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ON THE RINDLER'S TYPE MOTIONS BY MEANS OF A VARIATIONAL PRINCIPLE

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Abstract. Using a variational principle of Misner's type, based on a Lobachevsky plane metric treated as a Cayleian metric of the Euclidean plane, Rindler-type motions are analyzed. Within this framework and taking into account the principle of the independence of the simultaneous actions of the fields at a point, it can be replaced by the invariance of the field effects under a certain group and thus a theory of gravity can be formulated to eliminate the intrinsic inconsistencies present in the existing theory.

Keywords: Caylean metric, Rindler motion, variational principle.

1. Introduction

Martzner and Misner [3,4] showed that Einstein's field equations [1,3] can be linked to a variational principle for axially symmetric fields. While they did not directly derive the complex potential, their method can be applied to our context in the following way. Assume the field is represented by the variables (Y^j),, for which the metric has been determined:

$$h_{ij}dY^i dY^j \tag{1}$$

In an ambient space of metric:

$$\gamma_{\alpha\beta}dx^\alpha dx^\beta \tag{2}$$

By using variational principle associated with the Lagrangian [4], we obtain the field equations.

$$L = \gamma^{\alpha\beta}h_{ij}\frac{\partial Y^i}{\partial x^\alpha}\frac{\partial Y^j}{\partial x^\beta} \tag{3}$$

Consequently, if this variational principle is regarded as a foundational premise, the primary objective of this work is to derive metrics of the Lobachevsky plane (or those associated with it), which are intricately linked to Einstein's field equations, utilizing the formalism previously delineated. In this case, we cannot use the variational principle because it exceeds the theoretical limits of application of the Ricci tensor. Indeed, its application extends to scenarios unrelated to Einstein's field equations, such as Rindler's movements.

2. Metric of the Lobachevsky plane as Caylean metric of Euclidean plane

The metric of the Lobachevsky plane can be obtained as a Caleylean metric of a Euclidian plane but in this case the absoluteness is a circle of radius of 1, [4].

$$x^2 + y^2 = 1 \tag{4}$$

The Lobachevsky plane may be established in a one-to-one connection with the inside of this circle.

In general case, the metrization of a Caylean space assume to define the metric as a anharmonic ratio [4].

The Calyean metric can be written in following differential form under the assumption that the absoluteness of space is represented by the quadratic form Ω (X, Y) where X means any vector:

$$-\frac{ds^2}{k^2} = \frac{\Omega(dX,dX)}{\Omega(X,X)} - \frac{\Omega^2(X,dX)}{\Omega^2(X,X)} \tag{5}$$

In Equation (5), the term Ω (X, Y) is the duplication of Ω (X, X) and k is a constant which describes the space curvature.

For Lobachevsky plane, it is known that:

$$\Omega(X,X) = 1 - x^2 - y^2,$$
$$\Omega(X,dX) = -xdx - ydy \tag{6}$$
$$\Omega(dX,dX) = -dx^2 - dy^2$$

which results in:

$$\frac{ds^2}{k^2} = \frac{(1-y^2)dx^2 + 2xydxdy + (1-x^2)dy^2}{(1-x^2-y^2)^2} \tag{7}$$

Performing now the coordinate transformation:

$$x = \frac{h\bar{h}-1}{h\bar{h}+1}, \ y = \frac{h+\bar{h}}{h\bar{h}+1} \tag{8}$$

the metric (7) becomes:

$$\frac{ds^2}{k^2} = -4\frac{dhd\bar{h}}{\left(h-\bar{h}\right)^2} \tag{9}$$

We note that, the absoluteness $1 - x^2 - y^2 = 0$ turns into a straight line $\Im h = 0$. As a result, the lines in Euclidean space transform into circles centered on the real axis of the complex plane, (h).

3. Rindler's motions

The preceding theory applies to any real conic, as the Cayley metric associated with it corresponds directly to the metric defined by its invariance-preserving transformation group. Before analyzing metric (9) further, we provide an example illustrating the relevance of the Matzner-Misner variational principle to this measure. This example deliberately avoids reliance on the curvature tensor, focusing instead on an alternative approach.

W. Rindler [5,6] demonstrated that torsion-free trajectories with constant curvature in a universal space-time serve as a generalization of hyperbolic motions (constant acceleration) in Minkowski space-time. Notably, these motions are expressed in a coordinate system analogous to Kruskal coordinates [7].

