BULETINUL INSTITUTULUI POLITEHNIC DIN IAŞI
Publicat de
Universitatea Tehnică „Gheorghe Asachi" din Iaşi
Volumul 68 (72), Numărul 2, 2022
Secţia
CONSTRUCŢII. ARHITECTURĂ
DOI: 10.2478/bipca-2022-0013

A 2-D EXTENSION OF THE TREND CONCEPT APPLICABLE TO IMAGES, SURFACES AND SPACE-TIME SIGNALS

BY

JACQUES PADET*

ITheMM, Faculté des Sciences, Université de Reims, France

Received: May 2, 2022
Accepted for publication: July 3, 2022

Abstract. The analysis of highly irregular processes is often based on the search for a trend, an average curve representing the general pattern of the phenomenon observed. This concept seems intuitive, but determining it objectively raises a number of problems. As the moving average method is one of the most widely used for finding one-dimensional trends, we propose here to extend it to two-dimensional data sets, and to study some of the properties of these 2-D moving averages. We then show that they can be used to detect characteristic observation windows (uniform or self-adaptive), leading to structural trends in the analyzed signal. The method is first applied to functions *f(x,y)* whose variables have the same physical dimension (length,…). It is then generalized to cases of different dimensions (e.g. space-time signals). Applications cover a wide range of fields (turbulent structures in fluid mechanics, image analysis, characterization of surfaces, optimization of an observation process, multi-scale modelling, ...).

Keywords: Moving averages, Trends. Images, Surface analysis, Observation scales, Turbulence scales, Space-time signals.

*Corresponding author; *e-mail*: jacques.padet@univ-reims.fr

List of symbols

$f(x,y)$	function whose trend, or trends, we are looking for
$\overset{\frown}{f}(x,y,\tau)$	two-dimensional moving average of $f(x,y)$
$\overset{\frown}{f}(x,t,\tau,\theta)$	space-time moving average over a $2\tau\times2\theta$ window
$\overset{\frown}{f}_x(t,\theta)$	one-dimensional (time) moving average of $f(x=cst,t,\theta)$
$\overset{\frown}{f}_t(x,\tau)$	one-dimensional (spatial) moving average of $f(x, t=cst,\tau)$
MA	moving average (abbreviation)
τ	half-side of the spatial observation window
θ	half-side of the time observation window
$T_{ks}(x,y)$	stationary trend of order *ks*
x_1 , x_2 ; y_1, y_2	boundaries of the concerned domain

1. Introduction

Knowledge of a phenomenon can be provided by the values of a function $f(x)$ of a variable x, often called a signal. This signal appears highly irregular if it describes a complex phenomenon whose structure is little known, or even ignored. What's more, the signal may be interfered with by causes external to the phenomenon itself (noise in the broad meaning). There is therefore a need to eliminate accidental data and retain only useful information, but also to detect structural evolutions hidden in the complexity of the signal, i.e. *trends* (Padet C., 1996).

Various methods have been developed for trend detection. We will consider here only the moving average technique (Godichon-Baggioni, 2017). This involves replacing the value of the function at each point x with its average over an interval $[x-\tau,x+\tau]$. From a physical point of view, the phenomenon is thus observed using a technique with limited sensitivity, characterized by the observation window 2τ. From a mathematical point of view, we proceed with a local approximation of the function $f(x)$ (Padet C., 1987). In this way, an attempt will be made to highlight remarkable evolutions (smoothed or denoised signal, intermediate trends, ultimate trend, etc.) corresponding to increasingly larger window widths.

The major problem is that the choice of observation window must not be arbitrary. Authentic structural trends must be based on objective values of τ, representing characteristic observation scales. The difficulty lies in establishing clear and rigorous criteria for obtaining them.

The techniques used involve two different approaches, which are complementary:

- global methods, in which $\tau=cst$ over the entire domain of definition of $f(x)$. The selection criteria for τ are then either statistical (Padet J., 2024) or topological (Roche, 1992), or based on *a priori* knowledge (e.g. periodicity or pseudo-periodicity of the signal).

- local, self-adaptive methods, in which τ is variable as a function of x, the selection criteria then being statistical, topological or based on stationarity (Djeziri, 1997; Padet J., 2024).

All of the above relates to one-dimensional signals. But there is no reason to limit the concept of trend to these signals alone. By the way, its extension to images was proposed by Song (Song, 1998), based on a two-dimensional non-linearity index, but using *1-D* moving averages, and scanning the image along rows, columns or diagonals. It has also been used for edge detection and smoothing (Rachek, 2001, 2002, 2003).

In fact, a more fundamental generalization can be conceived, by designing multidimensional *n-D* trends *(n=2, 3…)* based on moving averages and on observation scales which are themselves *n-D* (note that *2-D* moving averages have already been used under the name of "averaging filter" (Girard, 1999; Rachek, 2003), but without reference to the notion of trend).

To do this, the first step is obviously to consider the *2-D* case. This includes images in the classical sense of the term, but also the roughness and shape of a surface in relation to a reference plane, and more generally any type of *2-D* file in which we are looking for characteristic structures and scales.

