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Existence of solutions to a noncontractive caputo fractional delay integral boundary value problem

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Summary

This paper establishes the existence of solutions for a class of Caputo fractional delay integral boundary value problems (CFDIBVPs) in the Banach space $AC^1([0,1],\mathbb{R}^n)$. The presence of nonlocal boundary conditions and delay arguments produces an associated integral operator that is generally noncontractive. Using a decomposition into condensing components and Sadovskii's fixed point theorem, we obtain existence results without invoking the Banach contraction principle. Precise estimates in the $AC^1([0,1])$-norm verified continuity, boundedness, and the self-map property of the operator. An explicit example illustrates how delay and boundary terms determined the operator constant and prevented classical contractivity. The analysis extends fixed-point methods for fractional systems with memory and delay, showing that the boundary and delay structures, rather than the fractional order alone, govern the solvability of the CFDIBVP.

Key words: Caputo fractional derivative, existence of solutions, Sadovskii's fixed point theorem, measure of noncompactness

1. Introduction

Caputo fractional differential equations with delay and nonlocal boundary conditions naturally arise in models of hereditary and memory-driven phenomena. A standard approach to proving the existence and uniqueness of solutions is to reformulate the problem as a fixed-point equation and apply Banach's contraction principle (Abbas, 2011; Ahmed et al., 2021; Almeida et al., 2018; Benchohra et al., 2023; Almezel et al., 2014). This approach requires the associated solution operator to be a global contraction in the chosen function space (Kochubei and Luchko, 2019; Miller and Ross, 1993;

Oldham and Spanier, 1975). However, many fractional delay systems fail to satisfy a strict contractive bound (Cui and Zhou, 2023; Henderson and Ouahab, 2020; Rajmane et al., 2024). The Caputo kernel introduces long-range memory, while the delay and integral boundary terms amplify growth in the operator norm, so that the natural operator constant typically exceeds one. Consequently, the classical Banach framework cannot be applied directly, even though the solution operator remains continuous and bounded.

Cui and Zhou (2023) established the existence of solutions for Caputo fractional delay differential equations with nonlocal and integral boundary conditions using the Banach contraction principle and the Burton–Kirk fixed point theorem without any attempt to analyze the influence of the boundary condition on the contractivity of the solution operator.

Udo and Eze (2025) investigated existence and uniqueness for a certain Caputo fractional delay integral boundary value problem using Sadovskii's fixed point theorem for existence and a fractional Grönwall inequality for uniqueness. However, they did not demonstrate the noncontractive nature of the solution operator or the influence of the delay function, leaving a gap in the theory that this work addresses.

Sivaranjani et al. (2025) examined the controllability of an implicit fractional system including state and control delay in terms of $\mathbb{ABC}$, a non-local derivative of the form

$$\begin{cases} {}_0^{\mathbb{ABC}}D_t^{\alpha}[s(t) - g(t, s(t))] = As(t) + Bc(t - \tilde{t}) + f(t, s_{t,0}\ {}^{\mathbb{ABC}}D^{\alpha}s(t), c(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0, T] = \mathcal{J}, 0 < \alpha \leqslant 1, \\ s(t) = \varphi(t), \quad t \in [-\tilde{t}, 0], \\ c(t) = c_0(t), \quad t \in [-\tilde{t}, 0] \end{cases}$$

where ${}_0^{\mathbb{ABC}}D_t^{\alpha}$ denotes the $\mathbb{ABC}$ fractional derivative with order α, $A : D(A) \subset \Omega \to \Omega$ is an infinitesimal generator of the α-resolvent family τ_α, and S_α for $t \geqslant 0$ is the solution operator on the Banach space $(\Omega, \|\cdot\|)$. k-set contraction mapping and the Darbo fixed point theorem were applied to obtain the controllability results.

Karthikeyan et al. (2023) demonstrated the existence, well-posedness and controllability of a second-order abstract differential system with state-delay using the combination of a measure of noncompactness and Sadovskii's fixed point theorem. However, they did not demonstrate the noncontractive nature of the solution operator utilized.

Karthikeyan et al. (2022a) investigated the existence and uniqueness of solutions to multi-point boundary value problems for sequential impulsive fractional differential equations involving the ψ-Hilfer fractional derivative using the Banach fixed point theorem, Krasnoselskii, and the nonlinear alternative of the Leray–Schauder fixed point theorem. Karthikeyan et al. (2022b) applied the contraction mapping concept and Schaefer's fixed point theorem to establish more generalized solutions to impulsive integro-differential equations involving Ulam–Hyers–Rassias stability results via a ψ- fractional derivative with boundary value conditions. Karthikeyan et al. (2021) studied a second-order functional abstract impulsive differential equation with state dependent delay. He applied a control to determine a controllability result, then applied Sadovskii's fixed point theorem to obtain sufficient conditions for controllability and well-posedness.

Manigandan et al. (2024) studied existence and uniqueness for boundary value problems defined by coupled sequential fractional differential inclusions:

$$\begin{cases} \left({}^cD^{\alpha_1} + K_1\,{}^cD^{\alpha_1-1}\right)X_1(t) = F_1\big(t, X_1(t), \Upsilon_1(t)\big), & t \in [0,1], \\ \left({}^cD^{\alpha_2} + K_2\,{}^cD^{\alpha_2-1}\right)\Upsilon_1(t) = F_2\big(t, X_1(t), \Upsilon_1(t)\big), & t \in [0,1], \end{cases}$$

with boundary conditions

$$\begin{cases} X_1(0) + \Upsilon_1(0) = -\big(X_1(T) + \Upsilon_1(T)\big), \\ \beta \frac{\rho^{1-q}}{\Gamma(q)} \int_0^{\xi} \frac{\varphi^{\rho-1}}{\xi^{\rho}-\varphi^{\rho}} (X_1 + \Upsilon_1)\varphi \, d\varphi = Q, \end{cases} \tag{1}$$

where F_1, F_2 are continuous, ρI^q is a generalized Riemann–Liouville fractional integral, and Υ_1, Υ_2 map into nonempty subsets of $\mathbb{R}$. They used Carathéodory functions and Lipschitz mappings with the Leray–Schauder fixed point theorem, but did not examine the noncontractive nature of the operator.

Similarly, Buvaneswari et al. (2024) applied Banach contraction and Schaefer's fixed point theorem to a coupled system of Caputo–Hadamard fractional differential equations, without investigating how noncontractivity and boundary conditions affected the system's local Lipschitz properties.

