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THE DETERMINATION OF THE ELEMENTS OF THE BASE CHANGE MATRIX IN THE KINEMATICS OF RIGID BODY IN A PARTICULAR CASE

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Abstract. The directional cosines matrix makes the link between the reference frame and the mobile frame in the kinematics of the rigid. This paper determines the matrix in the event that three of its elements are knowns, in a particular case.

Keywords: matrix, directional cosines, vectorial product, root.

1. Introduction

There is considered a solid rigid body into a general motion, related to an attached mobile frame and to a fixed reference frame. The frames are three-orthogonals, $OXYZ$ the reference frame and $OX'Y'Z'$ the mobile frame, with the specific notations for the unit vectors, $\overline{i}, \overline{j}, \overline{k}$, respectively $\overline{i}', \overline{j}', \overline{k}'$. In

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particular, there may be considered the same origin for the frames, the case of motion around a fixed point.

The link between the unit vectors of the reference frame and the unit vectors of the mobile frame is given by the matrix of directional cosines (Mangeron and Irimiciuc 1978; Rădoi and Deciu, 1993; Tocaci, 1985; Voinea *et al.*, 1983):

$$A^{-1} = A^{T} \tag{1}$$

Between the unit vectors of the reference frame and the unit vectors of the mobile frame there are the following relationships:

$$\begin{bmatrix} \overline{i'} \\ \overline{j'} \\ \overline{k'} \end{bmatrix} = A \cdot \begin{bmatrix} \overline{i} \\ \overline{j} \\ \overline{k} \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \tag{2}$$

respectively

$$\begin{bmatrix} \overline{i} \\ \overline{j} \\ \overline{k} \end{bmatrix} = A^{T} \cdot \begin{bmatrix} \overline{i'} \\ \overline{j'} \\ \overline{k'} \end{bmatrix}, \quad A^{T} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \tag{3}$$

with subunitary values for the elements of the matrix:

$$a_{\mu\sigma} \in [-1\,,\ 1]\ ,\ \mu, \sigma = 1, 2, 3 \tag{4}$$

The elements of the matrix satisfy the unitary norm conditions and the unit vectors perpendicularity conditions:

$$\sum_{\sigma=1}^{3} a_{\mu\sigma}^{2} = 1, \quad \mu = 1,\ 2,\ 3 \tag{5}$$

$$\sum_{\mu=1}^{3} a_{\mu\sigma}^{2} = 1, \quad \sigma = 1,\ 2,\ 3 \tag{6}$$

$$\sum_{\sigma=1}^{3} a_{1\sigma} a_{2\sigma} = 0, \quad \sum_{\sigma=1}^{3} a_{2\sigma} a_{3\sigma} = 0, \quad \sum_{\sigma=1}^{3} a_{1\sigma} a_{3\sigma} = 0 \tag{7}$$

$$\sum_{\mu=1}^{3} a_{\mu 1} a_{\mu 2} = 0, \quad \sum_{\mu=1}^{3} a_{\mu 2} a_{\mu 3} = 0, \quad \sum_{\mu=1}^{3} a_{\mu 1} a_{\mu 3} = 0 \tag{8}$$

Out of the nine elements of the matrix, only three are independents (Ibănescu and Rusu, 1998; Mangeron and Irimiciuc, 1978). Since A and A^T are orthogonal matrices, not all Eqs. (5)-(8) are independent. If three elements of the matrix are knowns, all the elements can be determinated according to Eqs. (5)-(8). These three elements given by hypothesis do not need to be all on the same line or on the same column.

2. Content

In this paperwork there are considered to be knowns by hypothesis three elements of the matrix, chosen so that each should by single on its line, without considering the diagonal case.

Thus, there are considered to be known

$$a_{11}, a_{23}, a_{32}, \tag{9}$$

under conditions:

$$a_{11}, a_{23}, a_{32} \neq 0, \quad a_{11}, a_{23}, a_{32} \in (-1, 1). \tag{10}$$

According to the first Eq. (5), respectively to the first Eq. (6), there results

$$a_{12}^2 + a_{13}^2 = 1 - a_{11}^2, \quad a_{21}^2 + a_{31}^2 = 1 - a_{11}^2 \tag{11}$$