2. Definitions

Consider an image (or more generally a representation) described by a function of two independent variables $f(x,y)$, with: $x \in [x_1, x_2]$ and $y \in [y_1, y_2]$. *Initially, the quantities x and y are assumed to be homogeneous (i.e. of the same physical dimension: most often, they are two lengths).*

We approximate $f(x,y)$ using a square observation window, centered on a point (x,y), with side equal to 2τ (Fig. 1), by the following application:

Two-dimensional moving average of f(x,y) (abbreviated MA-2D):

$$\overset{\frown}{f}(x,y,\tau) = \frac{1}{4\tau^2} \int_{x-\tau}^{x+\tau} \int_{y-\tau}^{y+\tau} f(u,v)\, du\, dv \tag{1}$$

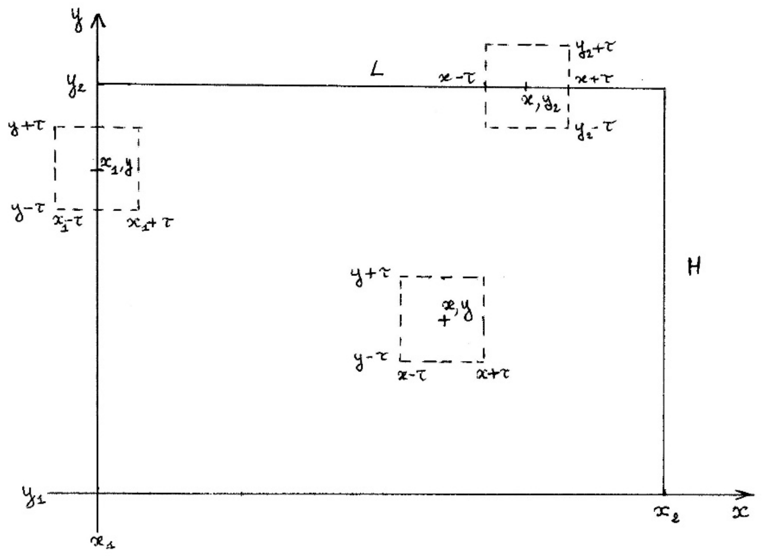

Fig. 1 – Observation windows of a function of two variables.

Note right now that, if $\tau \rightarrow 0$:

$$\overset{\frown}{f}\,(x, y, \tau) \sim \frac{1}{4\,\tau^2}\;\; 2\tau \times 2\tau\, f(x, y)$$

hence:

$$\overset{\frown}{f}\,(x,y,0)=f(x, y) \tag{2}$$

In addition, one-dimensional moving averages can be defined at each *y=cst* row, and *x=cst* column:

One-dimensional moving average of $f(x = cst, y, \tau)$ *(MA-1D over y):*

$$\overset{\frown}{f}_x\,(y,\tau)=\frac{1}{2\,\tau}\int_{y-\tau}^{y+\tau} f(x,v)\; dv \tag{3a}$$

One-dimensional moving average of $f(x, y = cst, \tau)$ *(MA-1D over x):*

$$\overset{\frown}{f}_y\,(x,\tau)=\frac{1}{2\,\tau}\int_{x-\tau}^{x+\tau} f(u,y)\; du \tag{3b}$$

3. Properties of the derivatives of the *MA* $\overset{\frown}{f}(x,y,\tau)$

3.1. Partial derivative of $\overset{\frown}{f}$ with respect to x or y

Let Δx be a small variation of x. According to (1), the corresponding variation of $\overset{\frown}{f}$ is:

$$\Delta \overset{\frown}{f} = \overset{\frown}{f}(x+\Delta x,y,\tau) - \overset{\frown}{f}(x,y,\tau) =$$

$$= \frac{1}{4\tau^2}\int_{x+\Delta x-\tau}^{x+\Delta x+\tau}\int_{y-\tau}^{y+\tau} f(u,v)\,du\,dv - \frac{1}{4\tau^2}\int_{x-\tau}^{x+\tau}\int_{y-\tau}^{y+\tau} f(u,v)\,du\,dv \qquad (4)$$

Let's decompose the first integral into three, according to x:

$$\int_{x+\Delta x-\tau}^{x+\Delta x+\tau} = \int_{x+\Delta x-\tau}^{x-\tau} + \int_{x-\tau}^{x+\tau} + \int_{x+\tau}^{x+\Delta x+\tau}$$

The first term of the right-hand side can also be written as: $-\int_{x-\tau}^{x+\Delta x-\tau}$. Transferring this to (4), we obtain:

$$\Delta \overset{\frown}{f} = -\frac{1}{4\tau^2}\int_{x-\tau}^{x+\Delta x-\tau}\int_{y-\tau}^{y+\tau} f(u,v)\,du\,dv + \frac{1}{4\tau^2}\int_{x+\tau}^{x+\Delta x+\tau}\int_{y-\tau}^{y+\tau} f(u,v)\,du\,dv$$

As Δx is small, we can write:

$$\Delta \overset{\frown}{f} = -\frac{1}{4\tau^2}\,\Delta x\int_{y-\tau}^{y+\tau} f(x-\tau,v)\,dv + \frac{1}{4\tau^2}\,\Delta x\int_{y-\tau}^{y+\tau} f(x+\tau,v)\,dv + O(\Delta x)$$

After dividing by Δx, tending to the limit $\Delta x \to 0$ gives the derivative:

