BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI
(BULLETIN OF THE POLYTECHNIC INSTITUTE FROM IAȘI)
Published by "Gheorghe Asachi" Technical University of Iași
Section Mathematics. Theoretical Mechanics. Physics

Volume 70(74), No. 3-4, 2024
DOI: 10.2478/bipmf-2024-0007

MOTION DYNAMIC SCENARIOS IN THE SCALE RELATIVITY THEORY

MONICA MOLCĂLUȚ[1], CRISTINA MARCELA RUSU[2,*]

[1]*Faculty of Physics, "Alexandru Ioan Cuza" Unversity of Iași, Romania*
[2]*Department of Physics, "Gheorghe Asachi" Technical University of Iași, Romania*
[*]*Corresponding authors: cristina-marcela.rusu@academic.tuiasi.ro*

MOLCĂLUȚ MONICA, RUSU CRISTINA MARCELA

MOTION DYNAMIC SCENARIOS IN THE SCALE RELATIVITY THEORY

BULLETIN OF THE POLYTECHNIC INSTITUTE OF IAȘI, 2024, 70(74), 3-4, 19-27;
DOI: 10.2478/bipmf-2024-0007

Received: December 20, 2024

Accepted: February 19, 2025

Publication: March 14, 2025

Abstract. Assuming that any complex system, both structural and functional, can be assimilated to a mathematical object of multifractal type, it is shown that its dynamics can be described by multifractal curves. In such a framework, based on the Multifractal Theory of Motion, the Schrödinger multifractal and the Madelung multifractal scenarios become not only compatible, but also complementary in the dynamic descriptions.

Keywords: multifractal, Schrödinger scenario, Madelung scenario, Markov type stochastic processes.

1. Introduction

If a complex system can be described structurally and functionally as mathematical entities with fractal or multifractal characteristics [1,2], then the dynamics associated with these entities—such as density fields, thermal fields, or tension fields—can be represented using the Multifractal Theory of Motion [3,4]. This theory integrates continuous yet non-differentiable curves (fractal/multifractal curves), making the scale covariant derivative [3,4] applicable.

$$\frac{\hat{d}}{dt} = \partial t + \hat{V}^e \partial_e + D^{ie} \partial_i \partial_e, \; i, e = 1,2,3 \quad (1)$$

In equation (1) we used the notations:

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_e = \frac{\partial}{\partial x^e}, \qquad \partial_i \partial_e = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^e}$$
$$\hat{V}^e = V_D^e - iV_F^e, \; i = \sqrt{-1} \quad (2\text{ a-g})$$
$$D^{ie} = \frac{1}{4}(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\left(d^{ie} + i\bar{d}^{ie}\right)$$
$$d^{ie} = \left(\lambda_+^i \lambda_+^e - \lambda_-^i \lambda_-^e\right), \qquad \bar{d}^{ie} = \left(\lambda_+^i \lambda_+^e + \lambda_-^i \lambda_-^e\right)$$

The spatial coordinates x^e that appear in equations (1) and (2) are described by continuous and non-differentiable functions that depend on the scale resolution.

The temporal coordinate, t, is defined by continuous and differentiable mathematical functions that remain unaffected by scale resolution.

The term V_D^e denotes the differentiable velocity, corresponding to the velocity at a differential scale resolution, while V_F^e represents the nondifferentiable velocity, corresponding to the velocity at a nondifferentiable scale resolution. The constants λ_+^e and λ_-^e are associated with the differentiable-to-nondifferentiable scale transition and the dynamics of forward and backward motion in the complex system, respectively.

The singularity spectrum of order α, denoted as (α), is defined by the relationship $\alpha = \alpha(D_F)$, where D_F signifies the fractal dimension of the motion curves of the structural units within the complex system. It is notable that the complex system maintains its complexity across all times [1,2,5].

Therefore, employing the singularity spectrum of order to examine the dynamics of complex systems offers several distinct advantages:

1. The behavior of structural units within a complex system may reveal a dominant fractal dimension, which assists in discerning a unified global pattern. This pattern is closely associated with the system's specific structural configurations and functional dynamics, providing insight into its overall organization and behavior.
2. The dynamics of the structural units may involve multiple fractal dimensions, enabling the recognition of localized patterns. These patterns reflect zone-specific structural and functional features of the complex system.
3. The singularity spectrum of order α, provides a means to classify universality classes in complex system dynamics, even when the attractors involved have distinct characteristics.

