ã la Brenier (see e.g. [19]).

3. Stably stratified fluids

Motivated by applications in oceanography, see for instance [20], we consider the non-homogeneous hydrostatic equations with gravity in their most general version:

$$\begin{aligned} \partial_t\rho + \mathbf{u}\cdot\nabla_{\mathbf{x}}\rho + w\partial_z\rho &= 0,\\ \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + w\partial_z\mathbf{u}) + \nabla_{\mathbf{x}}P &= 0,\\ \partial_z P + \rho &= 0,\\ \nabla_{\mathbf{x}}\cdot\mathbf{u} + \partial_z w &= 0,\\ \partial_t\zeta + \mathbf{u}|_{z=\zeta}\cdot\nabla_{\mathbf{x}}\zeta - w|_{z=\zeta} &= 0,\\ P|_{z=\zeta} = P_{\mathrm{atm}}, \quad w|_{z=-H} &= 0. \end{aligned} \tag{NHH}$$

We are interested in non-homogeneous fluids that are *stratified*. Namely, we consider perturbations of a steady density profile $\bar\rho_{\mathrm{eq}}(z)$ and a pressure $\bar P_{\mathrm{eq}}(z)$ that solve the non-homogeneous incompressible Euler equations with gravity under the

$$\textbf{hydrostatic balance}: \quad \partial_z\bar P_{\mathrm{eq}}(z) = -\bar\rho_{\mathrm{eq}}(z). \tag{3.2}$$

Compared to the homogeneous setting and in particular to [14], here we have:

- **NO** rigid-lid assumption: the top boundary of the fluid is described by a time-evolving free surface $\zeta(t,\mathbf{x})$ satisfying suitable *kinematic boundary conditions* (1.7).
- **NO** periodicity assumption in the horizontal direction.
- **NO** restriction to $x\in\mathbb{R}$ or $x\in\mathbb{T}$ (one horizontal dimension): in [14], this assumption is crucial since the proof relies on the hydrostatic version of the vorticity, which satisfies a scalar transport equation in two-space dimensions.

3.1. *Motivation and mathematical picture*

From a mathematical viewpoint, the question of ill-posedness or well-posedness in finite Sobolev regularity for the non-homogeneous hydrostatic equations without any diffusivity or viscosity is a long-standing open problem, even in the simplified case of the two-dimensional hydrostatic *Boussinesq* equations (for stably stratified fluids) described below (see Section 4.1):

$$
\begin{aligned}
\partial_t \rho + u\partial_x \rho + w\partial_z \rho &= 0, \\
\partial_t u + u\partial_x u + w\partial_z u &= -\partial_x P, \\
\partial_z P + \rho &= 0, \qquad\qquad (x,z) \in \mathbb{T}\times(0,1). \\
\partial_x u + \partial_z w &= 0, \\
w|_{z=0,1} &= 0,
\end{aligned}
\tag{3.3}
$$

This sharply contrasts with the results for non-hydrostatic equations, see [21], providing well-posedness for the inviscid, non-diffusive and non-hydrostatic equations in Sobolev spaces under the rigid-lid assumption. Although stable stratification stabilizes the system, it does not ensure a uniform lifespan of solutions with respect to the shallow-water parameter without additional smallness conditions on the initial data. For further developments in this direction, see the very recent work [22]. The main additional difficulty of the hydrostatic equations with respect to their non-hydrostatic counterpart is the *loss of horizontal derivative*, due to the absence of any dynamical equation for the vertical velocity w. Notice that extending the approach described in Section 2.3, due to Masmoudi & Wong [14], to the non-homogeneous setting, even in the simplified situation of the hydrostatic Boussinesq system with a rigid lid (3.3), is far from trivial. In particular:

- The functional (3.Key-HH) is well-defined under the Rayleigh condition $\partial_z \omega > 0$, which is a sufficient condition specifically designed to ensure spectral stability of the *homogeneous* equations (in the non-homogeneous setting, an analogous condition is known as the Miles-Howard criterion, on which we will come back below, see (MH)).
- The functional (3.Key-HH) is designed *ad hoc* to overcome the loss of derivatives in x in the nonlinear convection term for the *homogeneous* equations; however, the non-homogeneous hydrostatic equations display a non-trivial linear dynamics (in other words, such functional does not provide a closed energy estimate for system (3.3)).

One of the main observations in the work of the first author with V. Duchêne [23] is that for *stably stratified fluids*, the equations exhibit a *partial symmetric structure*. This symmetry can be exploited to partially, but not entirely, mitigate the difficulties associated with the loss of derivatives.

3.2. *The partial symmetric structure in Eulerian coordinates*

We start by considering the non-homogeneous hydrostatic equations *linearized* near a physically relevant class of steady states, known as **stratified shear flows**. The non-homogeneous hydrostatic equations, linearized near a *stably stratified density profile* and a *shear velocity*

$$
(\bar\rho_{\rm eq}(z), \bar{\mathbf{u}}_{\rm eq}(z), 0, \bar P_{\rm eq}(z)), \quad \bar\rho'_{\rm eq}(z) < 0, \quad \partial_z \bar P_{\rm eq} + \bar\rho_{\rm eq} = 0,
\tag{3.4}
$$

read as follows:

$$
\partial_t \rho + \bar{\mathbf{u}}_{\rm eq}(z)\cdot\nabla_{\mathbf{x}}\rho - \bar\rho'_{\rm eq}(z)\mathcal{L}^*\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0,
\tag{3.5}
$$

$$
\partial_t \mathbf{u} + \bar{\mathbf{u}}_{\rm eq}(z)\cdot\nabla_{\mathbf{x}}\mathbf{u} + w\bar{\mathbf{u}}'_{\rm eq}(z) + \frac{\nabla_{\mathbf{x}}\mathcal{L}\rho}{\bar\rho_{\rm eq}(z)} = 0,
\tag{3.6}
$$

where

$$
\mathcal{L}\mathrm{g}(z) := \int_z^{\zeta} \mathrm{g}(z')\,\mathrm{d}z'.
$$

The above-mentioned partial symmetric structure is clarified by the following identity

