BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI
(BULLETIN OF THE POLYTECHNIC INSTITUTE OF IAȘI)
Published by "Gheorghe Asachi" Technical University of Iasi
Section Mathematics Theoretical Mechanics. Physics

Volume 70(74), No. 1-2, 2024
DOI: 10.2478/bipmf-2024-0005

PHENOMENA IN BUBBLES CLUSTER

ION SIMACIU[1*], ZOLTAN BORSOS[2], VIOREL DRAFTA[3], GHEORGHE DUMITRESCU[4]

[1]*Retired lecturer, Petroleum-Gas University of Ploieşti, Ploiești 100680, Romania*
[2]*Petroleum-Gas University of Ploieşti, Ploiești 100680, Romania*
[3]*Independent researcher*
[4]*High School Toma N. Socolescu, Ploiești, Romania*
**Corresponding author; e-mail: isimaciu@yahoo.com*

Citation

SIMACIU ION, BORSOS ZOLTAN, DRAFTA VIOREL, DUMITRESCU GHEORGHE

PHENOMENA IN BUBBLES CLUSTER

BULLETIN OF THE POLYTECHNIC INSTITUTE OF IAȘI, 2024, 70(74), 1-2, 89-108;

DOI: 10.2478/bipmf-2024-0005

Received: January 18, 2024

Accepted: April 10, 2024

Publication: May 30, 2024

Abstract. In this paper, we continue to demonstrate the analogy between the acoustic world and the electromagnetic world. In the paper, we derive the expressions for: the electro-acoustic force between the bubbles in the cluster and an outer bubble, the gravito-acoustic forces between the bubbles in the cluster and between the cluster and an outer bubble, the temperature corresponding to the translational motion of the bubbles in the cluster (acoustic temperature) and the average acoustic radiation pressure. The most important result is the demonstration of the fact that the modulus of the average pressure of the acoustic radiation around a bubble is equal to the energy density of the electro-acoustic field, Eq. (10). These densities are the oscillation energy densities of the liquid around the bubble that are involved in the interaction phenomenon between two oscillating/pulsating bubbles. We have also demonstrated that the gravito - acoustic forces, at resonance, between the bubbles in the cluster, generated by the absorption of energy in the bubbles, are proportional to the square of the virtual masses of the bubbles, Eqs. (29) and (30).

Keywords: acoustic world, bubbles cluster, acoustic temperature, average pressure of the acoustic radiation, energy density of the electro - acoustic field.

1. Introduction

In the scientific literature, several systems and phenomena have been studied theoretically and experimentally, which prove the analogy between the systems and phenomena of the acoustic world and the electromagnetic world. The phenomena were studied: the interaction of two oscillating bubbles and the translational motion [1-5, 23, 30-34], the dynamics of a system of several bubbles (filament and spherical cluster) and the coupling of oscillations [6-8], the acoustic Casimir effect [9-12], the interaction between vortices and bubbles [13-15], the formation and properties of antibubbles [16, 17], the acoustic lens effect for an oscillating bubble and a cluster [18-20], the interaction between droplets, particles solids and bubbles [24], various other interactions involving bubbles [35, 36], the sonoluminescence phenomenon [37], etc. The purpose of this paper is to study other phenomena specific to a multi-bubble system. In the second part, we complete the study of the electro-acoustic interaction of bubbles in a cluster, demonstrating a Gaussian law for this interaction, and highlight the hydromechanical significance of the intensity of the electro-acoustic field. In the third sections we study the gravito - acoustic force between the bubbles in the cluster. In the fourth section we study the electro-acoustic and gravito - acoustic forces between an outer bubble and a cluster. In the fifth section we study the phenomenon of cluster compression and thermal effects and the corresponding cluster dumb hole.

2. Cluster electro-acoustic force: interpretation and significance

2.1. Electro-acoustic force and Gauss's law

We analyzed the electro-acoustic force between two bubbles in the cluster in the paper Mach's Principle in the Acoustic World [8]. According to the paper Interaction of a bubble and a bubble cluster in an acoustic field [7] the electro - acoustic force between two bubbles inside the cluster is,

$$\vec{F}_{2B}^{(1)} = \frac{\rho}{4\pi r^3}\left\langle\left(\dot{V}_2\right)^2\right\rangle\vec{r}, \tag{1}$$

that is Eq. (10) corrected, from paper [7]. The expressions of this force are given in Eqs. (27-29) from paper [8]. This summed force for the bubbles in the sphere of radius r_0 gives the force with which these bubbles act on a single bubble on the surface of the sphere, Eq. (11) from paper [7]. This expression is analogous to the electrostatic force with which charged particles in a sphere of radius r_0 act on a particle on the surface of the sphere, if we consider the quantity $\rho\left\langle V_2\ddot{V}_2\right\rangle/\left(4\pi\right) = e_{eain}^2$ to be the electro - acoustic charge on the inner bubbles, i.e.

$$\vec{F}_{2B}^{(2)} = \frac{N\rho r_0}{4\pi r_{clust}^3}\left\langle V_2\ddot{V}_2\right\rangle\vec{e}_{r0} = \frac{-Ne_{eain}^2 r_0}{r_{clust}^3}\vec{e}_{r0}. \tag{2}$$

The electrostatic force being a force proportional to the square of the distance, $\vec{F}_e = -q^2\vec{r}/\left(4\pi\varepsilon_0 r^3\right)$, the particles in the radius sphere r_0, $N_0 = \left(4\pi r_0^3/3\right)\left(3N/4\pi R_c^3\right) = Nr_0^3/R_c^3$, acts on a particle/charge on the surface of the sphere, according to Gauss's law [21, Ch. 4] as if it were a single-value charge $q_{N_0} = qN_0 = Nqr_0^3/R_c^3$ placed in the center of the sphere, $\vec{F}_{eN_0} = -qq_{N_0}\vec{r}_0/\left(4\pi\varepsilon_0 r_0^3\right) = -Nq^2\vec{r}_0/\left(4\pi\varepsilon_0 R_c^3\right) = -Ne^2\vec{r}_0/R_c^3$ with $e^2 = q^2/\left(4\pi\varepsilon_0\right)$ the square of the electrostatic charge in vacuum. Except for the sign (the forces between identical electric charges are repulsive) this electrostatic relation is analogous to Eq. (2). Also, according to Gauss's law, the force exerted by charged particles between the sphere of radius r_0 and the sphere of radius R_c on a particle on the sphere of radius r_0 or less, $r < r_0$, is zero. The Newtonian gravitational field and the gravito - acoustic field have the same properties. We will use this property to calculate the average potential energy of the bubbles in the cluster.

2.2. Electro - acoustic field intensity, mechanical significance

To solve the problem of the interaction force between two bubbles we used only the term proportional to the radial acceleration, $\ddot{R}$, from the expression of the pressure induced by the bubble oscillation [1]

$$, p_b(r,t) = \frac{\rho\ddot{V}}{4\pi r} - \frac{\rho\dot{R}^2}{2}\left(\frac{R}{r}\right)^4 = \frac{\rho R^2\ddot{R}}{r} + \frac{2\rho R\dot{R}^2}{r} - \frac{\rho\dot{R}^2}{2}\left(\frac{R}{r}\right)^4 \cong \frac{\rho R^2\ddot{R}}{r}. \tag{3}$$

to keep only the terms in the first order in the dimensionless amplitude of the oscillations $a \leftrightarrow \varepsilon$ [2, 4].