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,y,\tau) = \frac{1}{4\tau^2}\left\{\int_{y-\tau}^{y+\tau} f(x+\tau,v)\,dv - \int_{y-\tau}^{y+\tau} f(x-\tau,v)\,dv\right\} \qquad (5a)$$

We can recognize in the right-hand side the *MA-1D* defined by (3a), over the x-τ and x+τ columns, hence:

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,y,\tau) = \frac{1}{2\tau}\left\{\overset{\frown}{f}_{x+\tau}(y,\tau) - \overset{\frown}{f}_{x-\tau}(y,\tau)\right\} \qquad (5b)$$

Similarly, the derivative with respect to y has the expression:

$$\frac{\partial}{\partial y}\overset{\frown}{f}(x,y,\tau)=\frac{1}{4\,\tau^2}\left\{\int_{x-\tau}^{x+\tau} f(u,\,y+\tau)\,du-\int_{x-\tau}^{x+\tau} f(u,y-\tau)\,du\right\} \tag{6a}$$

or equally, with the *MA-1D* (3b) over the rows y-τ and y+τ:

$$\frac{\partial}{\partial y}\overset{\frown}{f}(x,y,\tau)=\frac{1}{2\,\tau}\left\{\overset{\frown}{f}_{y+\tau}(x,\tau)-\overset{\frown}{f}_{y-\tau}(x,\tau)\right\} \tag{6b}$$

These two first-order partial derivatives will be used to compute the second-order partial derivative, i.e. $\frac{\partial^2}{\partial x\,\partial y}\overset{\frown}{f}(x,y,\tau)$, which will identify the extremums of $\overset{\frown}{f}$ (cf. § 4.2).

As an aside, it is interesting to observe the formal consistency of (5b) or (6b) with the derivative of a one-dimensional *MA* (Padet J., 2024, equation (7)):

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,\tau)=\frac{1}{2\,\tau}\,\{f(x+\tau)-f(x-\tau)\} \tag{6c}$$

3.2. Partial derivative of $\overset{\frown}{\mathbf{f}}$ with respect to $\boldsymbol{\tau}$

Let us return to definition (1), for which a more compact formulation will be adopted in the following:

$$\overset{\frown}{f}(x,y,\tau)=\frac{1}{4\,\tau^2}\int_{x-\tau}^{x+\tau}\int_{y-\tau}^{y+\tau} f(u,v)\,du\,dv\equiv\frac{1}{4\,\tau^2}\int_{S_\tau} f(x,y)dS \tag{7}$$

where S_τ denotes the area of the observation window ($2\tau\times2\tau$), and dS the differential area element.

Let ΔS_τ be the area variation due to a small $\Delta\tau$ change in parameter τ. The corresponding variation of (7) is then:

$$\Delta\overset{\frown}{f}=\overset{\frown}{f}(x,y,\tau+\Delta\tau)-\overset{\frown}{f}(x,y,\tau)=\frac{1}{4\,(\tau+\Delta\tau)^2}\int_{S_\tau+\Delta S_\tau} f(x,y)dS-\frac{1}{4\,\tau^2}\int_{S_\tau} f(x,y)dS \tag{8}$$

An expansion limited to rank 1 gives:

$$\frac{1}{(\tau+\Delta\tau)^2}\cong\frac{1}{\tau^2+2\tau\,\Delta\tau}\cong\frac{1}{\tau^2}\left(1-2\frac{\Delta\tau}{\tau}\right) \tag{9}$$

Then:

$$\Delta \widetilde{f} = \frac{1}{4\tau^2}\left(1-2\frac{\Delta\tau}{\tau}\right)\left\{\int_{S_\tau} f(x,y)dS + \int_{\Delta S_\tau} f(x,y)dS\right\} - \frac{1}{4\tau^2}\int_{S_\tau} f(x,y)dS$$

so, by regrouping:

$$\Delta \widetilde{f} = \frac{1}{4\tau^2}\int_{\Delta S_\tau} f(x,y)dS - \frac{\Delta\tau}{2\tau^3}\int_{S_\tau} f(x,y)dS - \frac{\Delta\tau}{2\tau^3}\int_{\Delta S_\tau} f(x,y)dS$$

The last term is a second-order infinitesimal one ($\Delta\tau\times$ integral over ΔS_τ). The expression of $\Delta \widetilde{f}$ therefore reduces to:

$$\Delta \widetilde{f} = \frac{1}{4\tau^2}\int_{\Delta S_\tau} f(x,y)dS - \frac{\Delta\tau}{2\tau^3}\int_{S_\tau} f(x,y)dS \qquad (10)$$

According to (7), the last integral is the *MA-2D* of f, multiplied by a coefficient:

$$\int_{S_\tau} f(x,y)dS = 4\tau^2\,\widetilde{f}(x,y,\tau) \qquad (11)$$

Finally, the surface element ΔS_τ is a strip of width $\Delta\tau$ that surrounds the square window S_τ. It consists of four rectangular strips (two vertical strips at abscissas $x-\tau$ and $x+\tau$, two horizontal strips at ordinates $y-\tau$ and $y+\tau$), and four-square corners $\Delta\tau\times\Delta\tau$. The contribution of these in the first integral will be a second-order infinitesimal, so they will be neglected. As for the other four elements, take for example the vertical strip $x-\tau$: since $\Delta\tau$ is small, the integral of f over this surface element will have the value:

$$\Delta\tau \int_{y-\tau}^{y+\tau} f(x-\tau, v)\, dv \qquad (12a)$$

Here we can identify a one-dimensional moving average over y, at abscissa $x-\tau$, defined by (3a):

$$\int_{y-\tau}^{y+\tau} f(x-\tau,v)\, dv = 2\tau\,\widetilde{f}_{x-\tau}(y,\tau) \qquad (12b)$$