$$
\frac{\mathrm{d}}{\mathrm{dt}}\left((\rho, \frac{-1}{\bar\rho'_{\rm eq}(z)}\rho)_{L^2} + (\mathbf{u}, \bar\rho_{\rm eq}(z)\mathbf{u})_{L^2} \right) = -(\mathbf{u}, \bar\rho_{\rm eq}(z)\bar{\mathbf{u}}'_{\rm eq}(z)w)_{L^2},
$$

where the first term of the energy is well defined thanks to the stable stratification assumption, however the symmetric structure (and energy conservation) is disrupted by the term on the right-hand side, specifically due to the presence of the shear flow. In addition, there exists a famous criterion ensuring spectral stability of stratified states, namely the absence of an exponentially growing mode in the linearized equations. We say that $(\bar\rho_{\rm eq}(z), \bar u_{\rm eq}(z))$ satisfies the **Miles-Howard stability condition** [24,25] if

$$\forall z \in [0,1], \quad \frac{1}{4} \leq \frac{-\bar\rho_{\rm eq}'(z)}{\bar\rho_{\rm eq}(z)(\bar u_{\rm eq}'(z))^2}. \tag{MH}$$

The quotient on the right-hand side is known as the Richardson number, encoding potential stabilizing (resp. destabilizing) effect of the stable stratification (resp. shear velocity).

If (MH) holds, then the equations for 2d non-homogeneous incompressible flows (non-hydrostatic or hydrostatic, in the rigid-lid approximation) are spectrally stable [15]. We emphasize that this is only a sufficient condition. This condition should be compared to the well-known Rayleigh condition (R) for homogeneous fluids. However, in contrast to the homogeneous setting [14], to the best of our knowledge, no functional encoding the (MH) criterion is known to provide a closed energy estimate for the equations of 2d hydrostatic non-homogeneous incompressible flows (for the non-hydrostatic case, see [26]).

4. Instabilities of the hydrostatic approximation

In this section, we consider the equations for 2d non-homogeneous incompressible flows under the rigid lid assumption and the Boussinesq approximation (setting the dominant constant averaged density $\rho_c = 1$), that is

$$\begin{aligned} \partial_t\rho + u\partial_x\rho + w\partial_z\rho &= 0,\\ \partial_t u + u\partial_x u + w\partial_z u &= -\partial_x P,\\ \varepsilon^2\left(\partial_t w + u\partial_x w + w\partial_z v\right) &= -\partial_z P - \rho, \qquad (x,z)\in\mathbb{T}\times(0,1),\\ \partial_x u + \partial_z w &= 0,\\ w|_{z=0,1} &= 0, \end{aligned} \tag{4.1}$$

where, in particular, density variation in the momentum equation is only due to the buoyancy term. This simplified model is of paramount importance for applications to geophysical fluids [27]. The equations $(4.1)_{\varepsilon=0}$ correspond to the hydrostatic approximation: as pointed out in Section 3.1, the Cauchy theory for this system in finite regularity is still an open-problem. Not only does the targeted system appear to be poorly behaved, but there is also another heuristic explanation for why the limit $\varepsilon \to 0$ cannot be easily derived. Specifically, the change of variables

$$(t,x,z) \mapsto \left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, z\right) \tag{4.2}$$

(with an additional scaling for the vertical velocity w) formally transforms the non-hydrostatic system $(4.1)_{\varepsilon>0}$ into the same system but with $\varepsilon = 1$ in the momentum equation. In essence, this implies that deriving an energy estimate for $(4.1)_{\varepsilon>0}$ on a timescale of order $\mathcal{O}(1)$ and uniformly in ε is equivalent to proving the same type of estimate for $(4.1)_{\varepsilon=1}$ on a timescale of order $\mathcal{O}(\varepsilon^{-1})$. This naturally connects the hydrostatic limit to the issue of long-time dynamics for the 2d Euler-Boussinesq system. Unfortunately, understanding the latter remains an outstanding open problem (a key difference from the homogeneous 2d Euler equations), even in a perturbative setting (see [21,28] based on dispersive approaches, and [26,29] based on inviscid damping).

4.1. *Main results*

In another direction, this suggests that instabilities (in time) in the Euler-Boussinesq system can potentially be exploited to demonstrate that the hydrostatic limit $\varepsilon \to 0$ might fail. In the recent work

[30], the two authors and M. Coti Zelati leveraged this idea to prove that, even in a scenario of stable stratification, there exist smooth stationary solutions of $(4.1)_{\varepsilon>0}$ and $(4.1)_{\varepsilon=0}$ around which:

- The hydrostatic limit $\varepsilon \to 0$ from $(4.1)_{\varepsilon>0}$ to $(4.1)_{\varepsilon=0}$ does not hold.
- The hydrostatic system $(4.1)_{\varepsilon=0}$ is nonlinearly ill-posed in Sobolev spaces.

These results, that we briefly detail below, are exactly in the same spirit as that of [31,32], concerning the failure of the quasineutral limit and the ill-posedness of the singular systems (2.4) and (2.20) (around profiles that are unstable in the sense of Penrose). As we will see, they are also natural extensions to stratified fluids of the earlier results of Grenier [33] and Han-Kwan & Nguyen [32] about the homogeneous hydrostatic case.

For stratified fluids in the Boussinesq approximation, the steady states built in [30] belong to the important class of *stably stratified shear flows* (3.4). In order to give rise to some instabilities, they will *not satisfy* the Miles-Howard condition (see (MH)) which has to be slightly modified in that context in

$$(\mathrm{MH}_B) \qquad \forall z \in [0,1], \quad \frac{1}{4} \leq \frac{-\bar{\rho}'_{\mathrm{eq}}(z)}{(\bar{u}'_{\mathrm{eq}}(z))^2},$$

since the constant averaged density has been normalized.

The main results of [30] can be summarized as follows. The first one asserts that instabilities can prevent the hydrostatic limit from holding.