The average induced pressure determined by the bubble oscillations is

$$\left\langle p_b(r,t)\right\rangle = \frac{\rho\left\langle R^2\ddot{R}\right\rangle}{r} + \frac{2\rho\left\langle R\dot{R}^2\right\rangle}{r} - \frac{\rho\left\langle \dot{R}^2R^4\right\rangle}{2r^4}. \tag{4}$$

Since

$$R(t) = R_0\left(1 + x(t)\right) = R_0\left[1 + a\cos\left(\omega t + \varphi\right)\right], \tag{5}$$

result: $\dot{R} = R_0\dot{x} = -R_0\omega a\sin\left(\omega t + \varphi\right) = -R_0\omega a\sin\alpha$, $\ddot{R} = R_0\ddot{x} = -R_0\omega^2 a\cos\alpha$. With these, the expression of the averaged pressure (4) becomes:

$$\begin{aligned}\left\langle p_b(r,t)\right\rangle &= \frac{\rho R_0^3}{r}\left\langle\left(1+x\right)^2\ddot{x}\right\rangle + \frac{2\rho R_0^3}{r}\left\langle\left(1+x\right)\dot{x}^2\right\rangle - \frac{\rho R_0^6\left\langle\left(1+x\right)^4\dot{x}^2\right\rangle}{2r^4} = \\ &\frac{\rho R_0^3}{r}\left[\left\langle\ddot{x}\right\rangle + 2\left\langle x\ddot{x}\right\rangle + \left\langle x^2\ddot{x}\right\rangle + 2\left\langle\dot{x}^2\right\rangle + 2\left\langle x\dot{x}^2\right\rangle\right] - \\ &\frac{\rho R_0^6}{2r^4}\left[\left\langle\dot{x}^2\right\rangle + 4\left\langle x\dot{x}^2\right\rangle + 6\left\langle x^2\dot{x}^2\right\rangle + 4\left\langle x^3\dot{x}^2\right\rangle + \left\langle x^4\dot{x}^2\right\rangle\right].\end{aligned} \tag{6}$$

With $\langle \ddot{x} \rangle = -a\omega^2 \langle \cos\alpha \rangle = 0$, $\langle x\ddot{x} \rangle = -a^2\omega^2 \langle \cos^2\alpha \rangle = -a^2\omega^2/2 = -\langle \dot{x}^2 \rangle$, $\langle x\dot{x}^2 \rangle = a^3\omega^2 \langle \cos\alpha \sin^2\alpha \rangle = 0$, $\langle x^2\dot{x}^2 \rangle = a^4\omega^2 \langle \cos^2\alpha \sin^2\alpha \rangle = a^4\omega^2/8$, $\langle x^2\ddot{x} \rangle = -a^3\omega^2 \langle \cos^3\alpha \rangle = 0$, $\langle x^3\dot{x}^2 \rangle = a^5\omega^2 \langle \cos^3\alpha \sin^2\alpha \rangle = 0$ and $\langle x^4\dot{x}^2 \rangle = a^6\omega^2 \langle \cos^4\alpha \sin^2\alpha \rangle = a^6\omega^2/16$, substituting into (6), it follows

$$\langle p_b(r,t) \rangle = p_b(r) = \frac{-\rho R_0^6 a^2 \omega^2}{4r^4}\left(1 + \frac{3a^2}{2} + \frac{a^4}{8}\right) \cong \frac{-\rho R_0^6 a^2 \omega^2}{4r^4} < 0. \tag{7}$$

This pressure is the effect of the wave scattered by the oscillating bubble or, equivalently, the waves emitted by the oscillating bubble. The interpretation of this result is obtained if we take into account that the total pressure averaged in time outside the bubble is the sum of the hydrostatic pressure p_0 and the induced pressure averaged in time determined by the oscillations of the bubbles,

$$p_t = p_0 + p_b(r) = p_0 - \frac{\rho R_0^6 a^2 \omega^2}{4r^4} < p_0, \tag{8}$$

that is, the time-averaged total pressure around the oscillating bubble is less than the hydrostatic pressure p_0.

The total energy corresponding to this time-averaged scattered radiation pressure is

$$\langle E_b \rangle = \int_{R_0}^{R_c} \langle w_b \rangle 4\pi r^2 dr = \int_{R_0}^{\infty} p_b(r) 4\pi r^2 dr = \pi\rho R_0^6 a_N^2 \omega^2 \int_{R_0}^{\infty} \frac{dr}{r^2} \cong \pi\rho R_0^6 a_N^2 \omega^2 \left(\frac{1}{R_0}\right) = \pi\rho R_0^5 a_N^2 \omega^2 = E_{kb}, \tag{9}$$

with $w_b = -p_b(r)$ [21, 22] and is the kinetic energy of oscillation, of the liquid around the bubble, averaged over time.

The average oscillation energy density is equal to the energy density [21, 22] of the electro-acoustic field of a cluster bubble written in the CGS Gaussian units system

$$\langle w_{bin} \rangle = w_{eain} = \frac{E_{eain}^2}{8\pi} = \frac{e_{eain}^2}{8\pi r^4} = \frac{|F_{BN}|}{8\pi r^2} = \frac{\rho R_0^6 a_N^2 \omega^2}{4r^4}, \tag{10}$$

with e_{eain}^2 the square of the acoustic charge for the bubbles in the cluster, according to relation (2) in this paper:

$$\begin{aligned}
&F_{BNss}(r) = \frac{-2\pi\rho\omega^2 R_0^6 a_N^2}{r^2} \cong \frac{-2\pi\omega^2 A^2 R_0^2}{\rho r^2 \left[\left(\omega^2 N_c - \omega_0^2\right)^2 + 4\beta_s^2 \omega^2\right]} = \frac{-e_{eain}^2}{r^2},\\
&e_{eain}^2 = 2\pi\rho\omega^2 R_0^6 a_N^2 = \frac{-2\pi\omega^2 A^2 R_0^2}{\rho \left[\left(\omega^2 N_c - \omega_0^2\right)^2 + 4\beta_s^2 \omega^2\right]},\\
&F_{BNssr}(r) = \frac{-2\pi\rho\omega^2 R_0^6 a_{Nr}^2}{r^2} \cong \frac{-2\pi A^2 \rho u^2 R_0^4 N_c^2}{r^2 p_{eff}^2} = \frac{-e_{eainr}^2}{r^2},\\
&e_{eainr}^2 = \frac{2\pi A^2 \rho u^2 R_0^4 N_c^2}{p_{eff}^2}.
\end{aligned} \tag{11}$$

The total energy of the electro - acoustic field around a bubble, taking into account the energy density expression (10), is

$$E_{eain}=\int_{R_0}^{R_c} w_{eain}4\pi r^2 dr \cong \pi\rho R_0^5 a_N^2\omega^2 = E_{bin}=E_{bkin}. \tag{12}$$

Relation (12) highlights the significance of the electro - acoustic field energy as the average oscillation energy of the liquid around the oscillating bubble. This energy is interpreted by an outside observer as the rest energy of the oscillating bubble system, analogous to how the electrostatic field energy of a charged particle without spin is interpreted as the rest energy of the particle.

3. Cluster gravito-acoustic force

To derive the expression of the gravito-acoustic force between two bubbles in a cluster we use the expression of the absorption – scattering force between two bubbles (i.e. of the gravito-acoustic force), Eq. (12), from the paper [5] written under the conditions: $A\to A_N$, $\omega_0\to\omega_N$ and $\beta\to\beta_N$

$$F_{BNsa}(r)=\frac{-16\pi R_0^2\omega^4 A_N^2\beta_{sN}\beta_{aN}}{r^2\rho\left[\left(\omega^2-\omega_N^2\right)^2+4\beta_{sN}^2\omega^2\right]^2}. \tag{13}$$

Replacing parameters $A_N=A/(1+N_c), \omega_N=\omega_0/\sqrt{1+N_c}$ and $\beta_N=\beta/(1+N_c)$ given by the expressions (16) from the paper [8], results

$$F_{BNsa}(r)=\frac{-16\pi R_0^2\omega^4 A^2\beta_s\beta_a}{r^2\rho(1+N_c)^4\left[\left(\omega^2-\omega_0^2/(1+N_c)\right)^2+4\left[\beta_s^2/(1+N_c)^2\right]\omega^2\right]^2}= \\ \frac{-16\pi R_0^2\omega^4 A^2\beta_s\beta_a}{r^2\rho\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]^2} \tag{14}$$

With $\beta_s=\omega^2R_0/(2u)$, Eq. (14) gets

$$F_{BNsa}(r)=\frac{-16\pi R_0^2\omega^4 A^2\beta_s\beta_a}{r^2\rho\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]^2}= \\ \frac{-8\pi R_0^3\omega^6 A^2\beta_a}{r^2\rho u\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+R_0^2\omega^6/u^2\right]^2}. \tag{15}$$