Other authors (Ahmed et al., 2021; Mardanov et al., 2022; Zhou and Huang, 2018) focused their attention on solving fractional differential equations under boundary value conditions, but generally did not analyze the influence of the fractional kernel or boundary terms on noncontractivity conditions from our perspective. To the best of our knowledge, only a few studies have rigorously addressed this aspect of confirming the boundedness and noncontractive behavior of the solution operator with different delay and boundary conditions.

Motivated by the gaps in the aforementioned research publications, we examined the noncontractive regime of a Caputo fractional delay integral boundary value problem whose solution operator is continuous and bounded but may not necessarily be a contraction. By decomposing the operator into a contraction and a compact component, we apply Sadovskii's fixed point theorem to establish the existence of solutions beyond the reach of standard contraction arguments, and clarify the interplay between contractive and noncontractive dynamics in Caputo fractional delay systems in the Banach space of absolutely continuous functions with m-th derivative, $AC^m([0,1],\mathbb{R}^n)$:

$$\begin{cases} {}^cD^{\alpha}_{0+}u(t) = f\big(t, u(t), u(t-\tau)\big), \quad \alpha \in (0,1), \;\; t \in [0,1], \;\; \tau > 0, \\ u(0) = 0, \quad u(1) = \int_0^1 g(s, u(s), u(s-\tau))\, ds, \end{cases} \tag{2}$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo derivative of order α, $(t-\tau)$ is the delay term, and $f, g, u \in AC^m([0,1],\mathbb{R}^n)$ are continuous functions, with an associated norm in $AC^m([0,1],\mathbb{R}^n)$ defined by

$$\|u\|_{AC^m} := \sum_{k=0}^{m-1} \sup_{t\in[0,1]} \|u^{(k)}(t)\| + \int_0^1 \|u^{(m)}(t)\|\, dt, \quad k = 0, \dots, m-1. \tag{3}$$

This paper is organized as follows. Section 2 presents useful definitions and lemmas related to existence, fixed point theory, and Lipschitz contraction in $AC^1([0,1],\mathbb{R}^n)$. Section 3 establishes sufficient and necessary conditions for the existence of solutions. Section 4 provides examples illustrating our results. Finally, section 5 discusses and concludes the work.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.

Definition 1. *Kilbas et al. (2006). A function $f : [a,b] \to \mathbb{R} \in AC^m([a,b])$ is absolutely continuous if:*

i. $f^{(m-1)} \in AC([a,b])$,

ii. $f^{(m)}$ *exists almost everywhere and* $f^{(m)} \in L^1([a,b])$. *Hence,*

$$f \in AC^m([a,b]) \iff f^{(m-1)} \text{ is absolutely continuous on } [a,b].$$

In the vector-valued case $AC^m([a,b],\mathbb{R}^n)$, *each component satisfies the same conditions.*
A standard norm on $AC^m([a,b],\mathbb{R}^n)$ *is defined as*

$$\|f\|_{AC^m} := \sum_{k=0}^{m-1} \sup_{t\in[a,b]} \|f^{(k)}(t)\| + \int_a^b \|f^{(m)}(t)\|dt,$$

where $\|\cdot\|$ *is the Euclidean norm on* $\mathbb{R}^n$. *Equipped with this norm,* $AC^m([a,b],\mathbb{R}^n)$ *is a Banach space.*

Definition 2. *Podlubny (1999). The Riemann–Liouville fractional Integral of order* $\alpha \in (0,1)$ *of a function* $f : (0,\infty) \to \mathbb{R}$ *is defined as*

$$(I^{\alpha}_{0^+} f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad x \in [a,b].$$

Definition 3. *Caputo and Mainardi (1971). The Caputo fractional derivative of order* $\alpha \in (0,1)$ *for a function* $f \in AC^m([a,b],\mathbb{R}^n)$ *is defined by*

$$({}^cD^{\alpha}_{a^+} f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad x \in [a,b],$$

where $m = [\alpha]+1$ *if* $\alpha \notin \mathbb{N}_0$, *otherwise* $m = \alpha$ *and* $[\alpha]$ *is the integer part of* α *that is not greater than* α.

Definition 4. *Zeidler (1986). Let* $(X, \|\cdot\|)$ *be a normed linear space,* $x_0 \in X$ *and* $r > 0$; *the closed ball of radius* r *centered at* x_0 *is defined by*

$$\bar{B}_r(x_0) := \{y \in X : \|y - x_0\| \leqslant r\}. \tag{4}$$

Definition 5. *Samko et al. (1993). Let* $(X, \|\cdot\|)$ *be a Banach space and* $B \subset X$ *a bounded subset. The Kuratowski measure of noncompactness of* B *is defined as:*

$$\alpha(B) := \inf\Big\{\varepsilon > 0 : B \subset \bigcup_{i=1}^{n} B_i,\ diam(B_i) \leqslant \varepsilon,\ i = 1,\cdots,n,\ n \in \mathbb{N}\Big\},$$

where the diameter of a set $B_i \subset X$ is given by diam$(B_i) := \sup\{\|x - y\| : x, y \in B_i\}$

Definition 6. *Georgiev and Zennir (2020) An operator $T : D(T) \subset X \to X$ is called condensing if for every bounded set $A \subset D(T)$ with $\alpha(A) > 0$, we have*

$$\alpha(T(A)) < \alpha(A), \tag{5}$$

where α is the Kuratowski measure of noncompactness.

Theorem 7. *Sadovskii (1967). (Sadovskii's Fixed Point Theorem) Let B_r be a convex, closed, and bounded subset of a Banach space X. If $T : B_r \to B_r$ is condensing, then T has at least one fixed point in B_r.*

3. Main result

Theorem 8. *Udo and Eze (2025). (Existence of a solution)*
Let $X = AC^1([0,1], \mathbb{R}^n)$ be endowed with the norm:

$$\|u\|_{AC^1} := \sup_{t\in[0,1]} \|u(t)\| + \int_0^1 \|u^{'}(t)\|\, dt.$$

For fixed constants $\tau, r > 0$, consider $B_r := \{u \in X : \|u\|_{AC^1} \leqslant r\}$, a closed, bounded and convex subset.
Suppose the nonlinear functions $f, g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and let $\varphi : [-\tau, 0] \to B_r$, be a given delay function satisfying $u(0) = u_0 \in B_r$ and $\varphi(-\tau) = 0$. We extend the delay by $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$.
We define the operator $T : B_r \to X$

$$(Tu)(t) := I^\alpha[f(s, u(s), u(s-\tau))](t) + h(t)\Big(G(u) - A(u)\Big), \tag{6}$$

$$\text{where} \quad G(u) := \int_0^1 g(s, u(s), u(s-\tau))ds$$

$$A(u) := I^\alpha[F_u] \quad h \text{ is the given boundary functional. Assume:}$$

(H1) **Continuity**
The nonlinear functions $f, g : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous in all arguments and satisfy a local Lipschitz condition on bounded sets: for each $r > 0$ there exist $L_1, L_2 \geqslant 0$ such that

$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\| \leqslant L_1\|u_1-u_2\|+L_2\|v_1-v_2\|$$

for all $t\in[0,1]$ *and all* u_i, v_i *with* $\|u_i\|, \|v_i\| \leqslant r$.
The boundary functional $h:[0,1]\to\mathbb{R}^n$ *is continuous with* $h(0)=0$, $h(1)=1$.