The reasoning is similar for the other three strips, and the first integral of (10) is finally written as:

$$\int_{\Delta S_\tau} f(x,y)dS = 2\tau\,\Delta\tau\,\left\{\overset{\frown}{f}_{x-\tau}(y,\tau) + \overset{\frown}{f}_{x+\tau}(y,\tau) + \overset{\frown}{f}_{y-\tau}(x,\tau) + \overset{\frown}{f}_{y+\tau}(x,\tau)\right\} \tag{13}$$

Let us carry over (11) and (13) into (10). After dividing by $\Delta\tau$ and tending to the limit $\Delta\tau \to 0$, we get the partial derivative of $\overset{\frown}{f}$ with respect to τ:

$$\frac{\partial}{\partial\tau}\overset{\frown}{f}(x,y,\tau) = \frac{1}{2\,\tau}\left\{\overset{\frown}{f}_{x-\tau}(y,\tau) + \overset{\frown}{f}_{x+\tau}(y,\tau) + \overset{\frown}{f}_{y-\tau}(x,\tau) + \overset{\frown}{f}_{y+\tau}(x,\tau)\right\} - \frac{2}{\tau}\,\overset{\frown}{f}(x,y,\tau) \tag{14}$$

3.3. Partial derivative of the second order of $\overset{\frown}{f}$ with respect to x and y

Return to the expression (5b) for the derivative with respect to *x*:

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,y,\tau) = \frac{1}{2\,\tau}\left\{\overset{\frown}{f}_{x+\tau}(y,\tau) - \overset{\frown}{f}_{x-\tau}(y,\tau)\right\}$$

and now derive it with respect to *y*:

$$\frac{\partial^2}{\partial y\,\partial x}\overset{\frown}{f}(x,y,\tau) = \frac{1}{2\,\tau}\left\{\frac{\partial}{\partial y}\overset{\frown}{f}_{x+\tau}(y,\tau) - \frac{\partial}{\partial y}\overset{\frown}{f}_{x-\tau}(y,\tau)\right\} \tag{15}$$

In the second member, we have the derivatives of two *MA-1D*. Coming back to definition (3a) and property (6c), we see that:

$$\frac{\partial}{\partial y}\overset{\frown}{f}_{x+\tau}(y,\tau) = \frac{1}{2\tau}\,[f(x+\tau,\ y+\tau) - f(x+\tau,\ y-\tau)] \tag{16a}$$

$$\frac{\partial}{\partial y}\overset{\frown}{f}_{x-\tau}(y,\tau) = \frac{1}{2\tau}\,[f(x-\tau,\ y+\tau) - f(x-\tau,\ y-\tau)] \tag{16b}$$

Then, (15) becomes:

$$\frac{\partial^2}{\partial y\,\partial x}\overset{\frown}{f}(x,y,\tau) = \frac{1}{4\,\tau^2}\,\{f(x+\tau,\ y+\tau) - f(x+\tau,\ y-\tau) - f(x-\tau,\ y+\tau) + f(x-\tau,\ y-\tau)\} \tag{17}$$

In the same way, by deriving (6b) with respect to *x*, the commutativity of the derivation procedure can be verified:

$$\frac{\partial^2}{\partial y\,\partial x}\overset{\frown}{f}(x,y,\tau) \equiv \frac{\partial^2}{\partial x\,\partial y}\overset{\frown}{f}(x,y,\tau)$$

4. Trends of a 2-D file

As in the one-dimensional case, two options can be proposed for the search for *2-D* trends.

4.1. Statistical trends

The first approach has a statistical basis. The variance of the moving averages can show particular aspects for specific values of τ (Padet J., 2024). The corresponding *MA-2D* are then considered as trends for the image (or surface) analyzed.

However, this method relies on the conservation of the averages by the various *MA*, which implies a suitable extension beyond the edges of the image. The computations are then much more cumbersome than in the *1-D* case, and will not be undertaken here.

In addition, manipulating quantities averaged over the entire image results in a significantly greater loss of information than it would be on a one-dimensional file, which can reduce the quality of detection when we want to identify significant values of τ.

This does not mean that statistical 2-D trends should be rejected on principle, as they can be useful in some cases, but for the reasons given above, their search will not be given priority here.

4.2. Stationary trends

Stationary trend detection (Padet J., 2024), on the other hand, seems well suited to images and surfaces. Recall that they are based on the possible existence, at each point, of observation windows τ such that:

$$\frac{\partial}{\partial\tau}\overset{\frown}{f}(x,y,\tau)=0 \tag{18a}$$

In other words, given (14), we are looking for values $\tau_{ks}(x,y)$ ($ks=1, 2,\ldots$) satisfying equation:

$$\overset{\frown}{f}(x,y,\tau)=\frac{1}{4}\left\{\overset{\frown}{f}_{x-\tau}(y,\tau)+\overset{\frown}{f}_{x+\tau}(y,\tau)+\overset{\frown}{f}_{y-\tau}(x,\tau)+\overset{\frown}{f}_{y+\tau}(x,\tau)\right\} \tag{18b}$$

Then, the moving average:

$$\overset{\frown}{f}\{x,y,\tau_{ks}(x,y)\}=T_{ks}(x,y) \tag{18c}$$

can be considered as a stationary trend. But it is a variable-window, self-adaptive trend: at each point (x,y), the observation (or approximation) is adjusted so that a small variation in τ remains without effect on the observed value.