In accordance with the relationship (2), there are given the following equations:

$$\overline{i'} = a_{11}\overline{i} + a_{12}\overline{j} + a_{13}\overline{k} \tag{12}$$

$$\overline{j'} = a_{21}\overline{i} + a_{22}\overline{j} + a_{23}\overline{k} \tag{13}$$

$$\overline{k'} = a_{31}\overline{i} + a_{32}\overline{j} + a_{33}\overline{k} \tag{14}$$

In accordance with the definition of the vectorial product, there is given the following equation

$$\overline{j'} = \overline{k'} \times \overline{i'} \tag{15}$$

By applying the formula for the development of the vectorial product (15) with the Eqs. (12) and (14), there results

$$\overline{j'} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}, \tag{16}$$

respectively

$$\mathrm{pr}_{OZ}\left(\overline{j'}\right) = a_{31}a_{12} - a_{11}a_{32} \tag{17}$$

From Eqs. (13) and (17) there results

$$a_{23} = a_{31}a_{12} - a_{11}a_{32} \tag{18}$$

In accordance with the definition of the vectorial product, there is given the following equation

$$\overline{k'} = \overline{i'} \times \overline{j'} \tag{19}$$

By applying the formula for the development of the vectorial product (19) with the Eqs. (12) and (13), there results

$$\overline{k'} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}, \tag{20}$$

respectively

$$\mathrm{pr}_{OY}\left(\overline{k'}\right) = a_{21}a_{13} - a_{11}a_{23} \tag{21}$$

From Eqs. (14) and (21) there results

$$a_{31} = a_{21}a_{13} - a_{11}a_{23} \tag{22}$$

In accordance with Eqs. (11), (18) and (22), there is obtained an algebrical system with four equations in the unknows a_{12}, a_{21}, a_{13}, a_{31} :

$$\begin{cases} a_{12}^2 + a_{13}^2 = 1 - a_{11}^2 \\ a_{21}^2 + a_{31}^2 = 1 - a_{11}^2 \\ a_{12}a_{31} = a_{23} + a_{11}a_{32} \\ a_{21}a_{13} = a_{32} + a_{11}a_{23} \end{cases}, \tag{23}$$

with the hypothesis

$$a_{23} + a_{11}a_{32} \neq 0, \quad a_{32} + a_{11}a_{23} \neq 0 \tag{24}$$

The first Eq. (23) is multiplied by a_{31}^2, by which there is obtained the following equation

$$a_{12}^2 a_{31}^2 + a_{13}^2 a_{31}^2 = \left(1 - a_{11}^2\right) a_{31}^2 \ , \tag{25}$$

in which the first term is replaced according to the third Eq. (23):

$$a_{13}^2 a_{31}^2 + \left(a_{11}^2 - 1\right) a_{31}^2 + \left(a_{23} + a_{11}a_{32}\right)^2 = 0 \tag{26}$$

The second Eq. (23) is multiplied by a_{13}^2, by which there is obtained the following equation

$$a_{21}^2 a_{13}^2 + a_{31}^2 a_{13}^2 = \left(1 - a_{11}^2\right) a_{13}^2 \ , \tag{27}$$

in which the first term is replaced according to the fourth Eq. (23):

$$a_{31}^2 a_{13}^2 + \left(a_{11}^2 - 1\right) a_{13}^2 + \left(a_{32} + a_{11}a_{23}\right)^2 = 0 \tag{28}$$

The first Eq. (23) is multiplied by a_{21}^2, by which there is obtained the following equation

$$a_{12}^2 a_{21}^2 + a_{13}^2 a_{21}^2 = \left(1 - a_{11}^2\right) a_{21}^2 \ , \tag{29}$$

in which the second term is replaced according to the fourth Eq. (23):

$$a_{12}^2 a_{21}^2 + \left(a_{11}^2 - 1\right) a_{21}^2 + \left(a_{32} + a_{11}a_{23}\right)^2 = 0 \tag{30}$$

The second Eq. (23) is multiplied by a_{12}^2, by which there is obtained the following equation

$$a_{21}^2 a_{12}^2 + a_{31}^2 a_{12}^2 = \left(1 - a_{11}^2\right) a_{12}^2 \quad , \tag{31}$$

in which the second term is replaced according to the third Eq. (23):

$$a_{21}^2 a_{12}^2 + \left(a_{11}^2 - 1\right) a_{12}^2 + \left(a_{23} + a_{11} a_{32}\right)^2 = 0 \tag{32}$$