Substituted in Eq. (15), the damping coefficient of the oscillations

$$\beta_a = \beta_\upsilon + \beta_{th} = \frac{2\mu}{\rho R_0^2} + \frac{2\mu_{th}(\omega)}{\rho R_0^2}, \tag{16}$$

two types of gravito - acoustic forces result: the gravito - acoustic force determined by the absorption of acoustic energy in the liquid

$$F_{BNsa\mu}(r) = \frac{-8\pi R_0^3 \omega^6 A^2 \beta_\upsilon}{r^2 \rho u \left[\left(\omega^2 (1+N_c) - \omega_0^2 \right)^2 + R_0^2 \omega^6 / u^2 \right]^2} = \frac{-16\pi R_0 \omega^6 A^2 \mu}{r^2 \rho^2 u \left[\left(\omega^2 (1+N_c) - \omega_0^2 \right)^2 + R_0^2 \omega^6 / u^2 \right]^2} \tag{17}$$

and the gravito - acoustic force due to the absorption of acoustic energy in the vapor and/or gas in the bubbles

$$F_{BNsath}(r) = \frac{-8\pi R_0^3 \omega^6 A^2 \beta_{th}(\omega)}{r^2 \rho u \left[\left(\omega^2 (1+N_c) - \omega_0^2 \right)^2 + R_0^2 \omega^6 / u^2 \right]^2} = \frac{-16\pi R_0 \omega^6 A^2 \mu_{th}(\omega)}{r^2 \rho^2 u \left[\left(\omega^2 (1+N_c) - \omega_0^2 \right)^2 + R_0^2 \omega^6 / u^2 \right]^2}. \tag{18}$$

At resonance, $\omega = \omega_N$, the force expression (17) becomes

$$F_{BNsa\mu r}(r) = \frac{-16\pi A^2 \mu u^3}{r^2 \rho^2 R_0^3 \omega_N^6} = \frac{-16\pi (1+N_c)^3 A^2 \mu u^3}{r^2 \rho^2 R_0^3 \omega_0^6} = \frac{-16\pi (1+N_c)^3 (A^2 \rho u^2)}{p_{eff}^3} \left(\frac{R_0^3 u \mu}{r^2} \right), \tag{19}$$

With the virtual mass of the bubble $m_b = 2\pi R_0^3 \rho / 3$ [2], the force expression (19) becomes

$$F_{BNsa\mu}(r) = \frac{-24(1+N_c)^3 m_b A^2 u^3 \mu}{r^2 p_{eff}^3} = \frac{-24(1+N_c)^2 m_N A^2 u^3 \mu}{r^2 p_{eff}^3} \tag{20}$$

and therefore it is proportional to the virtual mass m_b or to the equivalent virtual mass (determined by the coupling of oscillations), according to Eq. (12) from the paper [8], $m_N = (1+N_c) m_b$.

Next we analyze the gravito-acoustic force determined by the absorption of acoustic energy in the vapor and/or gas in the bubbles. An analytical expression of the thermal damping constant $\beta_{th}(\omega) = 2\mu_{th}(\omega) / (\rho R^2)$ is given in the work Acoustic radiation forces: Classical theory and recent advances [24]

$$\beta_{thi} = \frac{3(\gamma-1)\left[X_i (\sinh X_i + \sin X_i) - 2(\cosh X_i - \cos X_i) \right] \omega_{0i}^2}{2X_i \left[X_i (\cosh X_i - \cos X_i) + 3(\gamma-1)(\sinh X_i - \sin X_i) \right] \omega}, \tag{21}$$

with $X = R_0 (2\omega/\chi)^{1/2}$, γ the ratio of specific heats, $\chi = K / (\rho_g c_P) \cong \mu_g / (\rho_g \gamma) = D_g / \gamma$ the thermal diffusivity of gas/vapor in bubbles, K the thermal conductivity, μ_g the dynamic viscosity of gas

/vapour and $D_g = l_g \upsilon_g / 3$ the diffusion coefficient (l_g the mean free path and υ_g is the mean thermal speed in gas/vapor).

In the paper Acoustic force of the gravitational type [25], relation (21) is approximated: a) for the case when $X = R_0 \left(2\omega/\chi\right)^{1/2} << 1$ ($\omega << \chi/\left(2R_0^2\right) = D_g/\left(2\gamma R_0^2\right)$)

$$\beta_{th} \cong \frac{(\gamma-1)X^2\omega_0^2}{60\gamma\omega} = \frac{(\gamma-1)R_0^2\omega_0^2}{30\gamma\chi} \tag{22}$$

and b) for the case when $X = R_0 \left(2\omega/\chi\right)^{1/2} >> 1$ ($\omega >> \chi/\left(2R_0^2\right) = D_g/\left(2\gamma R_0^2\right)$)

$$\beta_{th} \cong \frac{3(\gamma-1)\omega_0^2}{2X\omega} = \frac{3(\gamma-1)\omega_0^2\chi^{1/2}}{2\sqrt{2}\omega\sqrt{\omega}R_0}. \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (18), it results:

$$F_{BNsath}(r) = \frac{-4(\gamma-1)\pi R_0^5\omega_0^2\omega^6A^2}{15\gamma\chi r^2\rho u\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2 + R_0^2\omega^6/u^2\right]^2}, \omega << \chi/\left(2R_0^2\right) \tag{24}$$

and

$$F_{BNsath}(r) = \frac{-6\sqrt{2}(\gamma-1)\pi R_0^2\omega^4\sqrt{\omega}\omega_0^2\chi^{1/2}A^2}{r^2\rho u\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2 + R_0^2\omega^6/u^2\right]^2}, \omega >> \chi/\left(2R_0^2\right) \tag{25}$$

At resonance, $\omega = \omega_N$, the forces have the expressions:

$$F_{BNsathr}(r) = \frac{-8\pi u^3A^2\beta_{th}(\omega_N)}{r^2R_0\rho\omega_N^6} = \frac{-8\pi(1+N_c)^3u^3A^2\beta_{th}(\omega_N)}{r^2R_0\rho\omega_0^6} = \frac{-8\pi(1+N_c)^3R_0^5\rho^2u^3A^2\beta_{th}(\omega_N)}{r^2p_{eff}^3} \tag{26}$$

$$F_{BNsathr1}(r) = \frac{-4(\gamma-1)\pi R_0\omega_0^2A^2u^3}{15\gamma\chi r^2\rho\omega_N^6} = \frac{-4\pi(\gamma-1)(1+N_c)^3R_0A^2u^3}{15\gamma\chi r^2\rho\omega_0^4} = \frac{-4\pi(\gamma-1)(1+N_c)^3R_0^5\rho u^3}{15\gamma\chi r^2}\left(\frac{A}{p_{eff}}\right)^2, \omega << \chi/\left(2R_0^2\right). \tag{27}$$

and

$$F_{BNsathr2}(r) = \frac{-6\sqrt{2}(\gamma-1)\pi\omega_0^2\chi^{1/2}u^3A^2}{r^2R_0^2\rho\omega_N^7\sqrt{\omega_N}} = \frac{-6\sqrt{2}\pi(\gamma-1)(1+N_c)^{15/4}\chi^{1/2}u^3A^2}{r^2R_0^2\rho\omega_0^5\sqrt{\omega_0}} = \left[-6\sqrt{2}\pi(\gamma-1)(1+N_c)^{15/4}\right]\left(\frac{A}{p_{eff}}\right)^2\left(\frac{\rho u^2}{p_{eff}}\right)^{3/4}\frac{R_0^{7/2}\rho\chi^{1/2}u^{3/2}}{r^2}, \omega >> \chi/\left(2R_0^2\right) \tag{28}$$

At resonance, the variable $X(\omega_N) << 1$ ($X(\omega_N) = R_0\left(2\omega_N/\chi\right)^{1/2} = R_0\left(2\gamma\omega_0/\left(D_gN_c^{1/2}\right)\right)^{1/2}$ $= \left(6\gamma R_0 p_{eff}^{1/2}/\left(\rho^{1/2}l_g\upsilon_gN_c^{1/2}\right)\right)^{1/2} << 1$) for any value of the bubble radius R_0, because $l_g >> R_0$, $N_c >> 1$

and $p_{eff}/\left(\rho\upsilon_g{}^2\right)<1$. It follows that, at resonance, only expression (27) of the force is valid. With the virtual mass of the bubble $m_b=2\pi R_0^3\rho/3$, the expression (27) becomes,

$$F_{BNsa\mu_{th}r}\left(r\right)=\left(\frac{-m_b^2}{r^2}\right)\left[\frac{3\left(\gamma-1\right)\left(1+N_c\right)^3u^3A^2}{5\pi\gamma\chi\rho R_0 p_{eff}^2}\right]. \tag{29}$$

If we substitute in this expression the equivalent virtual mass of the bubble in the cluster, m_N, [8], it results

$$F_{BNsathr}\left(r\right)=\left(\frac{-m_N^2}{r^2}\right)\left[\frac{3\left(\gamma-1\right)\left(1+N_c\right)u^3A^2}{5\pi\gamma\chi\rho R_0 p_{eff}^2}\right]. \tag{30}$$

Both force expressions, at resonance, are proportional to the square of the virtual mass analogous to the gravitational force in the electromagnetic world.