Otherwise, if the function $\overset{\frown\frown}{f}(x,y,\tau)$ is monotonic along τ, no stationary trend can be locally defined at the point (x,y).

Finally, it should be noted that, for this set of approximations of the function f, the average $\overset{\frown\frown}{f}$ of the *MA* is not saved. Despite this, the question of extending the function beyond the edges of the image still arises, when the observation window extends beyond this frame. We can then adopt an extension of the convention proposed in the *1-D* case (Padet J., 2024), i.e. extend:

- on the left by $f(x_1,y)$, on the right by $f(x_2,y)$
- at the bottom by $f(x,y_1)$ and at the top by $f(x,y_2)$ (19)

It is hardly possible to identify general properties for stationary trends. However, their extremums are easy to characterize. They satisfy the condition:

$$\frac{\partial^2}{\partial x\,\partial y}\overset{\frown\frown}{f}[x,y,\tau(x,y)]=0 \tag{20}$$

With τ variable, the derivative of $\overset{\frown\frown}{f}$ with respect to x is written:

$$\frac{\partial}{\partial x}\overset{\frown\frown}{f}[x,y,\tau(x,y)]=\frac{\partial}{\partial x}\overset{\frown\frown}{f}[x,y,\tau=cte]+\frac{\partial}{\partial \tau}\overset{\frown\frown}{f}(x,y,\tau)\frac{\partial}{\partial x}\tau(x,y) \tag{21a}$$

But, since the trend is stationary, (18a) still works, leaving:

$$\frac{\partial}{\partial x}\overset{\frown\frown}{f}[x,y,\tau(x,y)]=\frac{\partial}{\partial x}\overset{\frown\frown}{f}[x,y,\tau=cte] \tag{21b}$$

Now let's derive with respect to y. In the same way, we get:

$$\frac{\partial^2}{\partial x\,\partial y}\overset{\frown\frown}{f}[x,y,\tau(x,y)]=\frac{\partial^2}{\partial x\,\partial y}\overset{\frown\frown}{f}[x,y,\tau=cte] \tag{22}$$

Consequently, formula (17) is valid here, and condition (20) for an extremum of the trend is expressed by the following relation, where $\tau=\tau_{sk}$ is the solution of (18b):

$$f(x+\tau,\ y+\tau)-f(x+\tau,\ y-\tau)-f(x-\tau,\ y+\tau)+f(x-\tau,\ y-\tau)=0 \tag{23}$$

5. Moving averages and space-time trends

5.1. Introducing the problem

At the beginning of section 2, we specified that the variables x and y were homogeneous, i.e. of the same nature (for example, lengths). But many phenomena are described using two parameters that do not have the same physical dimension. This is the case for all time-dependent phenomena, often referred to as variable states. For simplicity's sake, in what follows we will restrict ourselves to space-time evolutions $f(x,t)$, where x is a space variable and y is time t. This will lead us to characterize *space-time moving averages*, *optimal space-time observation windows*, and *space-time trends*.

Under these conditions, *the observation window is no longer a square surface $2\tau\times 2\tau$, but a space-time rectangle with sides 2τ (depending on x) and 2θ (depending on time t)*. Its area is $S_{\tau\theta}=4\,\tau\theta$.

The space-time moving average over the window $S_{\tau\theta}$ (*MA-ST* for short) is then:

$$\overset{\frown}{f}(x,t,\tau,\theta)=\frac{1}{4\,\tau\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)du\,dv \tag{24}$$

In addition, as with definitions (3), two one-dimensional moving averages of f will need to be considered, either on a time line ($x=cst$) or along x at a fixed time ($t=cst$):

One-dimensional moving average of $f(x=cst,\ t,\theta)$ (*MA-1D* over t):

$$\overset{\frown}{f}_x(t,\theta)=\frac{1}{2\,\theta}\int_{t-\theta}^{t+\theta} f(x,v)\,dv \tag{25a}$$

One-dimensional moving average of $f(x,\ t=cst,\tau)$ (MA-1D over x):

$$\overset{\frown}{f}_t(x,\tau)=\frac{1}{2\,\tau}\int_{x-\tau}^{x+\tau} f(u,t)\,du \tag{25b}$$

Some analytical properties of these *MA-ST* are discussed below.

