sciendo

ANNALS OF WEST UNIVERSITY OF TIMISOARA PHYSICS
Vol. LXVII, 2025

DOI: 10.2478/awutp-2025-0010

DO ALL THE THEORETICAL RESULTS OBTAINED IN THE CASE OF HOPFIELD NEURAL NETWORKS HAVE A COUNTERPART AT THE LEVEL OF THE HUMAN NEURAL SYSTEM?

Andreea V. Cojocaru[1], Stefan Balint[2]

[1]Independent researcher, Timisoara, Romania, candreeavalentina@gmail.com
[2]Department of Computer Science, West University of Timisoara, Blvd. V. Parvan 4, 300223 Timisoara, Romania, stefan.balint@e-uvt.ro
[*]Correspondence: candreeavalentina@gmail.com; stefan.balint@e-uvt.ro

Article Info	Abstract
Received: 27.02.2025 *Accepted: 20.10.2025* **Keywords:** Hopfield neural network, Nervous system, Stability in Liapunov sense, Structural stability in Smale sense	Continuous and discrete Hopfield type neural networks claim to be mathematical descriptions of electrical phenomena appearing in nervous system. We present a set of theoretical results having natural counterpart and another set of theoretical results concerning equilibriums that have no real counterpart at the level of the nervous system. The differentiation criterion of equilibriums is the local exponential asymptotic stability in Liapunov sense of the steady state voltage and the structural stability of the Hopfield system in Smale sense. The believe is that unstable equilibriums in Liapunov sense and equilibriums obtained in case of structurally unstable Hopfield systems, exist only in theory.

1. Continuous-time Hopfield neural networks

According to [1] pg.173 formula (5.1) a continuous-time Hopfield-type neural network, describing the voltage evolution in a nervous system of n biological neurons, is a system of nonlinear differential equations of the form:

$$\frac{dX}{dt} = A \cdot X + T \cdot G(X) + I \qquad (1)$$

$X = (x_1, x_2, \ldots x_n)^T \in R^n$; $;I = (I_1, I_2, \ldots I_n)^T \in R^n$; $G(X) = (g_1(x), g_2(x), \ldots g_n(x))^T$; x_i – the voltage of the neuron i ; $a_i > 0$ constant related to the neuron i membrane capacity; $T = (T_{ij})_{n \times n}$ is a constant matrix denote the interconnection matrix of neurons; $g_i : R \to R$ the input-output activations voltage of the neuron i; I_i are constants and denote the external electrical input of the neuron i. With function, $F(X,I)=A\cdot X+T\cdot G(X)+I$ equation (1) can be written in the form:

$$\frac{dX}{dt}=F(X,I) \tag{2}$$

An equilibrium of (2) is a solution of the equation:

$$F(X,I)=0 \tag{3}$$

In other words an equilibrium (X,I) of (2) is an element of $R^n \times R^n$ which verifies (3). An equilibrium (X,I) of (2) (if exist) then X is called steady state and I is called external input corresponding to the steady state X. The name steady state is justified by the fact that if (X^0,I^0) is an equilibrium, then for I equal to I^0 the solution of (2) which verifies the initial condition $X(t_0)=X^0$ is constant. On the other hand for a prior given steady state voltage X, according to the Theorem (5.1) pg.174. [1] there exists a unique external electrical input value:

$$I=-A\cdot X-T\cdot G(X) \tag{4}$$

such that (X,I) is an equilibrium for (2). Formula (4) give sense to the name "I is the external electrical input corresponding to the steady state voltage X". A natural neuro-psychological interpretation of the concept of equilibrium (X^0,I^0) is that if the neuronal system is in the equilibrium (X^0,I^0) then keeping the external electrical input value constant equal to I^0 the voltage X^0 of the neuronal system will rest constant equal to X^0. Hence, the idea that in the case when the neuronal voltage system reveal disorder, a new external electrical input has to be applied. If the desired new voltage is X^1 then it is natural to think that the new external electrical input value has to be taken according to the formula (4) i.e. $I^1=-A\cdot X^1-T\cdot G(X^1)$ hoping that after the change $I^0 \to I^1$ of the electrical input the voltage X^0 evolve to the voltage X^1. This though is correct if after the moment of time t_1 ,when the change $I^0 \to I^1$ takes place, the voltage state of the neuronal system described by the solution of the initial value problem

$\frac{dX}{dt} = F(X, I^1)$, $X(t_1) = X^0$ evolve to the voltage state X^1. This type of behavior is assured if the steady voltage state X^1 corresponding to the input I^1 is locally exponentially stable in Liapunov sense and the steady state voltage X^0 is in the region of attraction of X^1. Otherwise is not at all sure that the change $I^0 \to I^1$ of the external electrical input lead to the transfer of steady voltage state X^0 into the steady voltage state X^1 of neuronal system. We can illustrate the above facts in the case of the neural network (5.2) considered in [1] and defined by the system of differential equations:

$$\frac{dx_1}{dt} = -x_1 + \frac{17 \cdot \ln 4}{15} \cdot \tanh x_2 + I_1 \quad ; \quad \frac{dx_2}{dt} = -x_2 + \frac{17 \cdot \ln 4}{15} \cdot \tanh x_1 + I_2 \tag{5}$$