(H2) **Boundedness**
There exist nonnegative constants $L_{f_1}, L_{f_2}, L_{g_1}, L_{g_2}$ *(depending on* r*) such that for all* $t\in[0,1]$ *and all* u_1,u_2,$v_1, v_2\in B_r$*, we have the standard componentwise Lipschitz bounds*

$$\|f(t,u_1,u_2)-f(t,v_1,v_2)\| \leqslant L_{f_1}\left(\|u_1-v_1\|+L_{f_2}\|u_2-v_2\|\right),$$
$$\|g(t,u_1,u_2)-g(t,v_1,v_2)\| \leqslant L_{g_1}\left(\|u_1-v_1\|+L_{g_2}\|u_2-v_2\|\right).$$

(H3) **Self-map**
The operator $T:B_r\to B_r$ *maps the ball into itself: such that* $T(B_r)\subseteq B_r$.

(H4) **Condensing decomposition**
T *is* α*-condensing with respect to a Kuratowski measure of noncompactness* α*;*

$$\alpha(T(B))<\alpha(B), \quad \text{For every bounded } B\subseteq B_r \text{ with } \alpha(B)>0.$$

Moreover, T *admits the decomposition*

$$T=T_1+T_2,$$

where T_1 *is a contraction and* T_2 *is compact. If conditions (H1) and (H4) hold, then the CFDIBVP equation* (2) *admits at least one solution* $u\in AC^1([0,1],\mathbb{R}^n)$.

Proof of theorem (8)
Hypothesis [H1] – Continuity.
To prove hypothesis [H1] – **Continuity:**
Considering all the conditions of theorem (8), define the operator $T:B_r\to X$ by equation (6).

There exist nonnegative constants $M_{f_1}, M_{f_2}, M_{g_1}, M_{g_2}$ (depending on r) such that for all $t\in[0,1]$ and all u_1,u_2,$v_1, v_2\in B_r$, we have the standard componentwise Lipschitz bounds

$$\|f(t,u_1,u_2)-f(t,v_1,v_2)\| \leqslant M_{f_1}\left(\|u_1-v_1\|+M_{f_2}\|u_2-v_2\|\right),$$
$$\|g(t,u_1,u_2)-g(t,v_1,v_2)\| \leqslant M_{g_1}\left(\|u_1-v_1\|+M_{g_2}\|u_2-v_2\|\right).$$

We set the combined constants:

$M_f := M_{f_1}+M_{f_2}$,

$M_g := M_{g_1}+M_{g_2}$.

Proof: [H1] – Continuity:
Let $u_n \in B_r$ and $u_n \to u$ in $\|u\|_{AC^1}$. Then $\sup_{t\in[0,1]}\|u_n(t)-u(t)\| \to 0$, $\sup_{t\in[0,1]}\|u_n(t-\tau)-u(t-\tau)\| \to 0$
Define:

$$f_n(s) := f(s,u_n(s),u_n(s-\tau)),$$
$$g_n(s) := g(s,u_n(s),u_n(s-\tau)).$$

By continuity of f,g and uniform convergence of u_n and of the delay terms,

$$\|f_n - f(s,u(s),u(s-\tau))\|_{L^\infty(0,1)} \to 0$$
$$\|g_n - g(s,u(s),u(s-\tau))\|_{L^\infty(0,1)} \to 0.$$

Set $\quad M_n := \sup\limits_{s\in[0,1]}\|f_n(s)-f(s,u(s),u(s-\tau))\| \to 0$

and $\quad K_n := \sup\limits_{s\in[0,1]}\|g_n(s)-g(s,u(s),u(s-\tau))\| \to 0$

For the fractional integral term, we use the kernel estimate and linearity of I^α for each $t\in[0,1]$:

$$\begin{aligned}
&\|I^\alpha[f_n](t) - I^\alpha[f(s,u(s),u(s-\tau))]\| \\
&\qquad \leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f_n(s)-f(s,u(s),u(s-\tau))\|\,ds \\
&\qquad \leqslant \frac{M_n}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds = \frac{M_n t^\alpha}{\Gamma(\alpha+1)} \\
&\qquad \leqslant \frac{M_n}{\Gamma(\alpha+1)}
\end{aligned}$$

thus $\quad I^\alpha[f_n] \to I^\alpha[f(s,u(s),u(s-\tau))] \quad$ uniformly on $[0,1]$ (7)

For boundary functions G and A, using uniform convergence,

$$\begin{aligned} G(u_n) &= \int_0^1 g_n(s)ds \to \int_0^1 g(s, u(s), u(s-\tau))\, ds \\ &= G(u). \\ \text{And} \quad A(u_n) &= I^{\alpha} \Rightarrow I^{\alpha} = A(u) \end{aligned}$$

which confirms uniform convergence on $[0, 1]$ from equation (7).
We consider the boundary residual function (h), for $h \in AC^m([0, 1],\ \mathbb{R}^n)$, which is continuous and bounded.

$$\begin{aligned} &\text{Set} \quad H := \sup_{t\in[0,1]} \|h(t)\| < \infty, \quad \text{then for each t, we have:} \\ &\qquad \|h(t)\big(G(u_n) - A(u_n)\big) - h(t)\big(G(u_n) - A(u_n)\big)\| \\ &\qquad \leqslant \|h(t)\|\big(|G(u_n) - G(u)| + |A(u_n) - A(u)|\big) \\ &\qquad \leqslant H\big(|G(u_n) - G(u)| + |A(u_n) - A(u)|\big) \to 0 \\ &\qquad \text{uniformly in t, since} \\ &\qquad G(u_n) \to G(u) \\ &\quad \text{and} \quad A(u_n) \to A(u) \end{aligned}$$