In the expression (30) of the gravito-acoustic force we identify the gravitational acoustic constant

$$G_{aNthr}=\left(1+N_c\right)\frac{3\left(\gamma-1\right)u^3A^2}{5\pi\gamma\chi\rho R_0 p_{eff}^2}=\left(1+N_c\right)G_{0ath}, \tag{31}$$

where the factor G_{0ath} is the acoustic gravitational constant corresponding to the gravito-acoustic interaction of two free bubbles, according to relation (20a) from the paper [5], with $\beta_{a0}=\beta_{th}(\omega_0)$. With this constant, the gravito-acoustic force in the cluster, at resonance, becomes

$$F_{BNsathr}\left(r\right)=\frac{-G_{aNthr}m_N^2}{r^2}. \tag{32}$$

According to the paper Acoustic gravitational interaction revised [5], we can introduce the physical quantity the acoustic length of the resonant thermal damping R_{Nth0} starting from the ratio between the gravito-acoustic force, Eq. (13), and the electro-acoustic force in the cluster, Eq. (27), from the paper Mach's Principle in the Acoustic World [8]

$$\frac{F_{BNsa}\left(r\right)}{F_{BNss}\left(r\right)}=\frac{\dfrac{-8\pi R_0^3\omega^6A^2\beta_a}{r^2\rho u\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]^2}}{\dfrac{-2\pi R_0^2\omega^2A^2}{r^2\rho\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}}=\frac{4R_0\omega^4\beta_a}{\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]u}. \tag{33}$$

At resonance, $\omega=\omega_N$, Eq. (33) becomes

$$\begin{aligned}\lim_{\omega\to\omega_N}\left(\frac{F_{BNsa}\left(r\right)}{F_{BNss}\left(r\right)}\right)&=\frac{4R_0\omega_N^4u\beta_a(\omega_N)}{R_0^2\omega_N^6}=\frac{4u\beta_a(\omega_N)}{R_0\omega_N^2}=\frac{4\left(1+N_c\right)u\beta_a(\omega_N)}{R_0\omega_0^2}=\\&\frac{4\left(1+N_c\right)R_0\rho u\beta_a(\omega_N)}{p_{eff}}=\frac{4\left(1+N_c\right)\rho u^2}{p_{eff}}\frac{R_0\beta_a(\omega_N)}{u}.\end{aligned} \tag{34}$$

If the ratio limit is R_0/R_{Na0}, we can define the acoustic length of the resonant absorption damping R_{Na0}

$$R_{Na0}=\left(\frac{u}{\beta_a(\omega_N)}\right)\left(\frac{p_{eff}}{4(1+N_c)\rho u^2}\right) \tag{35}$$

For thermal absorption, we can define the acoustic length of the resonant thermal damping R_{Nth0}, with $\beta_a(\omega_N)=\beta_{Nth0}$

$$R_{Nth0}=\left(\frac{u}{\beta_{Nth0}}\right)\left(\frac{p_{eff}}{4(1+N_c)\rho u^2}\right) \tag{36}$$

According to Eq. (22), the thermal damping coefficient does not depend on the pulsation and therefore $\beta_{Nth0}=\beta_{th}$. Substituting into Eq. (36), it follows

$$\begin{gathered}R_{Nth0}=\left(\frac{u}{\beta_{th}}\right)\left(\frac{p_{eff}}{4(1+N_c)\rho u^2}\right)=\left(\frac{15\gamma\chi u}{(\gamma-1)R_0^2\omega_0^2}\right)\left(\frac{p_{eff}}{2(1+N_c)\rho u^2}\right)= \\ \frac{15}{2(1+N_c)(\gamma-1)}\left(\frac{\gamma\chi}{u}\right)=\frac{5\upsilon_g l_g}{2(1+N_c)(\gamma-1)u}.\end{gathered} \tag{37}$$

From Eqs. (31) and (37), results an expression of the acoustic gravitational constant, in the cluster, as a function of R_{Nth0} (the acoustic length of the resonant thermal damping)

$$G_{aNthr}=\frac{2(1+N_c)(\gamma-1)u}{15\gamma\chi}\frac{9u^2A^2}{2\pi\rho R_0 p_{eff}^2}=\left(\frac{R_0^2u^2}{R_{Nth0}m_b}\right)\left(\frac{3A^2}{p_{eff}^2}\right)=\left(\frac{R_0^2u^2}{R_{Nth0}m_N}\right)\left(\frac{3(1+N_c)A^2}{p_{eff}^2}\right). \tag{38}$$

This relation is analogous to Eq. (25) from paper [5].

4. Electro - acoustic and gravito - acoustic forces outside the cluster

To infer the expression of the electro - acoustic forces between a bubble in the cluster and a bubble outside the cluster (for identical bubbles, $R_{01}=R_{02}=R_0$), we use the expression of the Bjerknes secondary force [2, 4]

$$F_{B12}(r,\varphi)=-\frac{2\pi\rho\omega^2R_{01}^3R_{02}^3}{r^2}a_1a_2\cos(\varphi_2-\varphi_1), \tag{39}$$

and $\cos(\varphi_2-\varphi_1)$ given by Eq. (3) from paper [5]

$$\cos(\varphi_2-\varphi_1)=\frac{(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)+4\beta_1\beta_2\omega^2}{\sqrt{(\omega^2-\omega_1^2)^2+4\beta_{s1}^2\omega^2}\sqrt{(\omega^2-\omega_2^2)^2+4\beta_{s2}^2\omega^2}}. \tag{40}$$

In these expressions, according to the results in paper [7], subscript 1 refers to the parameters of the outer bubble and subscript 2 to the parameters of the cluster bubble. Taking into account the results of papers [5, 8], the dimensionless amplitude a_1 has the expression

$$a_1 = \frac{A}{\rho R_0^2\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]^{1/2}}, \tag{41}$$

and the dimensionless amplitude a_2 has the expression

$$a_2 \to a_N = \frac{A}{\rho R_0^2\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]^{1/2}}, \tag{42}$$

Eq. (40) gets

$$\cos\left(\varphi_2-\varphi_1\right)=\frac{\left(\omega^2-\omega_0^2\right)\left(\omega^2-\omega_N^2\right)+4\beta\beta_N\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2}\sqrt{\left(\omega^2-\omega_N^2\right)^2+4\beta_{sN}^2\omega^2}}. \tag{43}$$

Substituting into Eq. (43) the expressions $\omega_N^2=\omega_0^2/\left(1+N_c\right)$ and $\beta_N=\beta/\left(1+N_c\right)$, according to Eq. (16) from paper [8], it follows

$$\cos\left(\varphi_2-\varphi_1\right)=\frac{\left(\omega^2-\omega_0^2\right)\left(\omega^2-\omega_N^2\right)+4\beta^2\omega^2}{\sqrt{\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2}\sqrt{\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2}}. \tag{44}$$

Substituting into Eq. (39), Eqs. (41), (42) and (44), with $\beta=\beta_s+\beta_a$, results

$$\begin{aligned}F_{BNe}\left(r\right)&=\frac{-2\pi\omega^2R_0^2A^2\left[\left(\omega^2-\omega_0^2\right)\left(\omega^2-\omega_N^2\right)+4\left(\beta_s+\beta_a\right)^2\omega^2\right]}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}=\\&\frac{-2\pi\omega^2R_0^2A^2\left[\left(\omega^2-\omega_0^2\right)\left(\omega^2-\omega_N^2\right)+4\beta_s^2\omega^2\right]}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}+\\&\frac{-16\pi\omega^4R_0^2A^2\beta_s\beta_a}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}+\\&\frac{-8\pi\omega^4R_0^2A^2\beta_a^2}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}.\end{aligned} \tag{45}$$

In Eq. (45), the first relation is the expression of the electro-acoustic force

$$F_{BNess}\left(r\right)=\frac{-2\pi\omega^2R_0^2A^2\left[\left(\omega^2-\omega_0^2\right)\left(\omega^2-\omega_N^2\right)+4\beta_s^2\omega^2\right]}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]} \tag{46}$$

and the second relation is the expression of the gravito-acoustic force, between a bubble in the cluster and an identical bubble outside the cluster,

$$F_{BNesa}\left(r\right)=\frac{-16\pi\omega^4R_0^2A^2\beta_s\beta_a}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}. \tag{47}$$

Substituting into Eq. (47), Eq. (16) we obtain the expressions of the two types of gravito-acoustic forces:

$$F_{BNesa}(r)=\frac{-16\pi\omega^4R_0^2A^2\beta_s\left(\beta_\upsilon+\beta_{th}\right)}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}=$$
$$\frac{-32\pi\omega^4A^2\mu\beta_s}{r^2\rho^2\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}+ \tag{48}$$
$$\frac{-16\pi\omega^4R_0^2A^2\beta_s\beta_{th}}{r^2\rho\left[\left(\omega^2-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]\left[\left(\omega^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]}=$$
$$F_{BNesa\mu}(r)+F_{BNesath}(r),$$

with $F_{BNesa\mu}(r)$ the gravito - acoustic force generated by energy absorption in the liquid and $F_{BNesath}(r)$ the gravito - acoustic force generated by energy absorption in the gas/vapor of the bubble.