Theorem 4.1. *There exists an analytic stationary profile* $(\bar{\rho}_{eq}(z), \bar{u}_{eq}(z))$, *with* $\bar{\rho}'_{eq} < 0$, *which does not satisfy the Miles-Howard condition* (MH_B), *such that the following holds. For all* $p, k \in \mathbb{N}$, *there exist* $m > 0$, *a family of smooth solutions* $(\rho_\varepsilon, u_\varepsilon, w_\varepsilon)_{\varepsilon>0}$ *of* $(4.1)_{\varepsilon>0}$ *and times* $T_\varepsilon = O(\varepsilon|\log \varepsilon|) \to 0$ *such that*

$$\|(\rho_\varepsilon, u_\varepsilon, w_\varepsilon)_{|t=0} - (\bar{\rho}_{eq}, \bar{u}_{eq}, 0)\|_{H^p(\mathbb{T}\times(0,1))} \leq \varepsilon^k,$$

while

$$\sup_{t\in[0,T_\varepsilon]} \|(\rho_\varepsilon, u_\varepsilon, w_\varepsilon)(t) - (\bar{\rho}_{eq}, \bar{u}_{eq}, 0)\|_{L^2(\mathbb{T}\times(0,1))} \geq m.$$

As a byproduct, we can also recover the result stated in [33] for the constant density case (around shear flows that do not satisfy (R)). *Modulo* the scaling (4.2), this theorem has to be understood as a nonlinear Lyapunov instability (in time) for the original Euler-Boussinesq equation with $\varepsilon = 1$, which is also interesting in itself. Up to the identification of leading linear instabilities, the nonlinear part of the proof follows the robust scheme introduced by Grenier in [33]. We refer to [30] for more details.

Our second result is about the ill-posedness of the hydrostatic equation in the sense of Hadamard.

Theorem 4.2. *There exists an analytic stationary profile* $(\bar{\rho}_{eq}(z), \bar{u}_{eq}(z))$, *with* $\bar{\rho}'_{eq} < 0$, *which does not satisfy the Miles-Howard condition* (MH_B), *such that the following holds. For any* $p, k \in \mathbb{N}$ *and* $\alpha \in (0,1]$, *there exists* $0 < \delta_0 < 1$ *such that, for any* $0 < \delta \leq \delta_0$, *there exists a family of solutions* $(\rho_\delta, u_\delta, v_\delta)_{\delta<\delta_0}$ *to* $(4.1)_{\varepsilon=0}$ *, such that, for times* $T_\delta = \mathcal{O}(\delta|\log \delta|) \to 0$,

$$\lim_{\delta\to 0} \frac{\left\|(\rho_\delta, \omega_\delta) - \left(\bar{\rho}_{eq}, \bar{u}'_{eq}\right)\right\|_{L^2([0,T_\delta];H^1(\mathbb{T}\times(0,1)))}}{\left\|(\rho_\delta, \omega_\delta)_{|t=0} - \left(\bar{\rho}_{eq}, \bar{u}'_{eq}\right)\right\|^\alpha_{H^p(\mathbb{T}\times(0,1))}} = +\infty,$$

where $\omega_\delta := \partial_z u_\delta$ *is the hydrostatic vorticity.*

This theorem precisely asserts that, if it exists, the flow map of the hydrostatic equations cannot be HÃ¶lder continuous from H^s to H^1 for any s (i.e. arbitrary loss of derivatives), even for arbitrarily short times. As a consequence, it rules out the generic construction of finite regularity solutions for the hydrostatic equations through a standard procedure (like a fixed-point argument or an iterative scheme), which usually provides some regularity of the flow map. A similar result have been first obtained by Han-Kwan and Nguyen in [32] in the homogeneous case, around certain shear flows having an inflection point, and hence not satisfying the Rayleigh condition (R).

4.2. *Sketch of proof for the linear analysis*

We mainly focus on the construction of violent linear instabilities for the hydrostatic system $(4.1)_{\varepsilon=0}$ itself, which are crucial for the proof of Theorem 4.2. Once these are obtained, they can be easily lifted into suitable instabilities for $(4.1)_{\varepsilon=1}$ thanks to a long-wave approximation procedure (see [30] for more details) - paving the way for the proof of Theorem 4.1.

As it is usual in 2d cases, we first look at the formulation of the hydrostatic equation in terms of the (hydrostatic) vorticity

$$\omega := \partial_z u.$$

Applying ∂_z to the equation for u, we obtain the following new system on (ϱ, ω):

$$\begin{aligned} &\partial_t \rho + u\partial_x \rho + w\partial_z \rho = 0, \\ &\partial_t \omega + u\partial_x \omega + w\partial_z \omega = \partial_x \rho, \end{aligned} \tag{4.3}$$

where

$$u = -\partial_z \varphi, \qquad w = \partial_x \varphi, \qquad \text{and} \qquad \begin{cases} -\partial_z^2 \varphi = \omega, \\ \varphi_{|z=0,1} = 0. \end{cases} \tag{4.4}$$

The former relations stem from a degenerate Biot-Savart law arising in the limit $\varepsilon \to 0$. The lack of regularization in the x-variable is of course reminiscent of the loss of derivative that is at stake.

Following [32], the main idea is to look for some unbounded unstable spectrum of some linearized operator associated to (4.3)–(4.4). It essentially allows to build a solution of the form

$$e^{\lambda_{\text{eig}} t}(\rho_{\text{eig}}, u_{\text{eig}})(x, z) + R$$

where $(\lambda_{\text{eig}}, ((\rho_{\text{eig}}, u_{\text{eig}}))$ is a suitable maximal unstable eigenvalue/function (in particular $\text{Re}(\lambda_{\text{eig}}) > 0$) and where the remainder R is mainly built thanks to a Cauchy-Kovalevskaya argument.