Consequently, for $t \in [0, 1]$ we have:

$$\begin{aligned} &\|Tu_n(t) - Tu(t)\| \leqslant \|I^{\alpha}[f_n](t) - I^{\alpha}\big[f(s, u(s), u(s-\tau))\big]I^{\alpha}\| \\ &\leqslant \|h(t)\|\big(|G(u_n) - G(u)| + |A(u_n) - A(u)|\big). \\ &\text{By uniform convergence of both terms} \\ &\sup_{t\in[0,1]} \|Tu_n(t) - Tu(t)\| \to 0 \\ &\text{Thus} \quad Tu_n \to Tu \quad \text{uniformly} \end{aligned}$$

For convergence of derivatives:
Differentiating the integral term almost everywhere

$$\begin{aligned} &\text{For} \quad \int_0^1 \|(Tu_n)^{'}(t) - (Tu)^{'}\|\, dt \\ &\text{Let} \quad \phi_n(s) := f(s, u_n(s), u_n(s-\tau)) \in AC([0, 1], \mathbb{R}^n), \\ &\phi(s) := f(s, u(s), u(s-\tau)) \\ &\text{so} \quad \phi^{'}(s) = L^1([0, 1], \mathbb{R}^n), \quad \text{for } \phi \in AC([0, 1]), \end{aligned}$$

Applying integration by parts such that:

$u = \phi(s),\ dv = (t-s)^{\alpha-1}ds,\ v = \dfrac{-(t-s)^{\alpha}}{\alpha}$ and $du = \phi^{'}(s)ds$

Thus $\displaystyle\int_0^1 (t-s)^{\alpha-1}\phi(s)ds = \left[\frac{-\phi(s)(t-s)^{\alpha}}{\alpha}\right]_{s=0}^{s=t} + \frac{1}{\alpha}\int_0^t (t-s)^{\alpha}\phi^{'}(s)ds$

For $\phi^{'}(s) \in L^1([0,1],\mathbb{R}^n)$

Then $\displaystyle\frac{d}{dt}I^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}\phi^{'}(s)ds - \frac{\phi(0)t^{\alpha-1}}{\Gamma(\alpha)},$

therefore $\displaystyle(Tu_n)^{'}(t) - (Tu)^{'}(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}(\phi_n^{'}(s) - \phi^{'}(s))ds$

$\displaystyle - \frac{\phi_n(0) - \phi(0)}{\Gamma(\alpha)}t^{\alpha-1} + h^{'}(t)(G(u_n) - A(u_n))$

since $h(t)$ is independent of s and differentiates to $h^{'}(t)$ while $G(u)$, $A(u)$ are constants with respect to t. We take the norm and integrate over $t \in [0,1]$ using the triangle inequality:

$$\int_0^1 \|(Tu_n)^{'}(t) - (Tu)^{'}\|\, dt \leqslant \frac{1}{\Gamma(\alpha)}\int_0^1 \|\int_0^t (t-s)^{\alpha-1}(\phi_n^{'}(s) - \phi^{'}(s))ds\|dt$$
$$+ \frac{|\phi_n(0) - \phi(0)|}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}dt + \int_0^1 \|h^{'}(t)\||(G(u_n) - G(u)) - (A(u_n) - A(u))|dt$$

We now estimate the three terms on the right-hand side separately:
Term 1 (Double Integral Term):
Applying the Fubini theorem to exchange the order of integration:

$$\int_0^1 \|\int_0^t (t-s)^{\alpha-1}(\phi_n^{'}(s) - \phi^{'}(s))ds\|dt \leqslant \int_0^1\int_0^t (t-s)^{\alpha-1}\|\phi_n^{'}(s) - \phi^{'}(s)\|dsdt$$
$$\leqslant \frac{1}{\Gamma(\alpha+1)}\|\phi_n^{'}(s) - \phi^{'}(s)\|_{L^1(0,1)}$$

Term 2 (Delay Function Term): We compute the t-integral

$$\int_0^1 t^{\alpha-1}dt = \frac{1}{\alpha} \quad \text{Hence,} \quad \frac{|\phi_n(0)-\phi(0)|}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}dt = \frac{|\phi_n(0)-\phi(0)|}{\Gamma(\alpha+1)}$$

Term 3 (Boundary Residue Term): Factoring the t-independent difference:

$$\int_0^1 \|h^{'}(t)\|\big|\big(G(u_n)-G(u)\big)-\big(A(u_n)-A(u)\big)\big|dt$$
$$\|h^{'}(t)\|_{L^1(0,1)}\big(|G(u_n)-G(u)|-|A(u_n)-A(u)|\big)$$
$$G(u_n)\to G(u), \quad A(u_n)\to A(u) \quad \text{as } n\to\infty$$

Since f and g are continuous and I^α is continuous from $L^1[0,1]$ into $C([0,1])$, then:

$$\int_0^1 \|h^{'}(t)\|\big|\big(G(u_n)-G(u)\big)-\big(A(u_n)-A(u)\big)\big|dt \to 0$$

Collecting all three terms together, we obtain:

$$\begin{aligned}&\int_0^1 \|(Tu_n)^{'}(t)-(Tu)^{'}\|\,dt\\ &\leqslant \frac{1}{\Gamma(\alpha+1)}\|\phi_n^{'}(s)-\phi^{'}(s)\|_{L^1(0,1)}+\frac{|\phi_n(0)-\phi(0)|}{\Gamma(\alpha+1)}\\ &+\|h^{'}(t)\|_{L^1(0,1)}\big(|G(u_n)-G(u)|+|A(u_n)-A(u)|\big).\end{aligned}$$

We know that:

i. $u_n\to u$ in AC^1 implies $u_n\to u$ uniformly and $u_n^{'}\to u^{'}$ in $L^1([0,1])$
ii. For $f(s,u,v)\in C^1([0,1])$ on the relevant compact set:

$$\int_0^1 \|(Tu_n)^{'}(t)-(Tu)^{'}\|\,dt \to 0 \quad (\text{as } n\to\infty)$$

Together with the previously established uniform convergence $Tu_n\to Tu$ on $[0,1]$,
we obtain:

$$\|Tu_n-Tu\|_{AC^1} = \sup_{t\in[0,1]}\|Tu_n(t)-Tu(t)\|_{AC^1}+\int_0^1 \|(Tu_n)^{'}(t)-(Tu)^{'}\|\,dt \to 0$$