From the third Eq. (5), respectively from the third Eq. (6), there results the followings:

$$a_{31}^2 + a_{32}^2 = a_{13}^2 + a_{23}^2 = 1 - a_{33}^2 \quad , \tag{33}$$

or also

$$a_{13}^2 = a_{31}^2 + a_{32}^2 - a_{23}^2 \quad , \quad a_{31}^2 = a_{13}^2 + a_{23}^2 - a_{32}^2 \tag{34}$$

From the second Eq. (5), respectively from the second Eq. (6), there results the followings:

$$a_{21}^2 + a_{23}^2 = a_{12}^2 + a_{32}^2 = 1 - a_{22}^2 \quad , \tag{35}$$

or also

$$a_{12}^2 = a_{21}^2 + a_{23}^2 - a_{32}^2 \quad , \quad a_{21}^2 = a_{12}^2 + a_{32}^2 - a_{23}^2 \tag{36}$$

By replacing the first Eq. (34) in Eq. (26) and the second Eq. (34) in Eq. (28), respectively the first Eq. (36) in Eq. (30) and the second Eq. (36) in Eq. (32), there are obtained the following equations:

$$a_{31}^4 + \left(a_{11}^2 + a_{32}^2 - a_{23}^2 - 1\right) a_{31}^2 + \left(a_{23} + a_{11} a_{32}\right)^2 = 0 \tag{37}$$

$$a_{13}^4 + \left(a_{11}^2 + a_{23}^2 - a_{32}^2 - 1\right) a_{13}^2 + \left(a_{32} + a_{11} a_{23}\right)^2 = 0 \tag{38}$$

$$a_{21}^4 + \left(a_{11}^2 + a_{23}^2 - a_{32}^2 - 1\right) a_{21}^2 + \left(a_{32} + a_{11} a_{23}\right)^2 = 0 \tag{39}$$

$$a_{12}^4 + \left(a_{11}^2 + a_{32}^2 - a_{23}^2 - 1\right) a_{12}^2 + \left(a_{23} + a_{11} a_{32}\right)^2 = 0 \tag{40}$$

In accordance with the Eqs. (37) and (40) there results that the squares of elements a_{12} and a_{31} are obtained as reals and positive roots of the following equation

$$\alpha^2 + \left(a_{11}^2 + a_{32}^2 - a_{23}^2 - 1\right)\alpha + \left(a_{23} + a_{11}a_{32}\right)^2 = 0 \tag{41}$$

In accordance with the Eqs. (38) and (39) there results that the squares of elements a_{13} and a_{21} are obtained as reals and positive roots of the following equation

$$\beta^2 + \left(a_{11}^2 + a_{23}^2 - a_{32}^2 - 1\right)\beta + \left(a_{32} + a_{11}a_{23}\right)^2 = 0 \tag{42}$$

There is made the following calculation

$$\Delta_\alpha = \left(a_{11}^2 + a_{32}^2 - a_{23}^2 - 1\right)^2 - 4\left(a_{23} + a_{11}a_{32}\right)^2 \tag{43}$$

and there is obtained

$$\Delta_\alpha = \left[\left(a_{11} - 1\right)^2 - \left(a_{32} + a_{23}\right)^2\right] \cdot \left[\left(a_{11} + 1\right)^2 - \left(a_{32} - a_{23}\right)^2\right] \tag{44}$$

There is made the following calculation

$$\Delta_\beta = \left(a_{11}^2 + a_{23}^2 - a_{32}^2 - 1\right)^2 - 4\left(a_{32} + a_{11}a_{23}\right)^2 \tag{45}$$

and there is obtained

$$\Delta_\beta = \left[\left(a_{11} - 1\right)^2 - \left(a_{23} + a_{32}\right)^2\right] \cdot \left[\left(a_{11} + 1\right)^2 - \left(a_{23} - a_{32}\right)^2\right] \tag{46}$$

There is denoted the discriminant of Eqs. (41) and (42):

$$\Delta_\alpha = \Delta_\beta = \Delta \tag{47}$$

The roots of Eq. (41) are

$$\alpha_{1,2} = \frac{1}{2}\left(1 + a_{23}^2 - a_{32}^2 - a_{11}^2 \pm \sqrt{\Delta}\right) \tag{48}$$

The roots of Eq. (42) are

$$\beta_{1,2}=\frac{1}{2}\left(1+a_{32}^2-a_{23}^2-a_{11}^2\pm\sqrt{\Delta}\right) \tag{49}$$