If the pulsation of the exciting wave is the eigenpulsation/natural pulsation of the bubble outside the cluster, $\omega=\omega_0$, the expressions of the three forces, with $\beta_s=\omega^2R_0/(2u)$, become:

$$F_{BNess0}(r)=\frac{-2\pi R_0^2\omega_0^2A^2}{r^2\rho\left[\left(\omega_0^2\left(1+N_c\right)-\omega_0^2\right)^2+4\beta_{s0}^2\omega_0^2\right]}=$$
$$\frac{-2\pi R_0^2A^2}{r^2\rho\omega_0^2\left(N_c^2+R_0^2\omega_0^2/u^2\right)}=\frac{-2\pi R_0^4A^2}{r^2p_{eff}\left(N_c^2+p_{eff}/\left(\rho u^2\right)\right)}\cong \tag{49}$$
$$\frac{-2\pi R_0^4A^2}{r^2p_{eff}N_c^2};$$

$$F_{BNesa\mu0}(r)=\frac{-16\pi A^2u\mu}{r^2\rho^2R_0\omega_0^4\left(N_c^2+R_0^2\omega_0^2/u^2\right)}=\frac{-16\pi R_0^3A^2u\mu}{r^2p_{eff}^2\left(N_c^2+p_{eff}^2/\left(\rho u^2\right)\right)}\cong \tag{50}$$
$$\frac{-16\pi R_0^3A^2u\mu}{r^2p_{eff}^2N_c^2},$$

$$F_{BNesath0}(r)=\frac{-8\pi R_0^3A^2u\beta_{th}(\omega_0)}{r^2\rho\left(N_c^2+R_0^2\omega_0^2/u^2\right)}\cong\frac{-8\pi R_0^3A^2u\beta_{th}(\omega_0)}{r^2\rho N_c^2},\ N_c^2>>p_{eff}^2/\left(\rho u^2\right). \tag{51}$$

If the pulsation of the exciting wave is the eigenfrequency of the bubble inside the cluster, $\omega=\omega_N$, the three forces become:

$$F_{BNessN}(r)=\frac{-2\pi\omega_N^2R_0^2A^2}{r^2\rho\left[\left(\omega_N^2-\omega_0^2\right)^2+R_0^2\omega_N^6/u^2\right]}=\frac{-2\pi R_0^2A^2}{r^2\rho\omega_N^2\left[\left(1-\omega_0^2/\omega_N^2\right)^2+R_0^2\omega_N^2/u^2\right]}=$$
$$\frac{-2\pi\left(1+N_c\right)R_0^2A^2}{r^2\rho\omega_0^2\left[N_c^2+R_0^2\omega_0^2/\left(u^2N_c\right)\right]}\cong\frac{-2\pi R_0^2A^2}{r^2\rho\omega_0^2N_c}= \tag{52}$$
$$\frac{-2\pi R_0^4A^2}{r^2p_{eff}N_c},\quad N_c^2>>R_0^2\omega_0^2/\left(u^2N_c\right),\ N_c>>1;$$

$$F_{BNesa\mu N}\left(r\right)=\frac{-16\pi A^2u\mu}{r^2\rho^2R_0\left[\left(\omega_N^2-\omega_0^2\right)^2+4\beta_{sN}^2\omega_N^2\right]}=\frac{-16\left(1+N_c\right)^2\pi A^2u\mu}{r^2\rho^2R_0\omega_0^4\left[N_c^2+R_0^2\omega_0^2/\left[u^2\left(1+N_c\right)\right]\right]}\cong$$
$$\frac{-16\pi R_0^3A^2u\mu}{r^2p_{eff}^2},\ N_c^2>>R_0^2\omega_0^2/\left(u^2N_c\right); \tag{53}$$

$$F_{BNesathN}\left(r\right)=\frac{-4\pi\omega_N^2R_0^2A^2\beta_{th}}{r^2\rho\left[\left(\omega_N^2-\omega_0^2\right)^2+4\beta_{sN}^2\omega_N^2\right]\beta_s}=\frac{-4\left(1+N_c\right)^2\left(\gamma-1\right)\pi\omega_N^2R_0^3A^2}{15\gamma r^2\chi\rho\omega_0^2\left[N_c^2+R_0^2\omega_0^2/\left(u^2\left(1+N_c\right)\right)\right]\beta_s}\cong$$
$$\frac{-4\left(\gamma-1\right)\pi R_0^5A^2u}{15r^2\gamma\chi p_{eff}},\ N_c^2>>R_0^2\omega_0^2/\left(u^2N_c\right). \tag{54}$$

According to relations (53) and (54), the expressions of the gravito - acoustic forces between a cluster bubble and an outer bubble no longer depend on the acoustic Mach number.

The forces between the cluster and the outer bubble are obtained, according to Eq. (7) from paper [3], amplifying with the number of bubbles, N, the force expressions given by Eqs. (46 - 54).

5. Cluster compression, thermal effects

5.1. Average pressure determined by bubble oscillations

To find the average pressure in the cluster determined by the bubble oscillations, we will use the expression for the pressure around a bubble. The pressure around the bubble, due to bubble volume oscillations [1, 2, 8], for the bubbles in the cluster, is

$$p_{bN}(r,t)=\frac{\rho\ddot{V}_N}{4\pi r}-\frac{\rho\dot{R}_N^2}{2}\left(\frac{R_N}{r}\right)^4=\frac{\rho R_N^2\ddot{R}_N}{r}+\frac{2\rho R_N\dot{R}_N^2}{r}-\frac{\rho\dot{R}_N^2}{2}\left(\frac{R_N}{r}\right)^4. \tag{55}$$

The pressure generated by the N bubbles in the cluster is

$$p_{is}=\rho\sum_{j\neq i}\frac{R_{Nj}^2\ddot{R}_{Nj}}{r_{ij}}+2\rho\sum_{j\neq i}\frac{R_{Nj}\dot{R}_{Nj}^2}{r_{ij}}-\frac{\rho}{2}\sum_{j\neq i}\frac{\dot{R}_{Nj}^2R_{Nj}^4}{r_{ij}^4}. \tag{56}$$

Substituting $R(t)$ given by Eq. (5), $\dot{R}$ and $\ddot{R}$, the pressure expression (56) becomes:

$$p_{is}=\rho\sum_{j\neq i}\frac{R_0^3\left(1+x_N\right)^2\ddot{x}_N}{r_{ij}}+2\rho\sum_{j\neq i}\frac{R_0^3\left(1+x_N\right)\dot{x}_N^2}{r_{ij}}-\frac{\rho}{2}\sum_{j\neq i}\frac{R_0^6\left(1+x_N\right)^4\dot{x}_N^2}{r_{ij}^4}=$$
$$\rho R_0^2\left(1+x_N\right)^2\ddot{x}_N\sum_{j\neq i}\frac{R_0}{r_{ij}}+2\rho R_0^2\left(1+x_N\right)\dot{x}_N^2\sum_{j\neq i}\frac{R_0}{r_{ij}}-\frac{\rho R_0^2\left(1+x_N\right)^4\dot{x}_N^2}{2}\sum_{j\neq i}\frac{R_0^4}{r_{ij}^4}. \tag{57}$$