We thus consider the linearization of the previous system around a profile $(\bar{\rho}_{\text{eq}}(z), \bar{u}'_{\text{eq}}(z))$ (which is yet to be determined to provide instabilities). The linearized equations read as

$$\begin{aligned} &\partial_t \varrho + \bar{u}_{\text{eq}}(z)\partial_x \varrho + \bar{\rho}'_{\text{eq}}(z)\partial_x \varphi = 0, \\ &\partial_t \omega + \bar{u}_{\text{eq}}(z)\partial_x \omega + \bar{u}''_{\text{eq}}(z)\partial_x \varphi = \partial_x \varrho, \end{aligned} \tag{4.5}$$

together with (4.4). Roughly speaking, it is enough to perform a normal mode analysis by looking for some solution (say the stream function) of the form $\phi(z)e^{ikx}e^{ikct}$ where $k \in \mathbb{Z}$ $(k \neq 0)$ and $\text{Im}(c) > 0$. With this ansatz, we can reduce the analysis to the so-called (hydrostatic) Taylor-Goldstein equation on the function ϕ:

$$\begin{aligned} (\bar{u}_{\text{eq}} - c)^2 \partial_z^2 \phi - (\bar{u}_{\text{eq}} - c)\bar{u}''_{\text{eq}} \phi &= \bar{\rho}'_{\text{eq}} \phi, \\ \phi(0) = \phi(1) &= 0. \end{aligned} \tag{4.6}$$

which is a eigenvalue problem in (c, ϕ). Note that this problem is now independent of the frequency k: it is due to the homogeneity in time-space of the hydrostatic scaling[a]. It means that given a solution (c, ϕ_c) of the former problem (4.6), with c *independent* of k, all the multiples ikc (having real part positive) with $k \in \mathbb{Z}$ belong to the spectrum of the linearized operator.

Following an idea of Friedlander in [34], we now introduce a parameter $\alpha \in (1/2, 1)$ (to be determined) and look for a profile $(\bar{\rho}_{\text{eq}}(z), \bar{u}_{\text{eq}}(z))$ satisfying

$$-\bar{\rho}'_{\text{eq}}(z) = \alpha(1-\alpha)|\bar{u}'_{\text{eq}}(z)|^2. \tag{4.7}$$

[a] This simple remark also means that the hydrostatic limit can be somehow seen as a zero-dispersion limit.

With such constraint, the Miles-Howard stability condition (MH_B) cannot be satisfied (but the stratification is *stable* as $-\bar{\rho}'_{\mathrm{eq}}(z) > 0$). Note that the limit $\alpha \to 1$ corresponds to a degenerate homogeneous flow. By performing the change of unknowns

$$\phi = (\bar{u}_{\mathrm{eq}} - c)^{\alpha}\psi^c_{\alpha}, \tag{4.8}$$

for some new function $\psi^c_{\alpha}(z)$, we can transform the equation (4.6) into the new eigenvalue problem

$$\begin{aligned} \partial_z\big((\bar{u}_{\mathrm{eq}} - c)^{2\alpha}\partial_z\psi^c_{\alpha}\big) + (\alpha-1)(\bar{u}_{\mathrm{eq}} - c)^{2\alpha-1}\bar{u}''_{\mathrm{eq}}\psi^c_{\alpha} = 0, \\ \psi^c_{\alpha}(0) = \psi^c_{\alpha}(1) = 0. \end{aligned} \tag{4.9}$$

If $\alpha = 1$, the equation (4.9) is nothing but

$$\partial_z\big((\bar{u}_{\mathrm{eq}} - c)^2\partial_z\psi^c_1\big) = 0, \quad \psi_1(0) = \psi_1(1) = 0,$$

which can be solved provided that there exists $c \in \{\mathrm{Im} > 0\}$ satisfying

$$\int_0^1 \frac{\mathrm{d}z}{(\bar{u}_{\mathrm{eq}}(z) - c)^2} = 0. \tag{4.10}$$

In [35,36], explicit smooth functions $\bar{u}_{\mathrm{eq}}(z)$ providing solutions to this integral relation are found (this corresponds to instabilities in the constant density case). Being given such a profile, we maintain the constraint (4.7) and allow for α close to 1 but with $\alpha < 1$. This is a weakly inhomogeneous regime that we handle through a perturbative argument: we write (4.9) as

$$\psi^c_{\alpha}(z) = \int_0^z \frac{\mathrm{d}r}{(\bar{u}_{\mathrm{eq}}(r) - c)^{2\alpha}} + (1-\alpha)\int_0^z \frac{\mathrm{d}r}{(\bar{u}_{\mathrm{eq}}(r) - c)^{2\alpha}}\left\{\int_{-1}^r (\bar{u}_{\mathrm{eq}}(q) - c)^{2\alpha-1}\bar{u}''_{\mathrm{eq}}(q)\psi^c_{\alpha}(q)\mathrm{d}q\right\}, \tag{4.11}$$

that we solve through a fixed point procedure. Eventually, we have to find a couple (c, ψ^c_{α}) such that $\psi^c_{\alpha}(1) = 0$. In the spirit of [37], we then study the holomorphic sequence of functions $c \in \{\mathrm{Im} > 0\} \mapsto \psi^c_{\alpha}(1)$. Since we know that $c \mapsto \psi^c_1(1)$ has a zero in $\{\mathrm{Im} > 0\}$ thanks to (4.11), RouchÃ©'s theorem ensures that there exists $c^{\star} \in \{\mathrm{Im} > 0\}$ such that $\psi^{c^{\star}}_{\alpha}(1) = 0$, if α is sufficiently close to 1. This essentially allows to conclude the proof.

Remark 4.1. In the previous perturbative analysis, a suitable choice of unstable profile is for instance

$$\bar{\rho}_{\mathrm{eq}}(z) = \alpha(1-\alpha)\left(-\beta\tanh\left(\beta(z-1/2)\right) + \frac{\beta}{3}\tanh^3\left(\beta(z-1/2)\right)\right), \qquad \bar{u}_{\mathrm{eq}}(z) = \tanh\left(\beta(z-1/2)\right),$$

with $\beta > 0$ sufficiently large and for some $\alpha < 1$ close to 1. Note that $\bar{\rho}_{\mathrm{eq}}(z)$ is close to a decreasing affine function. It is actually far from straightforward to demonstrate that *any* profile failing to satisfy the Miles-Howard condition (MH_B) necessarily leads to instability. An even more intriguing challenge is to prove that a spectrally stable steady-state can still trigger instability in the hydrostatic limit, without any additional assumption.