The condition for the roots of Eqs. (41) and (42) to be reals is that the discriminant should be positive

$$\Delta>0\ , \tag{50}$$

i.e. the two squared parantheses from relationship (44) or from relationship (46) should be of the same sign:

$$\left(|a_{11}-1|-|a_{32}+a_{23}|\right)\cdot\left(|a_{11}+1|-|a_{32}-a_{23}|\right)>0 \tag{51}$$

$$\left(1-a_{11}-|a_{32}+a_{23}|\right)\cdot\left(1+a_{11}-|a_{32}-a_{23}|\right)>0 \tag{52}$$

The condition for the roots of Eq. (41) to be positive is

$$a_{11}^2+a_{32}^2-a_{23}^2-1<0\ ,\quad 1-a_{11}^2>a_{32}^2-a_{23}^2 \tag{53}$$

The condition for the roots of Eq. (42) to be positive is

$$a_{11}^2+a_{23}^2-a_{32}^2-1<0\ ,\quad 1-a_{11}^2>a_{23}^2-a_{32}^2 \tag{54}$$

According to the Ineqs. (53) and (54), the condition for the roots of Eqs. (41) and (42) to be positive is

$$1-a_{11}^2>\left|a_{32}^2-a_{23}^2\right|\ , \tag{55}$$

$$\left(1-a_{11}\right)\left(1+a_{11}\right)>|a_{32}+a_{23}|\cdot|a_{32}-a_{23}| \tag{56}$$

According to the Ineqs. (52) and (56), the condition for the roots of Eqs. (41) and (42) to be reals and positive are

$$1-a_{11}>|a_{32}+a_{23}|\ ,\ a_{11}\neq 1 \tag{57}$$

$$1+a_{11}>|a_{32}-a_{23}|\ ,\ a_{11}\neq -1 \tag{58}$$

The possible solutions for the directional cosines on the first line and on the first column of the matrix are

$$a_{12}^2=\alpha_1 \;\;,\;\; a_{31}^2=\alpha_2 \;\;,\;\; a_{21}^2=\beta_1 \;\;,\;\; a_{13}^2=\beta_2 \;\;, \tag{59}$$

or

$$a_{12}^2=\alpha_2 \;\;,\;\; a_{31}^2=\alpha_1 \;\;,\;\; a_{21}^2=\beta_2 \;\;,\;\; a_{13}^2=\beta_1 \;\;, \tag{60}$$

solutions in accordance with the Eqs. (23).

The signs of the solutions (59) and (60) are considered so that are verified the last two Eqs. (23), i.e.:

$$a_{12}a_{31}\left(a_{23}+a_{11}a_{32}\right)>0 \tag{61}$$

$$a_{21}a_{13}\left(a_{32}+a_{11}a_{23}\right)>0 \;. \tag{62}$$

The other two directional cosines of the matrix are determined from the conditions of orthogonality.

From the first Eq. (7) there is obtained

$$a_{22}=-\frac{1}{a_{12}}\left(a_{11}a_{21}+a_{13}a_{23}\right) \;\;,\;\; a_{12}\neq 0 \;\;, \tag{63}$$

respectively from the third Eq. (7):

$$a_{33}=-\frac{1}{a_{13}}\left(a_{11}a_{31}+a_{12}a_{32}\right) \;\;,\;\; a_{13}\neq 0 \;\;. \tag{64}$$

3. Conclusions

There is written the algorithm for the determination of the matrix elements in the hypothesis a_{11}, a_{23}, a_{32} :

$$a_{11},\ a_{23},\ a_{32}\in\left(-1\,,\ 1\right),\ a_{11},\ a_{23},\ a_{32}\neq 0 \tag{65}$$

The conditions for the existence of the solutions are given by the Ineqs. (57) and (58):

$$1-a_{11}>\left|a_{32}+a_{23}\right| \;,\; 1+a_{11}>\left|a_{32}-a_{23}\right| \tag{66}$$

One calculates with Eq. (43) or with Eq. (45):

$$\Delta = \left(a_{11}^2 + a_{32}^2 - a_{23}^2 - 1\right)^2 - 4\left(a_{23} + a_{11}a_{32}\right)^2 \tag{67.a}$$

$$\Delta = \left(a_{11}^2 + a_{23}^2 - a_{32}^2 - 1\right)^2 - 4\left(a_{32} + a_{11}a_{23}\right)^2 \tag{67.b}$$