Hence, T is continuous on B_r in the AC^1-norm □

Lemma 9. *(Mardanov et al. (2022) L^1-estimate for the derivative of a fractional integral) Let $\alpha \in (0,1)$, and $\varphi \in L^\infty(0,1)$ and set $(I^{\alpha\varphi})(t) := \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}\varphi(s)ds$, $t \in [0,1]$. Then $I^{\alpha\varphi}$ is absolutely continuous on $[0,1]$, and its derivative exists almost everywhere and is given by:*

$$\begin{aligned}
&\frac{d}{dt}(I^\alpha\varphi)(t) = I^{\alpha-1}\varphi(t),\\
&\text{there is a finite constant } C(\alpha) > 0, \text{depending only on } \alpha\\
&\int_0^1 \left|\frac{d}{dt}(I^\alpha\varphi)(t)\right| = \int_0^1 \left|I^{\alpha-1}\right|(t)dt \leqslant C(\alpha)\\
&\leqslant \frac{2}{\Gamma(\alpha+1)}\|\varphi\|_\infty
\end{aligned} \tag{8}$$

Proof of [H2] (Boundedness)
By the continuity of f and g from hypothesis $[H1]$, the functions f and g attain maxima on K. Hence, there exist finite constants:

$$\begin{aligned}
M_f &= \max_{(t,u,v)\in K}\|f(t,u,v)\| > 0\\
M_g &= \max_{(t,u,v)\in K}\|g(t,u,v)\| > 0
\end{aligned}$$

These constants are independent of the particular $u \in B_r$. Hence, we can now directly apply M_f and M_g in our proof of boundedness.
Step 1: Bound for $\int_0^1\|(Tu)(t)\|dt$
For each $t \in [0,1]$, the solution operator T is:

$$(Tu)(t) = I^\alpha[f(t,u(s),u(s-\tau))](t) + h(t)\Big(G(u) - A(u)\Big), \tag{9}$$

therefore the pointwise estimate is:

$$\begin{aligned}
\|(Tu)(t)\| \leqslant& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,u(s),u(s-\tau))ds\|\\
&+ \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\|f(s,u(s),u(s-\tau))\|ds\\
&+ t\int_0^1 \|g(s,u(s),u(s-\tau))\|ds
\end{aligned}$$

By applying the following compactness uniform bounds:

$$\|f(s,u(s),u(s-\tau))\|_\infty \leqslant M_f,$$
$$\|g(s,u(s),u(s-\tau))\|_\infty \leqslant M_g.$$

we obtain, for every $t \in [0,1]$,

$$\begin{aligned}
&\leqslant \frac{M_f}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{tM_f}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds + tM_g \\
&= M_f \left(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{2\Gamma(\alpha+1)} \right) + \frac{M_g}{2}.
\end{aligned}$$

Now applying the standard L^1-estimate for the derivative of a fractional integral:
Step 2: Bound for $\int_0^1 \|u^{'}(t)\| dt$ (Derivative of Tu using Lemma (9)):

$$(Tu)^{'}(t) = \Big(I^\alpha [f(s,u(s),u(s-\tau))] \Big)^{'}(t) + \Big(h(t)(G(u) - A(u)) \Big)^{'}$$

Then the norm estimate becomes:

$$\begin{aligned}
\|(Tu)^{'}(t)\| &\leqslant \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \|f(s,u(s),u(s-\tau))\| ds \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \|f(s,u(s),u(s-\tau))\| ds \\
&\quad + \int_0^1 \|g(s,u(s),u(s-\tau))\| ds \\
&\leqslant \frac{M_f(1-\alpha)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} ds + \frac{M_f}{\Gamma(\alpha)} \cdot \frac{1}{\alpha} + M_g \\
&\leqslant \frac{M_f t^{\alpha-1}}{\Gamma(\alpha)} + \frac{M_f}{\Gamma(\alpha+1)} + M_g
\end{aligned}$$

Integrating over $[0,1]$, from Lemma (9) and equation (8) we obtain:

$$\begin{aligned}
\int_0^1 \|(Tu)^{'}(t)\| \, ds &\leqslant \frac{M_f}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} dt + \left(\frac{M_f}{\Gamma(\alpha+1)} + M_g \right) \int_0^1 dt \\
&= \Big(\frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} \Big) M_f + M_g
\end{aligned}$$

$$= \frac{3M_f}{\Gamma(\alpha+1)} + M_g$$

Step 3: (Combine for AC^1-norm):

$$\begin{aligned}\|u\|_{AC^1} &= \int_0^1 \|u(t)\|dt + \int_0^1 \|u^{'}(t)\|dt \\ &\leqslant M_f\left(\frac{1}{\Gamma(\alpha+2)} + \frac{1}{2\Gamma(\alpha+1)}\right) + \frac{M_g}{2} + \frac{3M_f}{\Gamma(\alpha+1)} + M_g\end{aligned}$$

Therefore the final bound is

$$\leqslant M_f\left(\frac{1}{\Gamma(\alpha+2)} + \frac{7}{2\Gamma(\alpha+1)}\right) + \frac{3M_g}{2}$$

Hence, T is bounded, proving the boundedness hypothesis [H2] □

Proof: ([H3] Self-Map $T : B_r \to B_r$**)**
Since hypothesis [H1] (Continuity) and [H2] (Boundedness) hold, there exist constants
$M_f, M_g > 0$ for all $t \in [0,1]$. Let $h \in AC^m \subset C([0,1], \mathbb{R}^n)$ satisfy $h(0) = 0, h(1) = 1$.
We set

$$H := \sup_{t\in[0,1]} \|h(t)\| < \infty$$

Then there exists $r > 0$ depending only on M_f, M_g, α, H. From the proofs of [H1] and [H2] (Continuity and Boundedness), for every $u \in B_r$, we have $\|Tu\|_{AC^1} \leqslant r$, which implies that $Tu \in B_r$. Hence, $T(B_r) \subseteq B_r$. □

Proof: ([H4] − Condensing-Decomposition $T = T_1 + T_2$**)**
Fix $r > 0$, $h \in AC^m \subset C([0,1], \mathbb{R}^n)$ satisfying $h(0) = 0, h(1) = 1$, and set

$$H := \sup_{t\in[0,1]} \|h(t)\| < \infty$$

Then there exists $r > 0$ depending only on M_f, M_g, α, H.
Define the solution operator $T : B_r \to X$ by the solution operator (9)

decomposed to:

$$T = T_1 + T_2 \tag{10}$$

where T_1 is Lipschitz and T_2 is compact. Hence, T is condensing with respect to the Kuratowski measure of noncompactness α, for $t \in [0,1]$