Remark 4.2. What about the three-dimensional case? By considering initial data that are essentially two-dimensional, a uniqueness argument allows us to reduce the dynamics to the two-dimensional system (4.1), for which Theorems 4.1-4.2 apply. In essence, there exists a genuinely non-homogeneous two-dimensional instability that can be extended to a three-dimensional instability.

5. Gent & McWilliams parametrization of eddy diffusivity

The non-homogeneous hydrostatic equations with *regularization in the density equation* are motivated by physical application to modeling of the effective contributions of geostrophic eddy correlations in non-eddy-resolving systems, see [20,38]. The main distinction of this model compared to previous studies is the absence of any (regularizing) viscosity contribution added to the fluid-dynamics equations, while only thickness diffusivity effects are considered. The equations read as follows:

$$\text{(NHH)}\qquad \begin{aligned} \partial_t\rho + (\mathbf{u}+\mathbf{u}^*)\cdot\nabla_{\mathbf{x}}\rho + (w+w^*)\partial_z\rho = 0,\\ \rho(\partial_t\mathbf{u} + (\mathbf{u}+\mathbf{u}^*)\cdot\nabla_{\mathbf{x}}\mathbf{u} + (w+w^*)\partial_z\mathbf{u}) + \nabla_{\mathbf{x}}P = 0,\\ \partial_z P + \rho = 0,\\ \nabla_{\mathbf{x}}\cdot\mathbf{u} + \partial_z w = 0,\\ \partial_t\zeta + (\mathbf{u}|_{z=\zeta} + \mathbf{u}^*|_{z=\zeta})\cdot\nabla_{\mathbf{x}}\zeta - (w|_{z=\zeta} + w^*|_{z=\zeta}) = 0,\\ P|_{z=\zeta(t)} = P_{\mathrm{atm}},\quad w|_{z=-H} = 0, \end{aligned}$$

where $\mathbf{u}^*$ and w^* are the Gent & McWilliams correctors for the deterministic modeling of **eddy diffusivity**, representing the impact of mesoscale eddies that have an average dissipative effect on the large-scale flow in oceanography, see for instance [20] and the recent work [38]. The Gent & McWilliams correctors to the transport velocities are given below:

$$\mathbf{u}^* := -\kappa\partial_z\left(\frac{\nabla_{\mathbf{x}}\rho}{\partial_z\rho}\right),\quad w^* := \kappa\nabla_{\mathbf{x}}\cdot\left(\frac{\nabla_{\mathbf{x}}\rho}{\partial_z\rho}\right),\quad \kappa > 0. \tag{5.2}$$

In the recent work [23], the first author and V. Duchêne established the well-posedness of the hydrostatic and original (non-hydrostatic) equations for stably stratified fluids, along with their convergence in the limit of a vanishing shallow-water parameter ε. These results are obtained in high but finite Sobolev regularity and heavily rely on the *stably stratified assumption*. A key element of the analysis is the reformulation of the systems using *isopycnal coordinates*, which enables to provide careful energy estimates that are not readily apparent in the original Eulerian coordinate system.

5.1. *Isopycnal coordinates*

Assuming that the flow is *stably stratified*, i.e.

$$\inf(-\partial_z\rho) > 0,$$

the density $\rho : z \to \rho(\cdot,\cdot,z)$ is an invertible function of z. We denote its inverse $\mathcal{H} : \varrho \to \mathcal{H}(\cdot,\cdot,\varrho)$, so that

$$\rho(t,\mathbf{x},\mathcal{H}(t,\mathbf{x},\varrho)) = \varrho,\quad \mathcal{H}(t,\mathbf{x},\rho(t,\mathbf{x},z)) = z.$$

We also assume that $\rho(t,\mathbf{x},-H) = \rho_1,\ \rho(t,\mathbf{x},\zeta(t,\mathbf{x})) = \rho_0$ for $(t,\mathbf{x}) \in [0,T]\times\mathbb{R}^3$, where $\rho_0 < \rho_1$ are two fixed and positive constant reference densities (where the ordering of densities is preserved in time in the stably stratified setting). Then we have

$$\mathcal{H} : [0,T]\times\Omega \to \mathbb{R}\quad \text{with}\quad \Omega := \mathbb{R}^3\times(\rho_0,\rho_1)\quad \text{and}\quad h := -\partial_\varrho\mathcal{H} > 0, \tag{5.3}$$

the latter inequality accounting for the stable stratification assumption. We now introduce

$$\check{\mathbf{u}}(t,\mathbf{x},\varrho) = \mathbf{u}(t,\mathbf{x},\mathcal{H}(t,\mathbf{x},\varrho)),\quad \check{w}(t,\mathbf{x},\varrho) = w(t,\mathbf{x},\mathcal{H}(t,\mathbf{x},\varrho)),\quad \check{P}(t,\mathbf{x},\varrho) = P(t,\mathbf{x},\mathcal{H}(t,\mathbf{x},\varrho)).$$