The explicit configuration of the tensor D^{ie}, specifically that associated with the non-multifractal scale transition, is determined by multifractalization through stochastic processes. A widely used approach for fractalization is the application of stochastic processes [3,4].

In this framework, it is crucial to distinguish that the stochastic or chaotic processes employed in the analysis of complex system dynamics can be categorized into two distinct types:

1. **Homogeneous complex system dynamics:** These dynamics are characterized by a single global fractal dimension and display self-similar scaling properties across all time intervals, such as in the case of fractional Brownian motion.
2. **Multifractal processes:** These describe complex system dynamics using multiple fractal dimensions, enabling precise quantification of localized singularities.

Consequently, it is feasible to delineate the following categories of complex system dynamics:

i. Multifractal complex system dynamics governed by Markov stochasticity, determined by the ensuing boundaries [4,6]:

$$\lambda_+^i \lambda_+^e = \lambda_-^i \lambda_-^e = -2\lambda\delta^{ie} \tag{3}$$

In Equation (3), λ represents a diffusion-type coefficient corresponding to the multifractal-to-non-multifractal scale transition, and δ^{ie}denotes the Kronecker pseudo-tensor. Thus:

$$D^{ie} \to i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \tag{4}$$

follows as shown below from Equation (1):

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^e\partial_e - i\lambda(dt)^{\left[\frac{2}{f(\alpha)}\right]-1}\partial^e\partial_e \tag{5}$$

ii. Multifractal complex system dynamics governed by non-Markov stochasticity, defined by the following boundaries [12,13]:

$$d^{ie} = 4\alpha\delta^{ie},\ \bar{d}^{ie} = -4\beta\delta^{ie} \tag{6}$$

In Equation (60, diffusion-type coefficients, α and β are linked to the transition between multifractal and non-multifractal scales. Consequently, the following case arises:

$$D^{ie} = (\alpha - i\beta)(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \tag{7}$$

follows as shown below from Equation (1):

$$\frac{\hat{d}}{dt} = \partial_t + \hat{V}^e \partial_e - (\alpha - i\beta)\,(dt)^{\left[\frac{2}{f(\alpha)}\right]-1} \partial^e \partial_e \tag{8}$$

By assuming the concept of scale covariance, which asserts that the principles governing complex system physics remain unchanged across spatial, temporal, and scale transformations, various conservation laws can be derived. This work, demonstrates that, by applying the specific momentum conservation law to irrotational motions, two scenarios can be formulated to elucidate the dynamics of complex systems: the Schrödinger multifractal scenario and the Madelung multifractal scenario.

2. Complex system geodesics

Next, we will explore the principle of scale covariance, which states that the fundamental laws of physics governing complex systems remain unchanged under both space-time and scale transformations. Consequently, the transition from classical complex system physics to non-differentiable (fractal) complex system physics can be achieved by substituting the standard time derivative d/dt with the non-differentiable operator $\hat{d}/dt$ from Equation (1).

Consequently, this operator functions as the covariant derivative, serving to convey the fundamental equations of complex system dynamics in a manner analogous to the classical (differentiable) case. Given these assumptions, the application of operator (1) to the complex velocity field (2d), without any external constraints, results in the following form for the complex system geodesics.:

$$\frac{d\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i + \frac{1}{4}(dt)^{\left[\frac{2}{f(\alpha)}-1\right]} D^{lk} \partial_l \partial_k \hat{V}^i = 0 \tag{9}$$

This implies that the following terms: the local acceleration $\partial_t \hat{V}^i$, the convection $\hat{V}^l \partial_l \hat{V}^i$ and dissipation $D^{lk} \partial_l \partial_k \hat{V}^i$ make their balance in any point along the non-differentiable curve. Furthermore, the inclusion of the complex viscosity-type coefficient $\frac{1}{4}(dt)^{\left[\frac{2}{f(\alpha)}-1\right]} D^{lk}$ indicates that the complex system exhibits rheological behavior, implying that it possesses intrinsic memory encoded within its structural framework.