5.2. Partial derivative of the MA-ST with respect to x or t

The calculation of the partial derivative of (24) with respect to x is identical to that in paragraph 3.1, and need not be described in detail. The result is:

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,t,\tau,\theta)=\frac{1}{4\,\tau\,\theta}\left\{\int_{t-\theta}^{t+\theta} f(x+\tau,v)\,dv-\int_{t-\theta}^{t+\theta} f(x-\tau,v)\,dv\right\} \tag{26a}$$

Using definition (25a), this can be written a little more compactly:

$$\frac{\partial}{\partial x}\overset{\frown}{f}(x,t,\tau,\theta)=\frac{1}{2\tau}\left\{\overset{\frown}{f}_{x+\tau}(t,\theta)-\overset{\frown}{f}_{x-\tau}(t,\theta)\right\} \tag{26b}$$

For the derivative with respect to time, all we have to do is swap x and τ with t and θ:

$$\frac{\partial}{\partial t}\overset{\frown}{f}(x,t,\tau,\theta)=\frac{1}{4\,\tau\,\theta}\left\{\int_{x-\tau}^{x+\tau} f(t+\theta,u)\,du-\int_{x-\tau}^{x+\tau} f(t-\theta,u)\,du\right\} \tag{27a}$$

In the same, using (25b):

$$\frac{\partial}{\partial t}\overset{\frown}{f}(x,t,\tau,\theta)=\frac{1}{2\theta}\left\{\overset{\frown}{f}_{t+\theta}(x,\tau)-\overset{\frown}{f}_{t-\theta}(x,\tau)\right\} \tag{27b}$$

5.3. Partial derivatives with respect to observation parameters τ and θ

5.3.1. Derivative with respect to τ

Consider (24) again, and let $\Delta\tau$ be a small variation of τ. The corresponding variation of $\overset{\frown}{f}$ is:

$$\overset{\frown}{f}(x,t,\tau+\Delta\tau,\theta)-\overset{\frown}{f}(x,t,\tau,\theta)=\Delta\overset{\frown}{f}=$$

$$=\frac{1}{4(\tau+\Delta\tau)\theta}\int_{x-\tau-\Delta\tau}^{x+\tau+\Delta\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv-\frac{1}{4\,\tau\,\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv \tag{28}$$

Breaking down the first term:

$$\Delta\overset{\frown}{f}=\frac{1}{4\tau(1+\frac{\Delta\tau}{\tau})\theta}\left\{\int_{x-\tau-\Delta\tau}^{x-\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv+\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv+\right.$$
$$\left.\int_{x+\tau}^{x+\tau+\Delta\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv\right\}-\frac{1}{4\,\tau\,\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv$$

An expansion at first order gives:

$$\Delta\overset{\frown}{f}\sim\frac{1}{4\,\tau\,\theta}\left(1-\frac{\Delta\tau}{\tau}\right)\left\{\Delta\tau\int_{t-\theta}^{t+\theta} f(x-\tau,v)\,dv+\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv+\Delta\tau\right.$$
$$\left.\int_{t-\theta}^{t+\theta} f(x+\tau,v)\,dv\right\}-\frac{1}{4\,\tau\,\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv$$

or, after grouping the terms:

$$\Delta \overset{\frown}{f} \sim \frac{1}{4\,\tau\,\theta}\left\{\Delta\tau \int_{t-\theta}^{t+\theta} f(x-\tau,v)\,dv - \frac{\Delta\tau}{\tau}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv + \Delta\tau \int_{t-\theta}^{t+\theta} f(x+\tau,v)\,dv\right\} + O(\Delta\tau^2)$$

Dividing by $\Delta\tau$ and tending to the limit $\Delta\tau \to 0$, we get:

$$\frac{\partial}{\partial\tau}\overset{\frown}{f}(x,t,\tau,\theta) = \frac{1}{4\,\tau\,\theta}\left\{\int_{t-\theta}^{t+\theta} f(x-\tau,v)\,dv + \int_{t-\theta}^{t+\theta} f(x+\tau,v)\,dv\right\} - \frac{1}{4\,\tau^2\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)\,du\,dv \tag{29a}$$

Using definition (25a), this expression is written more briefly:

$$\frac{\partial}{\partial\tau}\overset{\frown}{f}(x,t,\tau,\theta) = \frac{1}{2\,\tau}\left\{\overset{\frown}{f}_{x-\tau}(t,\theta) + \overset{\frown}{f}_{x+\tau}(t,\theta)\right\} - \frac{1}{\tau}\overset{\frown}{f}(x,t,\tau,\theta) \tag{29b}$$

5.3.2. Derivative with respect to θ

As with (27), it only needs to swap x and τ with t and θ:

$$\frac{\partial}{\partial\theta}\overset{\frown}{f}(x,t,\tau,\theta) = \frac{1}{2\,\theta}\left\{\overset{\frown}{f}_{t-\theta}(x,\tau) + \overset{\frown}{f}_{t+\theta}(x,\tau)\right\} - \frac{1}{\theta}\overset{\frown}{f}(x,t,\tau,\theta) \tag{30}$$

5.3.3. Second-order derivative with respect to τ and θ

This second-order derivative of the MA-ST is important, as it will enable stationary space-time trends to be defined. It is obtained by deriving (30) with respect to τ (or (29b) with respect to θ):

$$\frac{\partial^2}{\partial\tau\,\partial\theta}\overset{\frown}{f}(x,t,\tau,\theta) = \frac{1}{2\,\theta}\left\{\frac{\partial}{\partial\tau}\overset{\frown}{f}_{t-\theta}(x,\tau) + \frac{\partial}{\partial\tau}\overset{\frown}{f}_{t+\theta}(x,\tau)\right\} - \frac{1}{\theta}\frac{\partial}{\partial\tau}\overset{\frown}{f}(x,t,\tau,\theta) \tag{31}$$