We are interested in small perturbations from stably stratified steady solutions as in (3.4). In isopycnal coordinates, such steady states read

$$(\bar{h}_{\mathrm{eq}},\bar{\mathbf{u}}_{\mathrm{eq}},\bar{w}_{\mathrm{eq}},\bar{P}_{\mathrm{eq}}) = (\bar{h}(\varrho),\bar{\mathbf{u}}(\varrho),0,\bar{P}(\varrho)),$$

satisfying the stable stratification assumption for some constants $0 < h_\star < h^\star < +\infty$

$$h_\star \le \bar{h}(\varrho) \le h^\star\qquad \left(-\partial_z\bar{\rho}_{\mathrm{eq}} = -\frac{1}{\partial_\varrho\bar{\mathcal{H}}_{\mathrm{eq}}} = \frac{1}{\bar{h}(\rho)}\right)$$

and the equilibrium condition below (hydrostatic balance in isopycnal coordinates):

$$\partial_\varrho \bar{P}(\varrho) = \varrho \bar{h}(\varrho).$$

Dropping the *check* and the subscript *eq*, the hydrostatic equations (NHH) in isopycnal coordinates read:

$$\begin{aligned} \partial_t h + \nabla_{\mathbf{x}} \cdot ((\bar{h}+h)(\bar{\mathbf{u}}+\mathbf{u})) &= \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + \big((\bar{\mathbf{u}}+\mathbf{u}-\kappa\tfrac{\nabla_{\mathbf{x}} h}{\bar{h}+h})\cdot\nabla_{\mathbf{x}}\big)\mathbf{u}\Big) + \nabla_{\mathbf{x}} \mathcal{M} &= 0, \end{aligned} \qquad (\mathbf{x},\varrho)\in\Omega, \tag{5.4}$$

where $\mathcal{M} = \mathcal{M}[h]$ is the so-called *Montgomery potential*, defined as

$$\begin{aligned} \mathcal{M}(t,\mathbf{x},\varrho) &:= \int_{\rho_0}^{\varrho} \varrho' h(t,\mathbf{x},\varrho')\,\mathrm{d}\varrho' + \varrho \int_{\varrho}^{\rho_1} h(t,\mathbf{x},\varrho')\,\mathrm{d}\varrho' \\ &= \rho_0 \int_{\rho_0}^{\rho_1} h(t,\mathbf{x},\varrho')\,\mathrm{d}\varrho' + \int_{\rho_0}^{\varrho}\int_{\varrho'}^{\rho_1} h(t,\mathbf{x},\varrho'')\,\mathrm{d}\varrho''\,\mathrm{d}\varrho'. \end{aligned} \tag{5.5}$$

On the other hand, the corresponding approximating equations are given by:

$$\begin{aligned} \partial_t h + \nabla_{\mathbf{x}} \cdot ((\bar{h}+h)(\bar{\mathbf{u}}+\mathbf{u})) &= \kappa \Delta_{\mathbf{x}} h, \\ \varrho\Big(\partial_t \mathbf{u} + \big((\bar{\mathbf{u}}+\mathbf{u}-\kappa\tfrac{\nabla_{\mathbf{x}} h}{\bar{h}+h})\cdot\nabla_{\mathbf{x}}\big)\mathbf{u}\Big) + \nabla_{\mathbf{x}} P + \frac{\nabla_{\mathbf{x}}\mathcal{H}}{\bar{h}+h}(\varrho\bar{h}+\partial_\varrho P) &= 0, \\ \varepsilon^2\varrho\Big(\partial_t w + (\bar{\mathbf{u}}+\mathbf{u}-\kappa\tfrac{\nabla_{\mathbf{x}} h}{\bar{h}+h})\cdot\nabla_{\mathbf{x}} w\Big) - \frac{\partial_\varrho P}{\bar{h}+h} + \frac{\varrho h}{\bar{h}+h} &= 0, \\ -(\bar{h}+h)\nabla_{\mathbf{x}}\cdot\mathbf{u} - \nabla_{\mathbf{x}}\mathcal{H}\cdot(\bar{\mathbf{u}}'+\partial_\varrho \mathbf{u}) + \partial_\varrho w &= 0, \\ P\big|_{\varrho=\rho_0} = 0, \qquad w\big|_{\varrho=\rho_1} &= 0, \end{aligned} \qquad (\mathbf{x},\varrho)\in\Omega, \tag{5.6}$$

where

$$\mathcal{H}(\varrho) = \int_{\varrho}^{\rho_1} h(\varrho')\,\mathrm{d}\varrho'.$$

Note that the second-to-last equation stems from the incompressibility condition, here written in isopycnal coordinates. Now, consider only the linear part of the hydrostatic system (5.4), in terms of the unknown $\mathcal{H}$:

$$\partial_t\mathcal{H} + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\mathcal{H} + \int_{\varrho}^{\rho_1} \bar{\mathbf{u}}'\cdot\nabla_{\mathbf{x}}\mathcal{H}\,\mathrm{d}\varrho' + \int_{\varrho}^{\rho_1} \bar{h}\nabla_{\mathbf{x}}\cdot\mathbf{u}\,\mathrm{d}\varrho' = \kappa\Delta_{\mathbf{x}}\mathcal{H}, \tag{5.7}$$

$$\varrho(\partial_t\mathbf{u} + (\bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}})\mathbf{u}) + \rho_0\nabla_{\mathbf{x}}\mathcal{H}(\rho_0) + \nabla_{\mathbf{x}}\int_{\rho_0}^{\varrho}\mathcal{H}\,\mathrm{d}\varrho' = 0. \tag{5.8}$$

For the above system, the following identity holds

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big((\mathcal{H},\ \mathcal{H})_{L^2_{\mathbf{x},\varrho}} + \rho_0\big(\mathcal{H}(\rho_0),\mathcal{H}(\rho_0)\big)_{L^2_{\mathbf{x}}} + (\mathbf{u},\bar{h}(\varrho)\mathbf{u})_{L^2_{\mathbf{x},\varrho}}\Big) + \kappa\Big((\nabla_{\mathbf{x}}\mathcal{H},\nabla_{\mathbf{x}}\mathcal{H}\ )_{L^2_{\mathbf{x},\varrho}} + \rho_0\big(\nabla_{\mathbf{x}}\mathcal{H}(\rho_0),\nabla_{\mathbf{x}}\mathcal{H}(\rho_0)\big)_{L^2_{\mathbf{x}}}\Big) \\ = -\big(\textstyle\int_{\varrho}^{\rho_1}\bar{\mathbf{u}}'\cdot\nabla_{\mathbf{x}}\mathcal{H}\,d\varrho',\mathcal{H}\big)_{L^2_{\mathbf{x},\varrho}} - \rho_0\big(\textstyle\int_{\rho_0}^{\rho_1}\bar{\mathbf{u}}'\cdot\nabla_{\mathbf{x}}\mathcal{H}d\varrho,\mathcal{H}(\rho_0)\big)_{L^2_{\mathbf{x}}}, \end{aligned}$$

showing that the Gent & McWilliams correctors (5.2) allow to restore the symmetric structure of the linearized system, providing a closed energy estimate at least for local times. The main advantages of isopycnal coordinates are listed below:

- The nonlinear diffusion terms $\mathbf{u}_\star$ and $w_\star$ in (5.2) take the simple form $\Delta_{\mathbf{x}}\mathcal{H}$.
- The free boundary is now fixed, with density ranging from ρ_0 to ρ_1.
- Transport occurs only in the horizontal direction $\mathbf{x}$.
- The loss of regularity in the velocity $\mathbf{u}$ is eliminated, although there is a loss of regularity in $\mathbf{x}$ for $\mathcal{H}$.

5.2. *Main results on the Gent & McWilliams model*

The first result in [23] is the well-posedness of the non-homogeneous hydrostatic equations with Gent & McWilliams eddy diffusivity, which is (roughly) stated below.

Theorem 5.1. *For any sufficiently regular initial data satisfying (in finite regularity) the stably stratified assumption*

$$-\partial_z\rho|_{t=0} \geq \alpha > 0,$$

and for any $\kappa > 0$, there exists the unique (classical) solution to (5.4)-(5.5) for times $t \in [0, T^h]$, with

$$(T^h)^{-1} = C\left(1 + \kappa^{-1}\left(\|\overline{\boldsymbol{u}}'\|^2_{L^2_z} + M_0^2\right)\right),$$

where M_0 is the size of the perturbation at the initial time, around the stably stratified shear flow *(steady solution)*

$$(\overline{\rho}(z), (\overline{\boldsymbol{u}}(z), 0))$$

and C only depends on M_0, α and the size (in $W^{k,\infty}$) of $(\overline{\rho}(z), \overline{\boldsymbol{u}}(z))$. Moreover,

$$\begin{aligned}(5.9)\qquad &\|\mathcal{H}^h(t,\cdot)\|_{H^{s,k}} + \|\boldsymbol{u}^h(t,\cdot)\|_{H^{s,k}} + \|\mathcal{H}^h\big|_{\varrho=\rho_0}(t,\cdot)\|_{H^s_{\boldsymbol{x}}} + \kappa^{1/2}\|h^h(t,\cdot)\|_{H^{s,k}} \\ &+ \kappa^{1/2}\|\nabla_{\boldsymbol{x}}\mathcal{H}^h\|_{L^2(0,T;H^{s,k})} + \kappa^{1/2}\|\nabla_{\boldsymbol{x}}\mathcal{H}^h|_{\varrho=\rho_0}\|_{L^2(0,T;H^s_{\boldsymbol{x}})} + \kappa\|\nabla_{\boldsymbol{x}}h^h\|_{L^2(0,T;H^{s,k})} \lesssim M_0,\end{aligned}$$

where $H^{s,k}$ are anisotropic Sobolev spaces allowing for different regularities in horizontal and vertical directions (see [23] for details).

Building upon the above result, [23] provides the rigorous justification of the hydrostatic limit from the non-homogeneous non-hydrostatic (5.6) towards the hydrostatic (5.4) equations with Gent & McWilliams eddy diffusivity.

Theorem 5.2. *For sufficiently regular initial data as before (but augmenting the regularity) satisfying*

$$-(\overline{h} + h_0)\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}_0 - (\nabla_{\boldsymbol{x}}\eta_0) \cdot (\overline{\boldsymbol{u}}' + \partial_\varrho \boldsymbol{u}_0) + \partial_\varrho w_0 = 0 \quad \textit{(div free)}$$

there exists the unique classical solution to (5.6) within the lifespan $[0, T^h]$ of the hydrostatic solution. Moreover, it converges towards the corresponding solution to the hydrostatic equations (5.4) and the convergence rate is of order of the shallowness parameter, namely

$$\|h^{nh} - h^h\|_{L^\infty(0,T^h;H^{s,k})} + \|\boldsymbol{u}^{nh} - \boldsymbol{u}^h\|_{L^\infty(0,T^h;H^{s,k})} \lesssim \varepsilon^2.$$

5.3. *Main ideas of the proofs*

First, it is easy to check that $\psi(t, \mathbf{x}, \varrho)$ in (5.5) is self-adjoint. Therefore, the natural energy functional associated with the hydrostatic equations (5.4) involves $\mathbf{u}$, $\mathcal{H} := \int_{\cdot}^{\rho_1} h\,\mathrm{d}\varrho$, and $\mathcal{H}|_{\varrho=\rho_0} = \int_{\rho_0}^{\rho_1} h\,\mathrm{d}\varrho$ (which represents the free surface deformation), along with their derivatives. Notice that we do *not* control $h = -\partial_\varrho\mathcal{H}$ in (5.4) in the same regularity class as $\mathbf{u}$ and $\mathcal{H}$ unless it is multiplied by $\kappa^{1/2}$. Consequently, to prove Theorem 5.1, we rely on a bootstrap argument allowing to control the functional (5.9). As a drawback, the lifespan T^h of solutions to the non-homogeneous hydrostatic equations is lower bounded by the diffusivity parameter κ and vanishes as $\kappa \to 0$. It is noteworthy that the indices of regularity with respect to the spatial variable, s, and the density variable, k, are decoupled, albeit only in the hydrostatic framework. This decoupling arises because the isopycnal coordinate transformation is semi-Lagrangian, with advection occurring only in the horizontal spatial directions. It would be highly interesting to reduce the regularity assumption concerning the density variable to allow for discontinuities representing density interfaces. Recent advances in these directions are contained in [39].