These coordinates showed the non-equivalence of second quantization between Minkovsky frames and uniformly accelerated frames [8]. Using this theory, we obtained an important result that an observer experiencing a uniform acceleration in the Minkovsky vacuum, is immersed in a thermal bath, with the temperature directly proportional to the acceleration [9]. For the one-dimensional case, the equation for Minkovsky hyperbolic motion is written as [6]:

$$Z^2 - c^2T^2 = \frac{c^4}{\alpha^2} \tag{10}$$

In Equation (10), the parameter α is the magnitude of acceleration and the other quantities having the usual significance. The equation (10) justifies the term "hyperbolic" corresponding to this motion. The quantity X defined as the following ratio:

$$X = \frac{c^2}{\alpha} \tag{11}$$

is called "Rindler coordinate", attached to hyperbolic motion. Equation (10) then yields:

$$Z^2 = c^2T^2 + X^2 \tag{12}$$

The Equation (12) can explain the rest condition $Z = const.$ in case of a particle in a hyperbolic motion an it can be understood as absoluteness in coordinates (*x,y*):

$$x = \frac{X}{Z},\ \ y = \frac{cT}{Z} \tag{13}$$

It is obtained the metric (9) by taking:

$$h = \frac{y \pm i\sqrt{1 - x^2 - y^2}}{1 - x}, i = \sqrt{-1} \tag{14}$$

or, using the equation (13):

$$h = \frac{cT \pm i\sqrt{z^2 - c^2T^2 - X^2}}{Z - X} \tag{15}$$

Now, the corresponding field equations connected to the Lagrangian:

$$L = \frac{\nabla h \nabla \bar{h}}{\left(h - \bar{h}\right)^2} \tag{16}$$

i.e.

$$\left(h - \bar{h}\right)\nabla^2 h = 2\nabla h \nabla h, \left(h - \bar{h}\right)\nabla^2 \bar{h} = 2\nabla \bar{h} \nabla \bar{h} \tag{17}$$

If we consider the term in the form, $h = u + iv$, then we obtain the following real equations (18):

$$v\nabla^2 u - 2\nabla u \nabla v = 0$$

$$v\nabla^2 v - (\nabla v)^2 + (\nabla u)^2 = 0 \tag{18}$$

In equations (18) the differential operators are considered in an arbitrary three-dimensional metric. It can be observed that, $T = 0$ leads to $h = iv$, where:

$$v = \frac{\sqrt{Z^2 - X^2}}{Z - X} \tag{19}$$

The Equation (19) shows that the Ernst potential [4,10] is real. The second equation of the system (18) describes the field.

By introducing the substitution $v = i\xi$, the equation simplifies to the Laplace equation:

$$\nabla^2 \xi = 0 \tag{20}$$

In spherical coordinates (r, θ, φ), equation (20) for spherical symmetry of ξ, takes the form:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\xi}{dr}\right) = 0 \tag{21}$$

with solution:

$$\xi(r) = C_1 - \frac{C_2}{r} \tag{22}$$

If we use Equation (19) and a suitable solution of the integration constants:

$$\frac{Z + X}{Z - X} = e^{-\frac{K}{r}} \tag{23}$$

From the Equation (23) the coordinate X can be obtained:

$$X = -Z\, tanh\left(\frac{K}{r}\right) \tag{24}$$

In the limit $\frac{K}{r} \rightarrow 0$, one can write:

$$X = -\frac{ZK}{r}$$

or, by means of (11):

$$\alpha = -k^2 r \tag{25}$$

with $k^2 = ZK/c^2$. Consequently, if we consider T = 0, the term α can be understood as the acceleration from the radial oscillatory motion or, more precisely—when considering the limiting case that leads to equation (25)—as the centripetal acceleration in uniform circular motion.

4. Conclusions

The variational principle that leads to the aforementioned result is based on metric (9), which retains its invariance under a specific group of rational coordinate transformations. Using the equivalence principle [3], we can state that at any given location, the Rindler coordinate X represents the gravitational field strength, while the associated group describes the transformations between different gravitational fields acting at that point.

Equation (25) describes the rotational movement obtained as a result of the simultaneous actions of the gravitational fields, a movement characterized by a centripetal force acting at a great distance from the common center of these field-generating sources, at the initial (Minkowski) moment.

Taking into account the principle of the independence of the simultaneous actions of the fields at a point, it can be replaced by the invariance of the field effects under a certain group and thus a theory of gravity can be formulated to eliminate the intrinsic inconsistencies present in the existing framework [4].

We thus focused on that distinct form of hyperbolic motion and not on the curvature tensor. But, if similar results derived from Ernst's equation—particularly the Matzner-Misner variational principle—prove effective in addressing certain inconsistencies in the current gravitational theory, it suggests that this variational principle is highly general and can produce outcomes that potentially contain the space curvature.

We affirm our confidence in the ultimate possibility and proceed with a comprehensive analysis of metric (9), its generalization, and the resulting extension of the variational principle it entails. Additionally, we will attempt to operationally define the fields that make up the Lagrangian.

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