Step 1: T_1 is Lipschitz on B_r in $\|u\|_{AC^1}$-norm

Assume f and g satisfy the Lipschitz condition:

i. Where f, g are continuous functions and satisfy Lipschitz conditions, then there exist non-negative constants $M_{f_1}, M_{f_2}, M_{g_1}, M_{g_2}$ (depending on r); for all $t \in [0,1]$ and all u_1,u_2,$v_1, v_2 \in B_r$, we have the standard componentwise Lipschitz bounds:

$$\|f(t,u_1,u_2) - f(t,v_1,v_2)\| \leqslant L_{f_1}\left(\|u_1 - v_1\| + L_{f_2}\|u_2 - v_2\|\right),$$
$$\|g(t,u_1,u_2) - g(t,v_1,v_2)\| \leqslant L_{g_1}\left(\|u_1 - v_1\| + L_{g_2}\|u_2 - v_2\|\right).$$

We set the combined constants:

$L_f := L_{f_1} + L_{f_2}$,

$L_g := L_{g_1} + L_{g_2}$.

ii. $u \in AC^1([0,T],\mathbb{R}^n)$ with $u(0) = 0$.

Evaluate:

$$\|T_1u(t) - T_1v(t)\| \leqslant \frac{L_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\|u(s) - v(s)\| + \|u(s-\tau) - v(s-\tau)\|)ds$$

Split the integral at $s = \tau$:

i. For $s \in [0, min(t,\tau)]$, $\|u(s-\tau) - v(s-\tau)\| = 0$ (since $u = v = \phi$ on $[-\tau, 0]$).

ii. For $s \in (0, min(t,\tau))$, $\|u(s-\tau) - v(s-\tau)\| \leqslant \|u - v\|_{AC^1}$

Thus:

$$\|T_1u(t) - T_1v(t)\| \leqslant \frac{L_f}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\|u(s) - v(s)\|ds + \int_{min(t,\tau)}^t (t-s)^{\alpha-1}\|u(s-\tau) - v(s-\tau)\|\,ds\Big)$$

$$\leqslant \frac{2L_f}{\Gamma(\alpha+1)}\|u-v\|_{AC^1}$$

Thus integrating over $[0,1]$:

$$\int_0^1 \|T_1u - T_1v\|_{AC^1}\, dt \ \leqslant \frac{2L_f}{\Gamma(\alpha+1)}\|u-v\|_{AC^1} \tag{11}$$

Evaluating the derivative of T_1u:

$$(T_1u)^{'}(t) = \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2} f(s, s(u), u(s-\tau))ds,$$

Applying the Lipschitz condition:

$$\begin{aligned}\|(Tu)^{'}(t) - (Tv)^{'}(t)\| &\leqslant \frac{(1-\alpha)L_f}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-2}\|u(s)-v(s)\| \\ &\quad + \|u(s-\tau)-v(s-\tau)\|\Big)ds \\ &\leqslant \frac{2L_f}{\Gamma(\alpha-1)}\|u-v\|_{AC^1}\int_0^t (t-s)^{\alpha-2}ds\end{aligned}$$

Evaluating and Integrating over $[0,1]$:

$$\int_0^1 \|(Tu)^{'} - (Tv)^{'}\|\, dt \leqslant \frac{2L_f}{\Gamma(\alpha+1)}\|u-v\|_{AC^1} \tag{12}$$

Combining equation (11) and equation (12), we obtain:

$$\|T_1u - T_1v\|_{AC^1}\, dt \ + \int_0^1 \|(Tu)^{'} - (Tv)^{'}\|$$

$$\begin{aligned}&\leqslant \left(\frac{2L_f}{\Gamma(\alpha+1)} + \frac{2L_f}{\Gamma(\alpha+1)}\right)\|u-v\|_{AC^1} \\ &\leqslant \frac{4L_f}{\Gamma(\alpha+1)}\|u-v\|_{AC^1}\end{aligned}$$

Thus T_1 is a contraction if:

$$\frac{4L_f}{\Gamma(\alpha+1)} < 1$$

Step 2: T_2 **(Compactness)**
Let $B \subset B_r$ be bounded. We show that $T_2(B)$ is relatively compact in X. We have to prove the boundedness of $\varphi(B)$. For B bounded in X and f, g continuous on B_r, a set $\nu := \{\varphi(u) : u \in B\} \subset \mathbb{R}^n$ is bounded.

Let $M_\varphi := \sup\{\|\varphi(u)\| : u \in B\} < \infty$
For $u \in B$, $(T_2u)(t) = h(t)\varphi(u)$, then
$$\frac{d}{dt}(T_2u)(t) = h'(t)\varphi(u)$$
Since h, h' are continuous on $[0,1]$, we set:
$$H_0 := \sup_{t\in[0,1]} \|h(t)\|,$$
$$H_1 := \sup_{t\in[0,1]} \|h'(t)\|.$$
Then for every $u \in B$ and $t \in [0,1]$,

$$\|T_2u(t)\| \leqslant H_0 M_\varphi,$$
$$\left\|\frac{d}{dt}T_2u(t)\right\| \leqslant H_1 M_\varphi$$

Hence, $\{T_2u : u \in B\}$ is equibounded and has uniformly bounded derivatives (equi-Lipschitz). In particular, the family is equicontinuous and uniformly bounded in $C([0,1], \mathbb{R}^n)$. **For finite-dimensional bound:**
For each $u \in B$:

$$T_2u(t) = h(t)\varphi(u)$$
then $\varphi(u) = \sum_{j=1}^{n} C_j(u)\ell_j$ (in the standard basis),

we have $T_2u(t) = \sum_{j=1}^{n} C_j(u)h(t)\ell_j,$

we have
$$T_2(B) \subset M := \{h(t)\ell_j, \cdots, h(t)\ell_n\} \in C([0,1], \mathbb{R}^n).$$

Consequently, bounded subsets of a finite-dimensional subspace are relatively compact in every Banach norm. In particular, they are relatively compact in $\|\cdot\|_{AC^1}$.