If fractalization is accomplished using Markov-type stochastic processes, which incorporate Lévy-type motions of the structural units of the complex system [1,2,5,6], then the equation for the geodesics of the complex system assumes a simple form.:

$$\frac{d\hat{V}^i}{dt} = \partial_t \hat{V}^i + \hat{V}^l \partial_l \hat{V}^i - i\lambda (dt)^{\left[\frac{2}{f(\alpha)}-1\right]} \partial^l \partial_l \hat{V}^i = 0 \tag{10}$$

If we separate the motions on differential and fractal scale resolutions we obtain:

$$\frac{d\hat{V}^i}{dt} = \partial_t V_D^i + V_D^l \partial_l V_D^i - [V_F^l - \lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial^l]\partial_l V_F^i = 0$$

$$\frac{d\hat{V}_F^i}{dt} = \partial_t V_F^i + V_D^l \partial_l V_F^i - [V_F^l - \lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial^l]\partial_l V_D^i = 0 \quad \text{(11 a, b)}$$

3. Schrödinger multifractal Scenario

For irrotational motions:

$$\varepsilon_{ikl}\partial^k \hat{V}^l = 0 \quad (12)$$

where ε_{ikl} is the Levi-Civita pseudo-tensor. Then, we choose $\hat{V}^i$ in the form [3.4]:

$$\hat{V}^i = -2i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial^i \ln\Psi \quad (13)$$

In Equation (13), $\ln\Psi$ is the scalar potential of the complex velocity field.

Substituting (13) in (10), we obtain:

$$\frac{\hat{d}\hat{V}^i}{dt} == -2\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\left\{\partial_t\partial^i \ln\Psi - i\left[2i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}(\partial^l \ln\Psi\,\partial_l)\partial^i \ln\Psi + \right.\right.$$
$$\left.\left. +\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial^l\partial_l\partial^i \ln\Psi\right]\right\} = 0 \quad (14)$$

Using the identities:

$$\partial^l\partial_l \ln\Psi + \partial_i \ln\Psi\,\partial^i \ln\Psi = \frac{\partial_l\partial^l\Psi}{\Psi} \quad (15)$$

$$\partial^i\left(\frac{\partial_l\partial^l\Psi}{\Psi}\right) = 2(\partial^l \ln\Psi\,\partial_l)\partial^i \ln\Psi + \partial^l\partial_l\partial^i \ln\Psi \quad (16)$$

the equation (14) becomes:

$$\frac{\hat{d}\hat{V}^i}{dt} = -2i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial_i[\partial_t \ln\Psi - 2i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\frac{\partial_l\partial^l\Psi}{\Psi} \quad (17)$$

We can integrate Equation (17) up to an arbitrary phase factor, which can be nullified through an appropriate selection of the phase in the resulting expression:

$$\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}-2\right]}\partial_l\partial^l\Psi + i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial_t\Psi = 0 \quad (18)$$

For motion on Peano-type curves, $D_F = 2$, at Compton scale, $\lambda = \hbar/2m_0$, where $\hbar$, is the reduced Planck's constant and m_0 the rest mass of the structural units, relation (17) reduces to the standard Schrödinger equation. This serves as the basis for the development of quantum measure theory [6].

Relation (18) constitutes a Schrödinger-type equation, representing complex system geodesics inside the Schrödinger framework. The inclusion of an external scalar potential *U* alters equation (18) as follows:

$$\lambda^2(dt)^{\left[\frac{4}{f(\alpha)}-2\right]}\partial_l\partial^l\Psi + i\lambda(dt)^{\left[\frac{2}{f(\alpha)}-1\right]}\partial_t\Psi - \frac{U}{2}\Psi = 0 \tag{19}$$

Equation (19) represents the equation of motion for the one-body problem within the Schrödinger-type formulation of the non-differentiable model [7].

According to the Scale Relativity Theory [6], the standard equation of motion for the "one body problem" is given by:

$$D^2\partial^l\partial_l\Psi + iD\partial_t\Psi - \frac{U}{2}\Psi = 0 \tag{20}$$

where this equation emerges from (19). Equation (20) characterizes the motions of the complex system structural units on Peano type curves, $D_F = 2$ and it establishes the correspondence $\lambda \equiv D$, where *D* denotes the coefficient governing the transition between fractal to non-fractal behavior from Scale Relativity Theory.