The average pressure, $\langle p_{is}\rangle$, generated by the coupling of bubble oscillations, is

$$\begin{aligned}\langle p_{is}\rangle &= \rho R_0^2\left\langle(1+x_N)^2\ddot{x}_N\right\rangle\sum_{j\neq i}\frac{R_0}{r_{ij}}+2\rho R_0^2\left\langle(1+x_N)\dot{x}_N^2\right\rangle\sum_{j\neq i}\frac{R_0}{r_{ij}}-\frac{\rho R_0^2\left\langle(1+x_N)^4\dot{x}_N^2\right\rangle}{2}\sum_{j\neq i}\frac{R_0^4}{r_{ij}^4}= \\ &\rho R_0^2\sum_{j\neq i}\frac{R_0}{r_{ij}}\left[\langle\ddot{x}_N\rangle+2\langle x_N\ddot{x}_N\rangle+\langle x_N^2\ddot{x}_N\rangle+2\langle\dot{x}_N^2\rangle+2\langle x_N\dot{x}_N^2\rangle\right]- \\ &\frac{\rho R_0^2}{2}\sum_{j\neq i}\frac{R_0^4}{r_{ij}^4}\left[\langle\dot{x}_N^2\rangle+4\langle x_N\dot{x}_N^2\rangle+6\langle x_N^2\dot{x}_N^2\rangle+4\langle x_N^3\dot{x}_N^2\rangle+\langle x_N^4\dot{x}_N^2\rangle\right].\end{aligned} \tag{58}$$

With the results obtained in estimating the averaged pressure expression for a free bubble, Eq. (6), the pressure expression (58) becomes

$$\langle p_{is}\rangle=\frac{-\rho R_0^2a_N^2\omega^2}{4}\left(1+\frac{3a_N^2}{2}+\frac{a_N^4}{8}\right)\cong\frac{-\rho R_0^2a_N^2\omega^2}{4}\sum_{j\neq i}\frac{R_0^4}{r_{ij}^4}. \tag{59}$$

For a uniform distribution of bubbles, the sum $\sum_{j\neq i}^{N}\left(R_0^4/r_{ij}^4\right)$ can be approached through integral calculation, according to the procedure used in the paper [8],

$$\sum_{j\neq i}\frac{R_0^4}{r_{ij}^4}\to R_0^4\int_{R_0}^{R_c}\frac{n_c dV}{r^4}=R_0^4\int_{R_0}^{R_c}\frac{4\pi n_c dr}{r^2}\cong\frac{4\pi R_0^4 n_c}{R_0}=\frac{3NR_0^3}{R_c^3}<<1. \tag{60}$$

The lower limit of the integral is the radius R_0 of the bubble because the waves scattered by the bubble depart from the surface of the bubble.

With this relation, the pressure expression (59) becomes

$$\langle p_{is}\rangle\cong\frac{-3N\rho R_0^5a_N^2\omega^2}{4R_c^3}=\frac{-3NR_0A^2\omega^2}{4\rho R_c^3\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+R_0^2\omega^6/u^2\right]}. \tag{61}$$

At resonance, $\omega=\omega_N$, the pressure (61) becomes

$$\langle p_{isr}\rangle=\frac{-3NA^2u^2}{4\rho R_0R_c^3\omega_N^4}=\frac{-3NN_c^2R_0^3A^2\rho u^2}{4R_c^3p_{eff}^2}=\frac{-27N^3R_0^6A^2\rho u^2}{16R_c^6p_{eff}^2}. \tag{62}$$

The interpretation of this result is obtained if we take into account that the average total pressure outside the bubbles in the cluster is the sum of the hydrostatic pressure p_0 and the average pressure determined by the oscillations of the bubbles

$$p_{tc}=p_0+\langle p_{is}\rangle=p_0-\frac{(3N)^{4/3}R_0^2A^2\omega^2}{4(4\pi)^{1/3}\rho R_c^4\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+R_0^2\omega^6/u^2\right]}<p_0, \tag{63}$$

that is, the average total pressure in the cluster is less than the hydrostatic pressure p_0 .

The total energy corresponding to this time-averaged scattered radiation pressure is

$$\langle E_{is}\rangle=\langle w_{is}\rangle V_c=-\langle p_{is}\rangle\frac{4\pi R_c^3}{3}=N\left(\pi\rho R_0^5a_N^2\omega^2\right)=N\left\{\frac{\pi R_0A^2\omega^2}{\rho\left[\left(\omega^2(1+N_c)-\omega_0^2\right)^2+R_0^2\omega^6/u^2\right]}\right\}, \tag{64}$$

with $w_{is} = -p_{is}$, if we consider that stress–energy tensors of the field and electro-acoustic radiation are analogous to those of the field and electromagnetic radiation [21 - Sch.31.8, 22 - Ch.8.]. Also, we will further prove that the energy corresponding to the averaged pressure for a bubble in the cluster is precisely the term in the brace. By averaging relation (55) over time, it follows

$$\left\langle p_{bN}(r,t)\right\rangle = \frac{\rho\left\langle R^2\ddot{R}_N\right\rangle}{r} + \frac{2\rho\left\langle R_N\dot{R}_N^2\right\rangle}{r} - \frac{\rho\left\langle \dot{R}_N^2 R_N^4\right\rangle}{2r^4} = -\frac{\rho R_0^6 a_N^2\omega^2}{4r^4}. \tag{65}$$

The energy density corresponding to this pressure is $\left\langle w_{bin}\right\rangle = -\left\langle p_{bN}\right\rangle = \rho R_0^6 a_N^2\omega^2 \big/ \left(4r^4\right)$. If we put this relation in the form $\left\langle w_{bin}\right\rangle = \left[R_0^4\left(\rho R_0^2 a_N^2\omega^2/4\right)\right]\big/ r^4 = R_0^4\left(w_{0kN}\right)\big/ r^4$ in which we highlighted an average density of kinetic energy of oscillation, $w_{0kN} = \rho R_0^2 a_N^2\omega^2/4 = \rho\left\langle\left(\dot{R}_N\right)^2\right\rangle\big/ 2$, the average energy of oscillation of the fluid around the bubble is

$$E_{bin} = \int_{R_0}^{R_c}\left\langle w_{bin}\right\rangle 4\pi r^2 dr = \pi\rho R_0^6 a_N^2\omega^2\int_{R_0}^{R_c}\frac{dr}{r^2} = \pi\rho R_0^6 a_N^2\omega^2\left(\frac{1}{R_0} - \frac{1}{R_c}\right) \cong \pi\rho R_0^5 a_N^2\omega^2 = E_{bkin}. \tag{66}$$

The average oscillation energy density is equal to the energy density of the electro-acoustic field [21 – Ch. 28,], written in the CGS-Gaussian system of units, of a cluster bubble

$$\left\langle w_{bin}\right\rangle = w_{eain} = \frac{E_{eain}^2}{8\pi} = \frac{e_{eain}^2}{8\pi r^4} = \frac{\left|F_{BN}\right|}{8\pi r^2} = \frac{\rho R_0^6 a_N^2\omega^2}{4r^4}, \tag{67}$$

with e_{eain}^2 the square of the acoustic load for the bubbles in the cluster, according to relation (11).

The total energy of the electro-acoustic field around a bubble, taking into account the energy density expression (67), is

$$E_{eain} = \int_{R_0}^{R_c} w_{eain} 4\pi r^2 dr \cong \pi\rho R_0^5 a_N^2\omega^2 = E_{bin} = E_{bkin}. \tag{68}$$

The relation (68) highlights the significance of the energy of the electro - acoustic field as the average oscillation energy of the liquid around the oscillating bubble. This energy is interpreted by an outside observer as the rest energy of the oscillating bubble system, analogous to how the electrostatic field energy of a charged particle without spin is interpreted as the rest energy of the particle.

The field energy of the N bubbles in the cluster is $E_{bNt} = NE_{bin} = N\left(\pi\rho R_0^5 a_N^2\omega^2\right)$, i.e. precisely the expression of the averaged total energy of the radiation, $\left\langle E_{is}\right\rangle$, Eq. (64). It follows, as expected, according to the oscillating bubble interaction model, that bubble-scattered radiation is involved (has an effect) in the coupled oscillators interaction phenomenon [23, 26].