One calculates with Eqs. (48) and (49):

$$\alpha_1 = \frac{1}{2}\left(1 + a_{23}^2 - a_{32}^2 - a_{11}^2 + \sqrt{\Delta}\right) \tag{68}$$

$$\alpha_2 = \frac{1}{2}\left(1 + a_{23}^2 - a_{32}^2 - a_{11}^2 - \sqrt{\Delta}\right) \tag{69}$$

$$\beta_1 = \frac{1}{2}\left(1 + a_{32}^2 - a_{23}^2 - a_{11}^2 + \sqrt{\Delta}\right) \tag{70}$$

$$\beta_2 = \frac{1}{2}\left(1 + a_{32}^2 - a_{23}^2 - a_{11}^2 - \sqrt{\Delta}\right) \tag{71}$$

There are written the solutions for the directional cosines on the first line and on the first column of the matrix as follows.

Case 1. $a_{23} + a_{11}a_{32} > 0,\ \ a_{32} + a_{11}a_{23} > 0$

a_{12}	a_{31}	a_{21}	a_{13}
$\pm\sqrt{\alpha_1}$	$\pm\sqrt{\alpha_2}$	$\pm\sqrt{\beta_1}$	$\pm\sqrt{\beta_2}$
$\pm\sqrt{\alpha_2}$	$\pm\sqrt{\alpha_1}$	$\pm\sqrt{\beta_2}$	$\pm\sqrt{\beta_1}$

(72)

Case 2. $a_{23} + a_{11}a_{32} > 0,\ \ a_{32} + a_{11}a_{23} < 0$

a_{12}	a_{31}	a_{21}	a_{13}
$\pm\sqrt{\alpha_1}$	$\pm\sqrt{\alpha_2}$	$\pm\sqrt{\beta_1}$	$\mp\sqrt{\beta_2}$
$\pm\sqrt{\alpha_2}$	$\pm\sqrt{\alpha_1}$	$\pm\sqrt{\beta_2}$	$\mp\sqrt{\beta_1}$

(73)

Case 3. $a_{23} + a_{11}a_{32} < 0,\ \ a_{32} + a_{11}a_{23} > 0$

a_{12}	a_{31}	a_{21}	a_{13}
$\pm\sqrt{\alpha_1}$	$\mp\sqrt{\alpha_2}$	$\pm\sqrt{\beta_1}$	$\pm\sqrt{\beta_2}$
$\pm\sqrt{\alpha_2}$	$\mp\sqrt{\alpha_1}$	$\pm\sqrt{\beta_2}$	$\pm\sqrt{\beta_1}$

(74)

Case 4. $a_{23} + a_{11}a_{32} < 0,\ \ a_{32} + a_{11}a_{23} < 0$

a_{12}	a_{31}	a_{21}	a_{13}
$\pm\sqrt{\alpha_1}$	$\mp\sqrt{\alpha_2}$	$\pm\sqrt{\beta_1}$	$\mp\sqrt{\beta_2}$
$\pm\sqrt{\alpha_2}$	$\mp\sqrt{\alpha_1}$	$\pm\sqrt{\beta_2}$	$\pm\sqrt{\beta_1}$

(75)

In the tables (72) ÷ (75) the signs for a_{12} and a_{31}, respectively a_{21} and a_{13} are considered on the same positions (up or down).

There are calculated the directional cosines on the main diagonal of the matrix with the Eqs. (63) and (64):

$$a_{22} = -\frac{1}{a_{12}}\left(a_{11}a_{21} + a_{13}a_{23}\right)\ \ ,\ \ a_{12} \neq 0\ \ , \tag{76}$$

$$a_{33} = -\frac{1}{a_{13}}\left(a_{11}a_{31} + a_{12}a_{32}\right)\ \ ,\ \ a_{13} \neq 0\ \ . \tag{77}$$

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DETERMINAREA ELEMENTELOR MATRICII SCHIMBĂRII DE BAZĂ ÎN CINEMATICA RIGIDULUI ÎNTR-UN CAZ PARTICULAR

(Rezumat)

Matricea cosinusurilor directoare face legătura între reperul de referinţă şi reperul mobil în cinematica rigidului. În această lucrare se determină matricea în situaţia în care se cunosc trei elemente ale acesteia, într-un caz particular.