4. The Madelung Multifractal Scenario

If $\Psi = \sqrt{\rho}exp\,(iS)$ with $\sqrt{\rho}$ the amplitude and *S* the phase of Ψ, the complex velocity field (13) takes the explicit form:

$$\hat{V}^i = -2i\lambda(dt)^{[\frac{2}{f(\alpha)}-1]}\partial^i\, ln\,\Psi$$

$$V_D^i = 2\lambda(dt)^{[\frac{2}{f(\alpha)}-1]}\partial^i S \tag{21 a-c}$$

$$V_F^i = \lambda(dt)^{[\frac{2}{f(\alpha)}-1]}\partial^i \rho$$

Substituting (21 a-c) in (10) and separating the real and imaginary parts, up to an arbitrary phase factor which may be set to zero by a suitable choice of the phase of Ψ, we obtain:

$$\partial_t V_D^i + \left(V_D^l\partial_l\right)V_D^i = -\partial^i(Q) \tag{22}$$

and

$$\partial_t\rho + \partial_l\left(\rho V_D^l\right) = 0 \tag{23}$$

with *Q* the specific non-differentiable (fractal) potential:

$$Q = -2\lambda^2(dt)^{(2/f(\alpha))-2}\,\frac{\partial^l\partial_l\sqrt{\rho}}{\sqrt{\rho}} = -\frac{V_F^l V_{Fl}}{2} - \lambda(dt)^{(2/f(\alpha))-1}\partial_i V_F^i \tag{24}$$

The external scalar potential *U* changes the equation (22) thus:

$$\partial_t V_D^i + \left(V_D^l\partial_l\right)V_D^i = -\partial^i(Q+U) \tag{25}$$

Equations (22) and (23) respectively represent the momentum conservation law and the state density conservation law, while equations (22), (23), and (24) collectively define the fractal hydrodynamic model. This model leads to the following insights:

i. Each structural unit of a complex system continuously interacts with a fractal medium via a distinct non-differentiable potential.
ii. The complex system can be modeled as a fractal fluid (or non-differentiable fluid), with its dynamics governed by the fractal hydrodynamic framework.
iii. The fractal velocity field, V_F^i, does not represent actual motion but rather enables the transfer of specific momentum and the concentration of energy. This is evident from the absence of V_F^i in the state density conservation law and its functional role within the variational principle [6,7].
iv. Any interpretation of the individual fractal potential must account for the intrinsic nature of specific momentum transfer. Fractal energy is conserved as a whole: part of it is stored as mass motion and fractal potential energy, while other portions remain available elsewhere. The conservation of fractal energy and fractal momentum guarantees fractal reversibility and the existence of fractal eigenstates. Additionally, this predicts the possibility of existence of a Lévy motion fractal force, which would suggest interaction with an external medium [6,7].
v. The specific fractal potential *Q* generates the tensor (fractal stress tensor—see [7] for details):

$$\hat{\sigma}_{il} = -2\lambda^2 (dt)^{\left(\frac{2}{f(\alpha)}\right)^{-2}} \rho \partial_i \partial_l \ln \rho \tag{26}$$

divergence of which corresponds to the usual force density associated with *Q*:

$$\partial^i \hat{\sigma}_{il} + \rho \partial_l Q = 0 \tag{27}$$

Under these conditions, the equations of the fractal hydrodynamics become:

$$\partial_t V_D^i + \left(V_D^l \partial_l\right) V_D^i = -\rho^{-1} \partial_l \hat{\sigma}_{il} + \partial^i U$$
$$\partial_t \rho + \partial_i \left(\rho V_D^i\right) = 0 \tag{28}$$

vi) Fractal stationary states can be categorized into two distinct types:

a) Fractal *dynamic states.* For $\partial_t = 0$ and $V_D^l \neq 0$, the equations (28 a, b) give:

$$\frac{1}{2} V_D^l V_{Dl} + U + Q = E$$
$$\rho V_D^i = \varepsilon^{ilk} \partial_l f_k \tag{29}$$

Thus, the sum of the specific kinetic energy $\frac{1}{2} V_D^l V_{Dl}$, the external specific scalar potential, *U,* and specific fractal potential, *Q*, remains constant, i.e., it is equal to the integration constant

$E \neq E(x^l)$ (refer to the first Eq. (29)). E≡<E> represents the total specific fractal energy of the fractal dynamic complex system.