The result on the approximating system (5.6) in Theorem 5.2 exploits the previously highlighted properties of the limiting hydrostatic equations together with other ingredients that allow to overcome the additional difficulties of the non-hydrostatic equations with respect to the limiting hydrostatic system. Specifically:

- (5.6) is a singular perturbation system;
- equations (5.6) contain an incompressible pressure P enforcing the divergence-free condition.

In particular, the last point suggests that careful *elliptic estimates* in isopycnal coordinates are needed in order to deal with equations (5.6). In fact, a key point in the proof of Theorem 5.2 is to solve the following elliptic boundary-value problem:

$$\frac{1}{\varepsilon^2}\begin{pmatrix}\varepsilon\nabla_{\mathbf{x}}\\ \partial_\varrho\end{pmatrix}\cdot\begin{pmatrix}\frac{h}{\varrho}\mathrm{Id} & \frac{\varepsilon\nabla_{\mathbf{x}}\mathcal{H}}{\varrho}\\ \frac{\varepsilon\nabla_{\mathbf{x}}^\top\mathcal{H}}{\varrho} & \frac{1+\varepsilon^2|\nabla_{\mathbf{x}}\mathcal{H}|^2}{\varrho h}\end{pmatrix}\begin{pmatrix}\varepsilon\nabla_{\mathbf{x}}\\ \partial_\varrho\end{pmatrix}P = \mathrm{RHS},$$
$$P\big|_{\varrho=\rho_0}=0,\qquad (\partial_\varrho P)\big|_{\varrho=\rho_1}=\rho_1 h\big|_{\varrho=\rho_1},$$

where RHS stands for right-hand side (see [23] for details). In compact formulation with $\nabla^\varepsilon_{\mathbf{x},\varrho}=(\varepsilon\nabla_{\mathbf{x}},\partial_\varrho)$

$$\nabla^\varepsilon_{\mathbf{x},\varrho}\cdot(A^\varepsilon\nabla^\varepsilon_{\mathbf{x},\varrho}P)=Q_0+\varepsilon\Lambda Q_1+\nabla^\varepsilon_{\mathbf{x},\varrho}\cdot\mathbf{R},\qquad\qquad P|_{\varrho=\rho_0}=0,$$

where A^ε is the above matrix and for some specific source terms Q_0, Q_1 and remainder $\mathbf{R}$ (see again [23] for details). For the above elliptic-value problem, we can obtain:

- uniform bound of $\nabla^\varepsilon_{\mathbf{x},\varrho}P$ from $H^{s,k}\to H^{s,k}$;
- bound of order ε^2 for $\nabla^\varepsilon_{\mathbf{x},\varrho}P$ as an operator from $H^{s,k}\to H^{s-2,k-2}$ (in other words, we gain smallness at the price of derivative loss).

In particular, the last point explains the need of *augmenting the regularity* in the justification of the hydrostatic limit of Theorem 5.2. The *uniform-in-ε* local existence of solutions to (5.6) is a byproduct of *uniform-in-ε* estimates on the difference

$$\mathbf{U}_{\mathrm{hydro}}-\mathbf{U}_{\mathrm{NONhydro}}$$

between the solution $\mathbf{U}_{\mathrm{hydro}}$ (in compact form) to the limiting system (5.4) and the solution $\mathbf{U}_{\mathrm{NONhydro}}$ to the approximating system (5.6), bootstrapping an energy functional that is a suitable modification of (5.9) (adding non-hydrostatic contribution). This approach provides a rate of convergence of order ε. The optimal rate of convergence ε^2 in Theorem 5.2 is obtained by relying on the previous result and taking the opposite viewpoint, namely viewing the non-hydrostatic solution $\mathbf{U}_{\mathrm{NONhydro}}$ as an approximation to the limiting hydrostatic equations (5.4).

6. Perspectives

In the context of modeling eddy diffusivity, an intriguing open problem, inspired by the specific form of viscosity contributions in the shallow water equations in [40], would be to consider the following system:

$$\begin{aligned}&\partial_t h+\nabla_{\mathbf{x}}\cdot((\bar h+h)(\bar{\mathbf{u}}+\mathbf{u}))=\kappa\Delta_{\mathbf{x}}h,\\ &\partial_t\mathbf{u}+\big((\bar{\mathbf{u}}+\mathbf{u}-\kappa\tfrac{\nabla_{\mathbf{x}}h}{\bar h+h})\cdot\nabla_{\mathbf{x}}\big)\mathbf{u}+\tfrac{1}{\varrho}\nabla_{\mathbf{x}}\mathcal{M}=\nu\frac{\mathrm{div}_{\mathbf{x}}((\bar h+h)\nabla_{\mathbf{x}}\mathbf{u})}{\bar h+h}.\end{aligned}\tag{6.1}$$

It can be written as

$$\begin{aligned}&\partial_t h+\nabla_{\mathbf{x}}\cdot((\bar h+h)(\bar{\mathbf{u}}+\mathbf{u}))=\kappa\Delta_{\mathbf{x}}h,\\ &\partial_t\mathbf{u}+\big((\bar{\mathbf{u}}+\mathbf{u}-(\kappa+\nu)\tfrac{\nabla_{\mathbf{x}}h}{\bar h+h})\cdot\nabla_{\mathbf{x}}\big)\mathbf{u}+\tfrac{1}{\varrho}\nabla_{\mathbf{x}}\psi=\nu\Delta_{\mathbf{x}}\mathbf{u},\end{aligned}\tag{6.2}$$

so that the difference with respect to (NHH) is minor, and the above analysis can easily be extended. However, it is reasonable to conjecture that the viscosity ν can be enough to establish the large-time well-posedness of the initial-value problem for (6.2) when $\kappa=0$, $\nu>0$, which we believe can be addressed in the spirit of the BD and κ entropy [41].

In the framework of fully inviscid and non-diffusive fluids, the long-standing open problem is to establish the well-posedness of the non-homogeneous hydrostatic equations, beginning with the simplified setting of the Boussinesq approximation with a rigid lid (3.3). The primary challenge lies in identifying a suitable energy functional that encodes spectral stability conditions, particularly the Miles-Howard criterion (MH).

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