The partial derivatives in the second member have already been calculated. As regards the first two, let us first recall the definition (25b):

$$\overset{\frown}{f}_t(x,\tau) = \frac{1}{2\,\tau}\int_{x-\tau}^{x+\tau} f(u,t)\,du$$

hence, in (31):

$$\overset{\frown}{f}_{t-\theta}(x,\tau) = \frac{1}{2\,\tau}\int_{x-\tau}^{x+\tau} f(u,t-\theta)du \text{ and } \overset{\frown}{f}_{t+\theta}(x,\tau) = \frac{1}{2\,\tau}\int_{x-\tau}^{x+\tau} f(u,t+\theta)du$$

The derivative of a one-dimensional *MA* has been calculated in (Padet J., 2024) (equation 11):

$$\frac{\partial}{\partial \tau}\overset{\frown}{f}(x,\tau)=\frac{1}{2\,\tau}\{f(x-\tau)+f(x+\tau)\}-\frac{1}{\tau}\overset{\frown}{f}(x,\tau)$$

That becomes here:

$$\frac{\partial}{\partial \tau}\overset{\frown}{f}_{t-\theta}(x,\tau)=\frac{1}{2\tau}\ \{f(x-\tau,\ t-\theta)+f(x+\tau\ ,t-\theta)\}-\frac{1}{\tau}\ \overset{\frown}{f}_{t-\theta}(x,\tau) \tag{33a}$$

$$\frac{\partial}{\partial \tau}\overset{\frown}{f}_{t+\theta}(x,\tau)=\frac{1}{2\tau}\ \{f(x-\tau,\ t+\theta)+f(x+\tau\ ,t+\theta)\}-\frac{1}{\tau}\overset{\frown}{f}_{t+\theta}(x,\tau) \tag{33b}$$

Carrying over (33), (32) and (29b) into (31), after a clustering of the terms, we have:

$$\frac{\partial^2}{\partial\tau\,\partial\theta}\ \overset{\frown}{f}(x,t,\tau,\theta)=$$

$$\frac{1}{\tau\,\theta}\left\{\begin{array}{c}\frac{1}{4}\ [\,f(x-\tau,t-\theta)+f(x+\tau,t-\theta)+f(x-\tau,t+\theta)\ +f(x+\tau,t+\theta)]\\ -\frac{1}{2}\left[\overset{\frown}{f}_{t-\theta}(x,\tau)+\overset{\frown}{f}_{t+\theta}(x,\tau)+\overset{\frown}{f}_{x-\tau}(t,\theta)+\overset{\frown}{f}_{x+\tau}(t,\theta)\right]\\ +\overset{\frown}{f}(x,t,\tau,\theta)\end{array}\right\} \tag{34}$$

5.4. Space, time and space-time trends

Compared with the basic two-dimensional case (x and y coordinates of the same nature), the case of space-time signals leads us to take a new look at the concept of trend. We are led to consider three types of moving averages, and therefore associated trends. These correspond to definitions (25a), (25b) and (24).

1 - At each time t let us associate the set of spatial *MA*:

$$\overset{\frown}{f}_t(x,\tau)=\frac{1}{2\,\tau}\int_{x\,-\,\tau}^{x+\tau} f(u,t)\ du \tag{35a}$$

In these, we can look for instantaneous, and therefore one-dimensional, spatial trends (statistical or stationary) according to the methods outlined in the reference (Padet J., 2024).

2 - On the other hand, to each abscissa x we can associate the set of time *M*A:

$$\overset{\frown}{f}_x\,(t,\theta)=\frac{1}{2\,\theta}\int_{t-\theta}^{t+\theta} f(x,v)\;dv \qquad (35b)$$

In a similar way, *local time trends*, also 1-D, statistical or stationary, can be detected there.

3 - *But the most interesting is to be found in the MA-ST* (24):

$$\overset{\frown}{f}\,(x,t,\tau,\theta)=\frac{1}{4\,\tau\,\theta}\int_{x-\tau}^{x+\tau}\int_{t-\theta}^{t+\theta} f(u,v)du\;dv$$

First of all, for reasons already explained in § 4.1, statistical trends will be ignored. On the other hand, the search for stationary trends offers new perspectives. Indeed, the *MA-ST* is locally stationary with respect to the observation parameters at *(x,t)* if the following condition is satisfied:

$$\frac{\partial^2}{\partial\tau\,\partial\theta}\overset{\frown}{f}\,(x,t,\tau,\theta)=0 \qquad (36a)$$

In that case, a small variation of τ and θ has no effect on the observation result $\overset{\frown}{f}\,(x,t,\tau,\theta)$. In other words, from (34), if there exist a pair $\{\tau_s(x,t);\theta_s(x,t)\}$ (or more) satisfying the equation:

$$\frac{1}{4}\left[f(x-\tau,\,t-\theta)+f(x+\tau,\,t-\theta)+f(x-\tau,t+\theta)+f(x+\tau,t+\theta)\right]-$$
$$\frac{1}{2}\left[\overset{\frown}{f}_{t-\theta}\,(x,\tau)+\overset{\frown}{f}_{t+\theta}\,(x,\tau)+\overset{\frown}{f}_{x-\tau}\,(t,\theta)+\overset{\frown}{f}_{x+\tau}\,(t,\theta)\right]+\overset{\frown}{f}\,(x,t,\tau,\theta)=0 \qquad (36b)$$

then, the $2\tau_s\times 2\theta_s$ window can be considered as an optimal space-time observation scale.