The fractal states density current, ρV_D^l, has no sources (see the second Eq. (29)), i.e. its fractal streamlines are closed.

b) *Fractal static states.* For $\partial_t = 0$ and $V_D^l = 0$ the equations (28 a, b) give:

$$U + Q = E \tag{30}$$

The sum of the external specific potential U and the specific fractal potential Q remains constant, equal to the integration constant $E \neq E(x^l)$ (see Eq. (30)).

Here, E≡<E> represents the total specific fractal energy of the static fractal complex system. Furthermore, the fractal states density conservation law is precisely satisfied.

i) Let us write the following logarithmic function which is call non-differentiable entropy:

$$S(x^l, t) = ln\, \rho(x^l, t) \tag{31}$$

The non-differentiable entropy evaluates the disorder degree of a complex system, (for details see [4,7].

On the contrary, the function:

$$I(x^l, t) = -\, ln\, \rho(x^l, t) \tag{32}$$

i.e., the negative non- differentiable entropy characterizes the order degree of a complex system. It will be called later non-differentiable information. The dynamics of both non-differentiable entropy and non-differentiable information in the aforementioned alternatives are governed by the unique fractal potential through the established relations.:

$$Q = -\lambda^2 (dt)^{\left[\frac{2}{f(\alpha)}-1\right]^{-2}} \left(\frac{1}{2}\partial_i S \partial^i S + \partial_i \partial^i S\right) \tag{33}$$

respectively:

$$Q = -\lambda^2 (dt)^{\left[\frac{2}{f(\alpha)}-1\right]^{-2}} \left(\frac{1}{2}\partial_l I \partial^i I + \partial_i \partial^i I\right) \tag{34}$$

Moreover, by derivation, the equation (34) in the form:

$$F^i = -\partial^i Q = \lambda^2 (dt)^{\left[\frac{2}{f(\alpha)}-1\right]^{-2}} \left(\frac{1}{2}\partial^i \partial_l I \partial^l I + \partial^i \partial_l \partial^l I\right) \tag{35}$$

specifies that the variations pf the non-differentiable information induce force fields.

These fields of forces depend on the type of interaction personalized by the constant structures of the transition fractal-nonfractal through the constant λ, but also on the fractalization "degree" given by the fractal dimension $f(\alpha)$.

5. Conclusions

The primary conclusions of this study are as follows:

i) The dynamics of structural units within any complex system are characterized by multifractal curves, leading to the derivation of a movement operator that functions as a covariant derivative;

ii) The assertion of a scale covariance principle necessitates the law of conservation of specific momentum;

iii) The elucidation of the velocity field in the context of irrotational dynamics necessitates the implementation of two analytical scenarios for the movement of any complex system: the multifractal Schrödinger scenario and the Madelung multifractal scenario.

iv) The two scenarios depicting the movement are not mutually incompatible but rather complementing. The Schrödinger multifractal scenario delineates "digital" behaviors in the dynamics of any complex system, while the Madelung multifractal scenario delineates "analog" behaviors in the same dynamics (for further details, see to [6,7]).

REFERENCES

[1] Bar Yam, Dynamics of Complex Systems, The Advanced Book Program, Addison-Wesley, Reading, Masachusetts, 1977.

[2] Badii R., Complexity: Hierarchical Structures and Scaling in Physics, Cambridge University Press, 1997.

[3] Mercheș I., Agop M., Differentiability and Fractality in dynamics of Physical Systems, World Scientific, New Jersey, 2016.

[4] Agop M., Paun V.P., On the New Perspectives of Fractal Theory. Fundaments and Applications, Romanian Academy Publishing House, Bucharest, 2017.

[5] Mitchell M., Complexity: A guided tour, Oxford University Press, Oxford UK, 2009.

[6] Nottale L., Scale Relativity and Fractal Space-Time, A new Approach to Unifying Relativity and Quantum Mechanics, Imperial College Press, London, 2011.

[7] Agop M., Irimiciuc Ș.A., The multifractal Theory of Motion From small to large Scale, Springer, Singapore, 2024.