Thus, $T_2(B)$ is relatively compact in X.
Therefore:

$$\alpha(T_2(B)) = 0.$$

We apply the Kuratowski inequality and the condensing property (subadditivity and the two results above). For any bounded $B \subset B_r$, we have:

$$\begin{aligned}\alpha(T(B)) &= \alpha(T_1(B) + T_2(B)) \leqslant \alpha(T_1(B)) + \alpha(T_2(B)) \\ &\leqslant k\alpha(B) + 0 \\ &= k\alpha(B).\end{aligned}$$

Since T_1 is a contraction ($k < 1$) and T_2 is compact ($\alpha(T_2) = 0$),
thus T is condensing. Hence, under the given conditions and assuming
f is Lipschitz with constant L_f satisfying

$$\frac{4L_f}{\Gamma(\alpha + 1)} < 1.$$

Therefore the operator T is condensing with decomposition equation (10) where T_1 is a contraction and T_2 is compact. □
By Sadovskii's fixed point theorem we conclude that for $T : B_r \to B_r$:

i. B_r is closed, bounded and convex.

ii. T is continuous and α-condensing on B_r, whenever $k < 1$.

iii. $T(B_r) \subseteq B_r$.

We therefore conclude that there exists at least one fixed point $u \in B_r$ such that:
$Tu = u$.
Hence, $u \in AC^1([0, 1], \mathbb{R}^n)$ is a solution of the generalized Caputo fractional delay integral boundary value problem as posed. Therefore a solution exists. □

Proof of noncontractivity

Theorem 10 (Noncontractive hypotheses)**.** *Assume that the nonlinearities $f, g : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the pointwise Lipschitz bounds on B_r:*

$$\begin{aligned}\|f(t, u_1, u_2) - f(t, v_1, v_2)\| &\leqslant L_{f_1}\|u_1 - v_1\| + L_{f_2}\|u_2 - v_2\|, \\ \|g(t, u_1, u_2) - g(t, v_1, v_2)\| &\leqslant L_{g_1}\|u_1 - v_1\| + L_{g_2}\|u_2 - v_2\|.\end{aligned}$$

For $L := L_{f_1} + L_{f_2}, \qquad G := L_{g_1} + L_{g_2}.$

Define the operator $T : B_r \to AC^1([0,1], \mathbb{R}^n)$ *by the fractional integral operator* (9) *for which the fixed point is a solution of the CFDIBVP. Then for all* $u, v \in B_r$ *we have*

$$\|Tu - Tv\|_{AC^1} \leqslant k_{\text{op}} \, \|u - v\|_{AC^1},$$

where the computed operator constant is

$$k_{\text{op}} = \Big(\frac{1}{\Gamma(\alpha+2)} + \frac{7}{2\Gamma(\alpha+1)}\Big) L \; + \; \frac{3}{2} G.$$

Moreover: **(Why k_{op} is generally not a contraction.)** *For* $\alpha \in (0,1)$ *we have*
$0 < \Gamma(\alpha+1) < 1$*, hence* $1/\Gamma(\alpha+1) > 1$*. Consequently*

$$\frac{1}{\Gamma(\alpha+2)} + \frac{7}{2\Gamma(\alpha+1)} \geqslant \frac{7}{2\Gamma(\alpha+1)} > \frac{7}{2},$$

and therefore the lower bound

$$k_{\text{op}} \geqslant \frac{7}{2} L + \frac{3}{2} G$$

holds. In particular, unless the combined Lipschitz constants L *and* G *are extremely small (so that* $\frac{7}{2}L + \frac{3}{2}G < 1$*), the value of* k_{op} *will* not *satisfy* $k_{\text{op}} < 1$*. For typical values of the local Lipschitz constants* L, G *the computed operator constant* k_{op} *will not satisfy* $k_{\text{op}} < 1$*. Thus Banach's contraction condition is usually too restrictive for this problem on* $[0,1]$*. This motivates a Volterra/fractional Grönwall approach that exploits the convolutional kernel instead of a global operator norm estimate.*

System analysis

Let $\alpha = 0.7$
$\Gamma(\alpha+1) = \Gamma(1.7) \approxeq 0.908$
$\Gamma(\alpha+2) = \Gamma(2.7) \approxeq 1.54$

Basic constants: $L_0 = 0.4,\; G_0 = 0.3,\; \beta = 1.$
The operator constant is:

$$\Big(\frac{1}{\Gamma(\alpha+2)}+\frac{7}{2\Gamma(\alpha+1)}\Big)L(\tau)+\frac{3}{2}\beta G(\tau)$$

where $L(\tau)=0.4(1+\tau)$ and $G(\tau)=0.3(1+\tau^2)$

Because $\dfrac{1}{\Gamma(\alpha+2)}\approx 0.67 \quad \dfrac{7}{(2\Gamma(\alpha+1))}\approx 3.85.$

The prefactor for $L(\tau)$ is about $0.65=3.85\approx 4.5$

so $k_{\text{op}}(\tau)\approx 4.5\times 0.4(1+\tau)+1.5\times 0.3(1+\tau^2)$

$\approx 1.8(1+\tau)+0.45(1+\tau^2)$

For $\tau\in[0,1]$**:**

i. At $\tau=0, k_{\text{op}}\approx 1.8+0.45=2.25.$

ii. At $\tau=1, k_{\text{op}}\approx 1.8\times 2+0.45\times 2=4.5$

Therefore: $[\min_{k_{\text{op}}},\max_{k_{\text{op}}}]=[2.25,4.5]$.
Unless the combined Lipschitz constants L and G are extremely small (i.e., $\frac{7}{2}L+\frac{3}{2}G<1$), the inequality $k_{\text{op}}<1$ is typically *not* satisfied. For standard or realistic values of L and G, T does not satisfy Banach's contraction condition.

Remark: This demonstrates that the CFDIBVP operator is generally noncontractive in $AC^1([0,1],\mathbb{R}^n)$, which motivates the use of Sadovskii's fixed point theorem rather than Banach's fixed point theorem for establishing the existence of solutions.

4. Example: a noncontractive Caputo fractional delay BVP

Consider the Caputo fractional delay integral boundary value problem

$$\begin{cases} {}^{c}D^{\alpha}u(t)=t^2u(t)+\sin\big(u(t-\tau)\big), & t\in[0,1],\ 0<\alpha<1,\\ u(t)=\varphi(t), & t\in[-\tau,0],\\ u(1)=\displaystyle\int_0^1\frac{u(s-\tau)}{2+s^2}\,ds, & \end{cases}\tag{13}$$

where $\tau\in(0,1)$ is a fixed delay and φ is a given continuous initial function.

i. Evaluate the noncontractivity condition of the system equation (13).