5.2. Acoustic temperature

To calculate the temperature in the cluster (the acoustic temperature, T_a, different from the substrate/liquide temperature, T, i.e. the temperature of the liquid which is close to the boiling temperature) we use the virial theorem to estimate the translational kinetic energy of a bubble in the cluster. According to the virial theorem [27, Sch.10], if the bubbles, in translational motion, do not radiate, the translational kinetic energy of a bubble in the cluster is equal to half the module of the potential energy of the bubble in acoustic interaction with the other bubbles in the cluster

$$E_{bk} = \frac{\left|E_{bp}\right|}{2} = \frac{\left|E_{bpN}\right|}{2N}. \tag{69}$$

To calculate the potential energy of the electro-acoustic interaction of the bubbles in the cluster with radius R_c containing N uniformly distributed bubbles ($N = \left(4\pi R_c^3/3\right)n_c$), we use the gravitational potential energy expression (Eq. 60.11) from the paper [28, p. 87]) written for the electro - acoustic interaction

$$E_{bpN} = \int_0^{R_c} \frac{\left(-N\left(r\right)e_{eain}\right)d\left(e_{eain}N(r)\right)}{r} = \frac{-\left(4\pi\right)^2 n_c^2 e_{eain}^2}{3}\int_0^{R_c} r^4\, dr = \frac{-3e_{eain}^2}{5R_c}\left(\frac{4\pi R_c^3 n_c}{3}\right)^2 = \frac{-3N^2 e_{eain}^2}{5R_c}. \tag{70}$$

Substituting the electro - acoustic charges, Eq. (11), in the expression of the total potential energy (70), it follows:

$$\begin{gathered} E_{bpN} = \frac{-3N^2 e_{eain}^2}{5R_c} = \frac{-6\pi N^2 A^2 R_0^2 \omega^2}{5\rho R_c\left[\left(\omega^2 N_c - \omega_0^2\right)^2 + 4\beta_s^2\omega^2\right]}, \\ E_{bpNr} = \frac{-3N^2 e_{eainr}^2}{5R_c} = \frac{-6\pi N^2 N_c^2 R_0^4 \rho u^2 A^2}{5R_c p_{eff}^2}. \end{gathered} \tag{71}$$

The average translational kinetic energy of a bubble in the cluster is

$$E_{bk} = \frac{3kT_a}{2}. \tag{72}$$

Substituting Eqs. (71) and (72) in Eq. (69), with $N_c = 3NR_0/\left(2R_c\right)$ [8], results

$$\begin{gathered} T_a = \frac{Ne_{eain}^2}{5kR_c} = \frac{2N_c e_{eain}^2}{15kR_0} = \frac{4\pi N_c \omega^2 A^2 R_0}{15k\rho\left[\left(\omega^2 N_c - \omega_0^2\right)^2 + 4\beta_s^2\omega^2\right]}, \\ T_{ar} = \frac{2N_c e_{eainr}^2}{15kR_0} = \frac{4\pi N_c^3 A^2 \rho u^2 R_0^3}{15kp_{eff}^2} = \left(\frac{m_b u^2}{k}\right)\left(\frac{N_c^3 A^2}{5p_{eff}^2}\right), m_b = \frac{2\pi R_0^3 \rho}{3} \end{gathered} \tag{73}$$

or

$$T_a = \frac{Ne_{eain}^2}{5kR_c} = \frac{2\pi N\omega^2 A^2 R_0^2}{5kR_c\rho\left[\left(3NR_0\omega^2/\left(2R_c\right)-\omega_0^2\right)^2+4\beta_s^2\omega^2\right]},$$
$$T_{ar} = \left(\frac{m_b u^2}{k}\right)\left(\frac{27N^3R_0^3A^2}{40R_c^3 p_{eff}^2}\right). \tag{74}$$

If the energy emitted by acoustic radiation is also considered, part of the potential energy variation is transformed into a thermal background of acoustic radiation. At thermal equilibrium, the temperature of this thermal acoustic radiation is equal to the temperature of the "gas" of bubbles in the cluster. In this case, each bubble in the cluster oscillates under the action of a thermal background having a Planck distribution and not a monochromatic background of mean pressure, $\langle p_{is}\rangle$, generated by the coupling of bubble oscillations. Thermalization of acoustic radiation is done by the Doppler effect [29, Sch. 67] generated by the translational motion of the bubbles. We will study the phenomenon of thermalization of acoustic radiation in the cluster in another paper.

5.3. Dumb Hole corresponding to a collapsed cluster

In the paper Acoustic Black Hole Generated by a Cluster of Oscillating Bubbles [20], we studied the conditions under which the liquid around a cluster becomes an acoustic lens. Also in this paper we hypothesized that a collapsing cluster can become a dumb hole.

In the case of a cluster of identical bubbles, the collapse is done under the action of electro - acoustic forces. Because by collapsing the "gas" of bubbles heats up, generating a thermal pressure, the collapse takes place until a balance is reached between the thermal pressure and the pressure of the electro - acoustic forces, $p(T_a) = p_{ea}$.

We can calculate the pressure caused by the electro - acoustic forces, in the case of the uniform distribution of the bubbles, $n_c = \left(3N/4\pi R_c^3\right)$, starting from the pressure determined by the electro - acoustic force between the bubbles in the sphere of radius r and the bubbles in the spherical shell of radius r and thickness dr

$$dp = \frac{\left(4\pi r^3 n_c e_{eain}\right)\left(4\pi r^2 n_c e_{eain} dr\right)}{3r^2}\frac{1}{4\pi r^2} = \left(\frac{4\pi n_c^2 e_{eain}^2}{3}\right) r dr. \tag{75}$$

Integrating Eq. (75) and substituting the bubble density yields

$$p_{eac} = \frac{4\pi n_c^2 e_{eain}^2}{3}\int_0^{R_c} r dr = \frac{4\pi n_c^2 e_{eain}^2 R_c^2}{6} = \frac{3N^2 e_{eain}^2}{8\pi R_c^4}. \tag{76}$$

Also substituting the expression for the modulus of the total potential energy (70), it results

$$p_{eac} = \frac{3N^2 e_{eain}^2}{8\pi R_c^4} = \frac{5\left|E_{pbN}\right|}{8\pi R_c^3}. \tag{77}$$

The thermal pressure, $p(T_a)$, is determined by the translational energy and is equal to one third of the kinetic energy density

$$p(T_a) = \frac{w_k}{3} = \frac{n_{bc}E_{bk}}{3} = \frac{NE_{bk}}{4\pi R_c^3} = \frac{N\left|E_{bp}\right|}{8\pi R_c^3} = \frac{\left|E_{bpN}\right|}{8\pi R_c^3}. \tag{78}$$

Substituting the expressions of the potential energies (71) into the expression (78), it results:

$$\begin{gathered} p(T_a) = \frac{\left|E_{bpN}\right|}{8\pi R_c^3} = \frac{3N^2 R_0^2 A^2 \omega^2}{20\rho R_c^4\left[\left(\omega^2 N_c - \omega_0^2\right)^2 + 4\beta_s^2\omega^2\right]}, \\ p_r(T_a) = \frac{\left|E_{bpNr}\right|}{8\pi R_c^3} = \frac{3N^2 N_c^2 R_0^4 A^2 \rho u^2}{20 R_c^4 p_{eff}^2} = \frac{27N^4 R_0^6 A^2 \rho u^2}{80 R_c^6 p_{eff}^2}. \end{gathered} \tag{79}$$

Comparing these pressures, it turns out that the thermal pressure of the translational moving bubbles is less than the pressure of the electro - acoustic forces by the factor of 5 that appears from how we calculated the potential energy.

Comparing the bubble thermal pressure relations, Eq. (9), with the cluster averaged internal radiation pressure expressions, Eqs. (61) and (62), it follows, except for a factor of $1/5$, that the thermal pressure (76) is N times higher than this. The factor $1/5$ arises from the different ways in which we calculated the averages of the internal radiation pressure and potential energy, and for $N >> 1$, it is insignificant for the difference between the two quantities. The mean internal radiation pressure is the result of the oscillating/pulsating motion of the bubbles. The translational movement of the bubbles corresponds to a translational thermal radiation with a much higher intensity. The energy density of this translational radiation is of the order of the translational kinetic energy density (bubbles and translational radiation are at equilibrium) and so, correspondingly, the translational radiation pressure is of the order of the translational kinetic energy density. At the level of our knowledge of the phenomena in the cluster, at this moment, we can state that it is possible for a cluster to collapse and become a dumb hole, that is, the radius of the collapsed cluster to be smaller than the acoustic radius.

Another problem that arises when studying thermal equilibrium is that of relativistic effects determined by the fact that systems and phenomena in the acoustic world interact and correlate through acoustic waves. If the translational speed of the bubbles approaches the speed of sound in the liquid, relativistic effects occur, and energy and momentum are calculated in relativistic case.

The same relativistic limitation appears for the bubble oscillation/pulsating energy when the excitation wave amplitude is large. We will address these issues in a future paper.