For obtain the prior given steady voltage state $X = (1,2)^T$ the external input which has to be applied according to (4) is $I = (-0.514616132, 0.803433824)^T$. In order to see the voltage of the neural network corresponding to the external input $I = (-0.514616132, 0.803433824)^T$ the following initial value problem has to be solved:

$$\frac{dx_1}{dt} = -x_1 + \frac{17 \cdot \ln 4}{15} \cdot \tanh x_2 - 0.514616132 \;; \frac{dx_2}{dt} = -x_2 + \frac{17 \cdot \ln 4}{15} \cdot \tanh x_1 + 0.803433824 \tag{6}$$

$$x_1(0) = 1 \qquad\qquad x_2(0) = 2$$

The solution of the initial value problem (6) is represented on the figures Fig.1 and Fig.2.

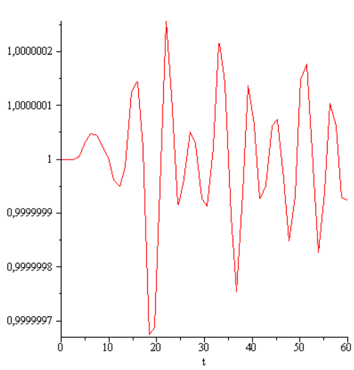

Fig. 1. x_1 versus t

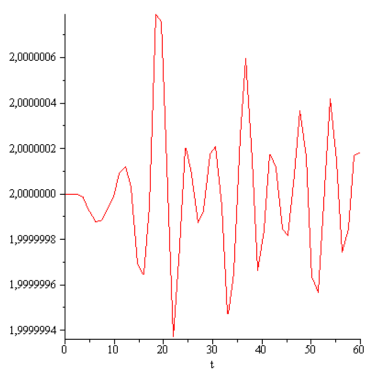

Fig. 2. x_2 versus t

These figures show that, in the first five seconds the voltage of the neuron network is constant equal to the initial value, after that, oscillate around the initial value with an amplitude less than 10^{-6}. This means that practically there is no change in the voltage of the neural network. This type of the voltage behavior indicates a steady state. If from medical point of view, the steady state $X=(1,2)^T$ is a “bed one“, then a neurological or psychological intervention is required. The change of the external electrical input represents a type of intervention.

Assume now that the purpose of the medical intervention is to lead the patient from the steady state voltage $X^1=(1,2)^T$ into the steady state voltage $X^0=(0,0)^T$. In order to realize this transformation by changing the external input a new external input has to be computed corresponding to the steady state $X^0=(0,0)^T$. The new external input, obtained by computation using formula (4), is $(I_1^{\,0}, I^0{}_2)^T=(0,0)^T$.

The effect of the external input change can be seen solving the following initial value problem:

$$\frac{dx_1}{dt}=-x_1+\frac{17\cdot\ln 4}{15}\cdot\tanh x_2\,;\ \frac{dx_2}{dt}=-x_2+\frac{17\cdot\ln 4}{15}\cdot\tanh x_1\,;\ x_1(0)=1;\ \ x_2(0)=2\,. \qquad (7)$$

The solution of the initial value problem (7) is represented on the figures Fig.3. and Fig.4.

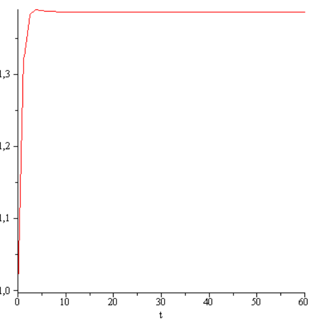

Fig. 3. x_1 versus t

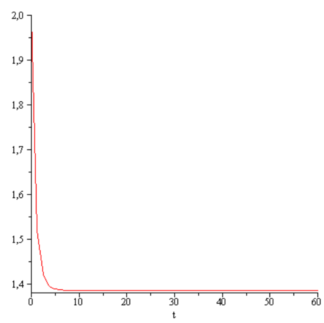

Fig. 4. x_2 versus t

Figures 3 and 4 show that the effect of the change of the external electrical input is the transfer of the steady state $X^1=(1,2)^T$ into the steady state $X^2=(\ln 4,\ln 4)^T$ and not into the steady state voltage $X^0=(0,0)^T$,as were expected. The mathematical explanation is: the steady state $X^2=(\ln 4,\ln 4)^T$ is locally exponentially stable in Liapunov sense and the steady state $X^1=(1,2)^T$ belongs to the region of attraction of the steady state $X^2=(\ln 4,\ln 4)^T$. On the other hand, the steady voltage state $X^0 = (0,0)^T$ is unstable and it is repulsive. The meanings of the concepts unstable and repulsive are obtained by negating the meanings of the concepts stable and attractive respectively. The repulsive character of the steady voltage state is illustrated on the figures Fig.5.and Fig.6.