It is important to note that τ_s and θ_s are linked, since they both depend on x and t. As a result, the stationarity of the observation does not relate separately to space or time, but to a space-time window that combines them.

Consider then a set of points *(x,t)* where (36b) has a solution $\{\tau_s\,;\theta_s\}$. If it is convex, the corresponding observational results $\overset{\frown}{f}\,(x,t,\tau_s,\theta_s)$ will constitute a *stationary space-time trend* of the phenomenon described by *f(x,t)* (N.B. Isolated points or points arranged in a line do not seem to be in adequacy here with the notion of a trend).

Finally, it should be remembered that the preceding approach is not limited to space and time, and can be applied to any pair of different physical quantities.

6. Conclusion

This article presents a generalization of the trend concept, based on a two-dimensional extension of moving averages.

In a first stage, the *2-D* extension concerned *f(x,y)* files where *x* and *y* have the same physical dimension (generally a length), so essentially images and surfaces. This led to the characterization of locally stationary (self-adapting) observation windows and two-dimensional stationary trends.

A second level of generalization was then addressed, when the variables *x* and *y* are not of the same nature, a situation largely represented by space-time signals *f(x,t)*. This has made it possible to define optimal space-time observation windows (which are also self-adapting), from which stationary space-time trends can be extracted. In addition, these signals are likely to contain one-dimensional partial trends: time trends at a given position *x*, or space trends at a fixed instant *t*.

The proposed method can be applied to a wide range of situations, in all areas of physics, mechanics and information, to find characteristic or optimal observation scales. It also provides another basis for edges detection, by associating them with trends (in particular, the intersection of two stationary trends leads to the notion of contours that are themselves stationary). Finally, in numerical calculations that require many iterations, the stepwise replacement of a sequence of results by a previous trend could help to speed up convergence.

REFERENCES

Djeziri S., Petit E., Lemoine J., *Une méthode adaptative pour le lissage de courbes. Application à la détection de contours*, Traitement du signal, Vol. 14, N°1 (1997).

Girard M-C., Girard C.M., *Traitement des données de télédétection*, Dunod (1999).

Godichon-Baggioni A., *Cours de statistiques*, Ed. INSA de Rouen, France (2017).

Padet C., Padet J., *Etude d'une classe d'approximations pour un ensemble continu de donnée*, Congrès ICIAM. Paris, France (1987).

Padet C., Roche J.-M., *New methods in data analysis: potentials offered for the study of variable phenomena*, ICHMT Int. Symposium on Convective Heat Transfer, CONV96, Cesme, Turkey (1996).

Padet J., *Quelques propriétés analytiques des moyennes mobiles, et leur application à la détection de tendances dans un signal*, Entropie, Vol. 24, N°2, p. 1-20, DOI : 10.21494/ISTE.OP.2024.1079 (2024).

Rachek A., El Wakil N., Padet C., Thomas L., *Self-adapting model for edge detection and image smoothing. Convective heat transfer applications*, 2nd Int. Conf. on Computational Heat and Mass Transfer, Rio de Janeiro, Brazil (2001).

Rachek A., Padet J., Stoian M., Padet C., *A new definition of contours in images. Application in heat and mass transfer*, Annals of the New-York Academy of Sciences, 972 (1), p. 29-35, (2002).

Rachek A., *Analyse de données et analyse d'images appliquées à la mécanique des fluides et à la thermique*, Thèse de Doctorat, Université de Reims, France (2003).

Roche J.-M., Padet C., Padet J., *Non-linearity index and trends of irregular phenomena*, Int. Conf. on Information Processing and Management of Uncertainty IPMU'92, Mallorca, Spain, (1992).

Song H., *Analyse de données bidimensionnelles : tendances, structures et contours*, Thèse de Doctorat, Université de Reims, France (1998).

O EXTENSIE BI-DIMENSIONALĂ A CONCEPTULUI DE TENDINȚĂ APLICABILĂ IMAGINILOR, SUPRAFEȚELOR ȘI SEMNALELOR SPAȚIO-TEMPORALE

Analiza proceselor cu un grad mare de iregularitate se bazează pe căutarea unei tendințe, a unei curbe medii ce reprezintă tiparul general al fenomenului observant. Acest concept pare intuitiv, dar determinarea sa obiectivă prezintă o serie de probleme. Întrucât metoda mediilor mobile este una dintre cele mai utilizate pentru găsirea tendințelor unidimensionale, propunem aici extinderea acesteia la seturi de date bidimensionale și studierea unor proprietăți ale acestor medii mobile 2-D. Se arată că acestea pot fi utilizate pentru detectarea intervalelor caracteristice de observație (uniforme sau auto-adaptive), conducând la tendințe structurale în semnalul analizat. Metoda este aplicată initial funcțiilor *f(x,y)* ale căror variabile au aceeași dimensiune fizică (lungime, …). Este apoi generalizată pentru cazuri cu dimensiuni diferite (ex. semnale spațio-temporale). Aplicațiile acoperă o gamă largă de domenii (structuri turbulente în mecanica fluidelor, analiza imaginilor, caracterizarea suprafețelor, optimizarea unui process de observație, modelare multi-scală, …).