For the nonlinearities

$$f(t,u,v) = t^2 u + \sin v, \qquad g(t,u,v) = \frac{v}{2+t^2}.$$

Bounds on B_r For the closed ball

$$B_r := \{u \in AC^1([0,1]) : \|u\|_{AC^1} \leqslant r\},$$

$$\|u\|_{AC^1} = \sup_{t\in[0,1]} |u(t)| + \int_0^1 |u'(t)|\, dt,$$

we have

$$|f(t,u,v)| \leqslant t^2|u| + |\sin v| \leqslant r+1 =: M_f,$$

$$|g(t,u,v)| \leqslant \frac{|v|}{2} \leqslant \frac{r}{2} =: M_g.$$

Lipschitz constants:

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leqslant |u_1-u_2| + |v_1-v_2|, \quad L_f = 2,$$

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leqslant \tfrac{1}{2}|v_1-v_2|, \quad L_g = 0.5.$$

Operator constant:

For the integral operator T defined in the existence proof,

$$\|Tu - Tv\|_{AC^1} \leqslant k_{\text{op}}\|u-v\|_{AC^1}, \quad k_{\text{op}} = \Big[\tfrac{1}{\Gamma(\alpha+2)} + \tfrac{7}{2\Gamma(\alpha+1)}\Big] L_f + \tfrac{3}{2}L_g.$$

Because $0<\alpha<1$ implies $1/\Gamma(\alpha+1) > 1$, it follows that

$$k_{\text{op}} \geqslant \tfrac{7}{2}L_f + \tfrac{3}{2}L_g = \tfrac{7}{2}(2) + \tfrac{3}{2}(0.5) > 1.$$

Hence, Banach's contraction principle is not applicable.
Remark 2: All hypotheses of Sadovskii's fixed point theorem are satisfied, and so the problem (13) admits at least one solution $u \in AC^1([0,1])$ even though the associated operator is noncontractive.

5. Discussion and conclusion

The analysis confirms that the integral operator associated with the Caputo fractional delay integral boundary value problem (CFDIBVP) is noncontractive under realistic parameter ranges.
From Figure 1: The rising (upward) curve means that as the delay parameter τ increases the operator constants k_{op} also increase, hence influencing contractivity as follows:

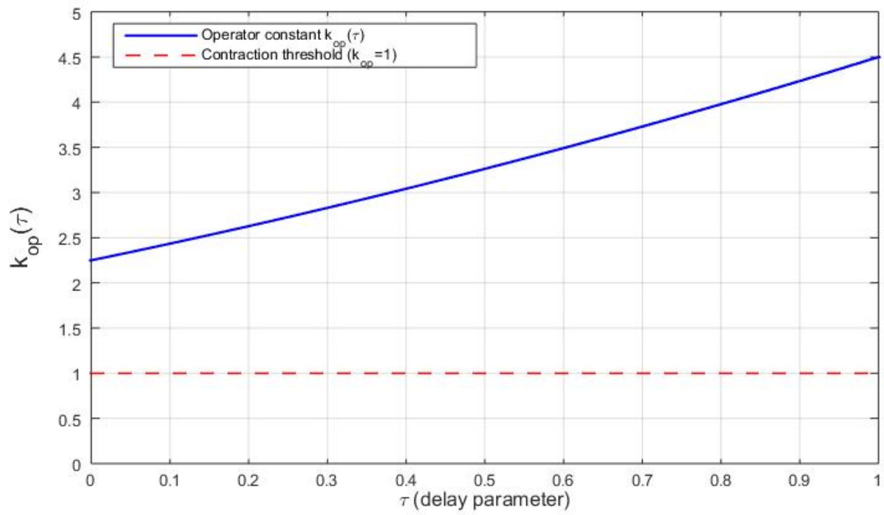

Figure 1. Operator constant vs. delay parameter

i. Banach's contraction principle requires $k_{\mathrm{op}} < 1$.
ii. Our computed $k_{\mathrm{op}}(\tau)$ started well above 1; (≈ 2.25 when $\tau = 0$) and increased to about 4.5 at $\tau = 1$.
iii. From our analysis, the delay term $u(t-\tau)$ contributed directly to the Lipschitz bounds $L(\tau)$ and $G(\tau)$ respectively. Hence, as τ increases, the effective Lipschitz bounds also increase, thereby amplifying the noncontractive behavior of the CFDIBVP.
iv. The existence of solutions cannot rely on Banach's fixed point theorem, but instead on the condensing-operator framework of Sadovskii's fixed point theorem, as convincingly illustrated by our analysis.
v. This reinforces our theoretical conclusion that the Caputo fractional delay integral boundary value problem is noncontractive.

This methodology provides a template for analyzing other fractional functional equations with complex boundary data or memory effects.

Main contribution. This study establishes the existence of solutions for a Caputo fractional delay integral boundary value problem in $AC^1([0,1],\mathbb{R}^n)$ without invoking contraction assumptions. by demonstrating that the associated integral operator is inherently noncontractive; consequent on the combined effects of delay terms and nonlocal boundary conditions. By employing Sadovskii's condensing operator framework, the work extends fixed point theory to a class of fractional functional equations where Banach's contraction principle is inapplicable, thereby offering a new analytical route for fractional delay systems with complex boundary interactions.

Recommendations for future work.

1. **Relaxed boundary conditions:** Extend the analysis to more general nonlocal or multi-point boundary conditions, including integral and impulsive types, to test the robustness of the Sadovskii framework.
2. **Higher-order regularity and functional settings:** To investigate Caputo fractional delay equations of order $\alpha > 1$ or systems with multiple fractional orders to capture richer memory effects, extend the analysis beyond AC^1 to AC^n, Sobolev-type fractional spaces, or weighted spaces, which could yield sharper compactness estimates and regularity of solutions.
3. **Uniqueness and multiplicity framework:** Future work may employ weak contractions, mixed monotone operators, topological degree, or other sophisticated fixed point theorems like Darbo–Sadovskii generalizations, cone-theoretic fixed points, or monotone iterative sequences tailored to fractional delay kernels to derive uniqueness and multiplicity results.
4. **Redefined condensing decompositions:** Improve the operator splitting approach to reduce the measure of noncompactness and identify explicit thresholds at which the operator becomes strictly condensing or ceases to be condensing when delay or boundary parameters vary.
5. **Generalization to broader fractional models:** Future work can be extended to Caputo–Hadamard, Atangana–Baleanu, and distributed-order derivatives, or systems with state-dependent delays.

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