6. Conclusions, discussions

Through the results of this paper, namely the deduction of: the expression of the electro-acoustic force between the bubbles in the cluster and an outer bubble, the gravito-acoustic forces in the cluster and the cluster with an outer bubble, the acoustic temperature corresponding to the translational motion of the bubbles in the cluster and the of the average acoustic radiation pressure, we have demonstrated a better similarity between the acoustic world and the electromagnetic world. The most important result is the demonstration that the modulus of the average acoustic radiation pressure around a bubble is equal to the energy density of the electrostatic field, Eq. (10). These densities are the oscillation energy densities of the liquid around the bubble that are involved in the interaction phenomenon between two oscillating bubbles. This property supports the hypothesis that the electro - acoustic charge property and the corresponding interaction measure the ability of bubbles to oscillate radially and to absorb and scatter acoustic waves [23, 26]. By analogy, electrostatic charge measures the ability of charged particles to absorb and scatter electromagnetic radiation and is therefore characterized by an internal oscillatory motion [38]. It follows that elementary particles must be nonpoints. The model of a non-point particle, as an oscillating/pulsating bubble in the electromagnetic vacuum, must be completed with an internal orbital motion (bubble captured by a vortex) for the particle to have a nonzero spin. We will address this model, for bubbles in liquid, in a future paper.

REFERENCES

[1] Prosperetti A., *Bubble phenomena in sound fields: part two*, Ultrasonics 22, 115-123, 1984.

[2] Barbat T., Ashgriz N. and Liu C.S., *Dynamics of two interacting bubbles in an acoustic field*, J. Fluid Mech. 389, 137-168, 1999.

[3] Wang X., Chen W., Zhou M., Zhang Z. and Zhang L., *Dynamics of double bubbles under the driving of burst ultrasound*, Ultrasonics Sonochemistry, Vol. 84, 10595, 2022.

[4] Simaciu I., Borsos Z., Dumitrescu Gh., Silva G.T. and Bărbat T., *Acoustic scattering-extinction cross section and the acoustic force of electrostatic type*, arXiv:1711.03567, 2017.

[5] Simaciu I., Borsos Z. and Dumitrescu Gh., *Acoustic gravitational interaction revised*, arXiv:2206.00435v1, 2022.

[6] An Y., *Formulation of multibubble cavitation*, Physical Review E 83, 066313, 2011.

[7] Wang C.H. and Cheng J.C., *Interaction of a bubble and a bubble cluster in an ultrasonic field*, Chin. Phys. B, Vol. 22, No. 1, 014304, 2013

[8] Simaciu I., Dumitrescu Gh. and Borsos Z., *Mach's Principle in the Acoustic World*, arXiv: 1907.05713, 2019; Buletinul Institutului Politehnic din Iaşi, Secţia Matematică. Mecanică Teoretică. Fizică, Vol. 67 (71), No. 4, 59-69, 2021.

[9] Larraza A. and Denardo B., *An acoustic Casimir effect*, Phys. Lett. A, 248, 151, 1998.

[10] Bárcenas J., Reyes L. and Esquivel-Sirventa R., *Acoustic Casimir pressure for arbitrary media*, J. Acoust. Soc. Am. 116 (2), 2004.

[11] Larraza A., *A demonstration apparatus for an acoustic analog to the Casimir effect*, Am. J. Phys. 67, 1028, 1999.

[12] Jaskula J.C., Partridge G.B., Bonneau M., Lopes R., Ruaudel J., Boiron D. and Westbrook C.I., *Acoustic Analog to the Dynamical Casimir Effect in a Bose-Einstein Condensate*, Phys. Rev. Lett., 109, 220401, 2012.

[13] Cui B., Ni B. and Wu Q., *Bubble–bubble interaction effects on dynamics of multiple bubbles in a vortical flow field*, Advances in Mechanical Engineering 8 (2), 2016.

[14] Narsing K. Jha and Govardhan R.N., *Interaction of a vortex ring with a single bubble: Bubble and vorticity dynamics*, Journal of Fluid Mechanics, Vol. 773, pp. 460-497, 2015.

[15] Wang Z., Uchiyama T. and Chena B., *Numerical simulation of the interaction between vortex ring and bubble plume*, Applied Mathematical Modelling, Vol. 37, Issue 24, pp.10007-10026, 2013.

[16] Zia R., Nazir A., Poortinga A.T. and van Nostrum C.F., *Advances in antibubble formation and potential applications*, Advances in Colloid and Interface Science, Volume 305, 102688, 2022.

[17] Dorbolo S., Caps H. and Vandewalle N., *Fluid instabilities in the birth and death of antibubbles*, New Journal of Physics, 5, 161, 2003.

[18] Simaciu I., Dumitrescu Gh., Borsos Z. and Brădac M., *Interactions in an Acoustic World: Dumb Hole*, Advances in High Energy Physics, Article ID 7265362, Volume 2018.

[19] Simaciu I., Dumitrescu Gh., Borsos Z. and Drafta V., *Acoustic lens associated with a radial oscillating bubble*, Buletinul Institutului Politehnic din Iaşi, Secţia Matematică. Mecanică Teoretică. Fizică, Vol. 66 (70), No. 2, pp. 9-15, 2020.

[20] Borsos Z., Simaciu I., Dumitrescu Gh. and Drafta V., *Acoustic Black Hole Generated by a Cluster of Oscillating Bubbles*, Buletinul Institutului Politehnic din Iaşi, Secţia Matematică. Mecanică Teoretică. Fizică, Volume 67(71), No. 4, pp. 89-96, 2021; arXiv:2007.04067v1.

[21] Feynman R.P., Leighton R.B. and Matthew S., *The Feynman Lectures on Physics*, vol 2, Massachusetts: Addison-Wesley, 1964.

[22] Griffiths D.J., *Introduction to electrodynamics*, 3rd ed., Prentice Hall, 1999.

[23] Doubochinski D. and Tennenbaum J., *On the General Nature of Physical Objects and their Interactions, as Suggested by the Properties of Argumentally-Coupled Oscillating Systems*, arXiv:0808.1205

[24] Doinikov A., *Acoustic radiation forces: Classical theory and recent advances*, Recent Res. Devel. Acoustics, 1, 39-67, 2003.

[25] Simaciu I., Dumitrescu Gh. and Borsos Z., *Acoustic force of the gravitational type*, arXiv:1905.03622.

[26] Zavtrak S.T., *A classical treatment of the long-range radiative interaction of small particles*, Journal of Physics A: Mathematical and General, 1990, 23 9 1493-1499

[27] Landau L.D., Lifshitz E.M., *Mechanics*, Volume 1, Second Edition, 1976.

[28] Eddington A.S., *The internal constitution of the star*, Cambridge Univ. Press, 1926.

[29] Landau L.D., Lifshitz E.M., *Fluid Mechanics*, Vol. 6, Third Rev. Ed., 1966.

[30] Leighton T.G., *The Acoustic Bubble*, The Journal of the Acoustical Society of America 96, 2616, 1994.

[31] Mettin R., Akhatov I., Parlitz U., Ohl C.D. and Lauterborn W., *Bjerknes forces between small cavitation bubbles in a strong acoustic field*, Physical Review E, Vol. 56, No 3, 1997.

[32] Ida M., *A characteristic frequency of two mutually interacting gas bubbles in an acoustic field*, Physics Letters A 297 210-217, 2002.

[33] Louisnard O., *Analytical expressions for primary Bjerknes force on inertial cavitation bubbles*, Phys. Rev. E, 2008, 78, 036322.

[34] Doinikov A.A. and Bouakaz A., *Theoretical model for coupled radial and translational motion of two bubbles at arbitrary separation distances*, Physical Review E, 92, 2015, 043001.

[35] Doinikov A.A., Gerlt M.S. and Dual J., *Acoustic Radiation Forces Produced by Sharp-Edge Structures in Microfluidic Systems*, Physical Review Letters 124, 154501, 2020.

[36] Combriat T., Rouby-Poizat F., Doinikov A.A., Stephan O. and Marmottant F., *Acoustic interaction between 3D-fabricated cubic bubbles*, Soft Matter, 16, 2829-2835, 2020.

[37] Yasui K., *Multibubble Sonoluminescence from a Theoretical Perspective*, Molecules, 26, 4624, 2021.

[38] Simaciu I., Borsos Z., Drafta V. and Dumitrescu Gh., *Electrostatic Interaction in Stochastic Electrodynamics*, arXiv:2106.04401v1, Bulletin of the Polytechnic Institute of Iași, Section Mathematics. Theoretical Mechanics. Physics, 2022, Volume 68 (72), No. 4, 29-40.