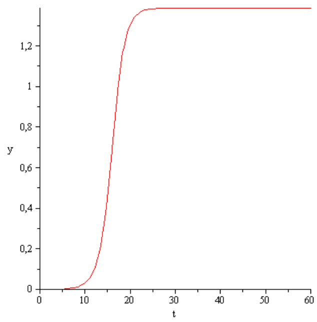

Fig. 5. x_1 versus t

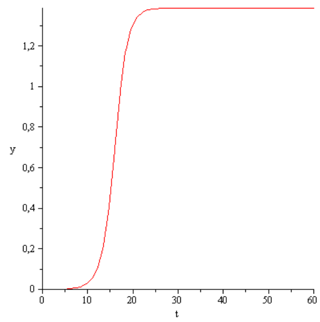

Fig. 6. $x_2$versus t

Figures 5, 6 show that a small deviation (less than 10^{-3}) of the initial condition from the steady state $(0,0)^T$ imply that the solution of the initial value problem tends to the steady state $(\ln 4, \ln 4)^T$.

The fundamental difference between the equilibriums (X^0, I^0) and (X^2, I^0) consists in the fact that, in the case of the equilibrium (X^0, I^0) after a small perturbation of the steady state voltage X^0 the equilibrium (X^0, I^0) is not recovered and in case of the equilibrium (X^2, I^0) after a small perturbation of the steady state voltage X^2 the equilibrium (X^2, I^0) is recovered .This difference make that the equilibrium (X^0, I^0) has no counterpart in the corresponding neuron system (exist just in model) and the equilibrium (X^2, I^0) has counterpart in the corresponding neuron system (exist in reality).

Encephalogram represent exponentially stable steady state voltage of neuron system. It is used for diagnosis, treatment and rehabilitation.

Theoretical results regarding existence of locally exponentially stable equilibriums, for the general continuous-time Hopfield neural network (1) reported in [1] are Theorem.5.4. pg.176, Theorem 5.6 pg.177, Theorem 5.9. pg.179, Theorem 5.11. pg.180. These Theorems permit to identify locally asymptotically stable equilibriums, to build up path of such equilibriums, along which by small successive changes, the nervous system is conducted, through the regions of attraction of intermediary asymptotically stable steady states to the prior planed steady state voltage.

2. Discrete-time Hopfield neural networks.

According to [1] pg.189 formula (5.24) a discrete-time Hopfield-type neural network, describing the voltage evolution in a nervous system of n biological neurons, is a semi-dynamical system of the form:

$$X_{p+1} = B \cdot X_p + \hat{T} \cdot G(X_p) + \hat{I} \quad , \quad p \in N \tag{8}$$

$X_p = (x^p{}_1, x^p{}_2, \dots, x^p{}_n)^T \in R^n$, $B = diag(b_1, b_2, \dots, b_n) \in M_{n\times n}$, $\hat{I} = (\hat{I}_1, \hat{I}_2, \dots, \hat{I}_n)^T \in R^n$,

$G(X_p) = (g_1(x^p{}_1), g_2(x^p{}_2), \dots, g_n(x^p{}_n))^T$, $x^p{}_i$ – the voltage of the neuron i at the moment $p \in N$, $1 > b_i > 0$ constant related to the neuron i membrane capacity;

$\hat{T} = (\hat{T}_{ij})_{n\times n}$ is a constant matrix denote the interconnection matrix of neurons; $\hat{I}_i$ - are constants and denote the external electrical input of the neuron i. With function, $F(X, \hat{I}) = B \cdot X + \hat{T} \cdot G(X) + \hat{I}$ the semi- dynamical system (8) can be written in the form:

$$X_{p+1} = F(X_p, \hat{I}) \quad , \quad p \in N \tag{9}$$

An equilibrium of the semi-dynamical system (8) by definition is a solution of the nonlinear equation

$$F(X, \hat{I}) = X \tag{10}$$

This means that an equilibrium is a couple $(X, \hat{I}) \in R^n \times R^n$ which verifies (10). If $(X, \hat{I}) \in R^n \times R^n$ is an equilibrium of (8) then X is called steady state voltage and $\hat{I}$ external electrical input corresponding to X. The name steady state voltage is justified by the fact that : if $(X^0, \hat{I}^0) \in R^n \times R^n$ is an equilibrium of (8) then, keeping $X^1 = X^0$ and $\hat{I} = \hat{I}^0$ formula (9) provide a constant sequence in which each term is equal to X^0. On the other hand remark, that for a prior given steady state voltage X, according to the Theorem 5.15 pg.190. [1] there exists a unique external electrical input value:

$$\hat{I} = (I_d - B) \cdot X - \hat{T} \cdot G(X) \tag{11}$$

such that the couple $(X,\hat{I})$ is an equilibrium of the semi-dynamical system (8).In formula (11), $I_d \in M_{n\times n}$ is the identity matrix and, formula (11) give sense to the name "$\hat{I}$ is the external electrical input corresponding to the steady state X". A natural neuro-psychological interpretation of the concept of equilibrium $(X^0,\hat{I}^0)$ is that if the neuronal system is in the equilibrium $(X^0,\hat{I}^0)$ then for the external electrical input value constant equal to $\hat{I}^0$ the neural system voltage is constant equal to X^0. Therefore in order to modify the neural system voltage a new external electrical input has to be applied. If the desired new steady state voltage of neural system is X^2 then it is natural to think that the new external input value has to be taken according to the formula (11) i.e. $\hat{I}^2 = (I_d - B)\cdot X^2 - \hat{T}\cdot G(X^2)$ hoping that after the change $\hat{I}^0 \to \hat{I}^2$ the steady state voltage X^0 evolve to the steady state voltage X^2. This though is correct if after the moment of time p_1,when the change $\hat{I}^0 \to \hat{I}^2$ takes place, the voltage state of the neural network (described by the solution of the initial value problem $X_{p+1} = F(X_p,\hat{I}^2)$, $X^1 = X^{p_1}$) evolve to the voltage state X^2. This type of behavior is assured if the steady voltage state X^2 corresponding to the input $\hat{I}^2$ is locally exponentially stable in Liapunov sense and the steady state voltage X^0 is in the region of attraction of X^2. The above presented concepts and facts are illustrated in the example 5.5 pg.197. [1]. This is the discrete semi-dynamical system obtained from the system (2.7) by the semi -discretization technique given in [3].

Because the high similarities with the case of continuous Hopfield type neural network, we emphasize only that also in the case of discrete-time Hopfield neural network there exist equilibriums which are locally exponentially stable, in Liapunov sense, and equilibriums which are instable, in Liapunov sense. Instable equilibriums have no counterpart in the corresponding neuron system. Only locally exponentially stable equilibriums have counterpart in the corresponding neuron system.

Theoretical results regarding existence of locally exponentially stable equilibriums, for the general discrete-time Hopfield neural network (8) reported in [1] are: Theorem.5.17. pg. 191, Theorem (5.18) pg.191. Theorem 5.20. pg.192., Theorem.5.21.pg.193.

In the following we present different types of bifurcations and chaotic behavior of the voltage steady states in case of particular discrete-time Hopfield neural networks. In these examples the external electrical input is equal to zero. The network behavior is due to the variation of an aggregated internal parameter of the network. Due to that, the neural network from the stable zero steady voltage arrive in chaos. On the way towards chaos the neural network goes through different bifurcations and different orbits appear that are not diffeomorphic. Therefore, the neural network is not structurally stable at the point of bifurcations. Thus, the orbits at bifurcation point only exist theoretically, but they have no counterpart in the neuron system.

In the paper [3], a bifurcation analysis is undertaken for a discrete-time Hopfield neural network with a single delay:

$$x_{p+1} = a \cdot x_p + T_{11} \cdot g_1(x_{p-k}) + T_{12} \cdot g_2(y_{p-k}),\; y_{p+1} = a \cdot y_p + T_{21} \cdot g_1(x_{p-k}) + T_{22} \cdot g_2(y_{p-k}) \quad (12)$$

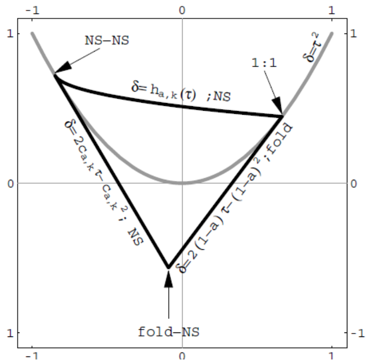

Fig .7. Stability region of the zero voltage steady state

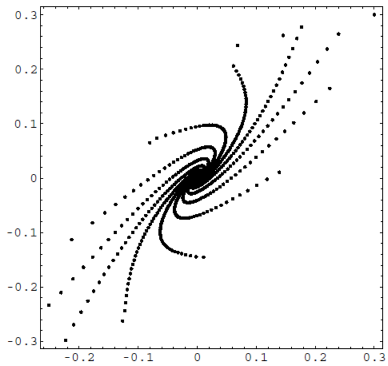
Fig. 8. Orbits before bifurcation

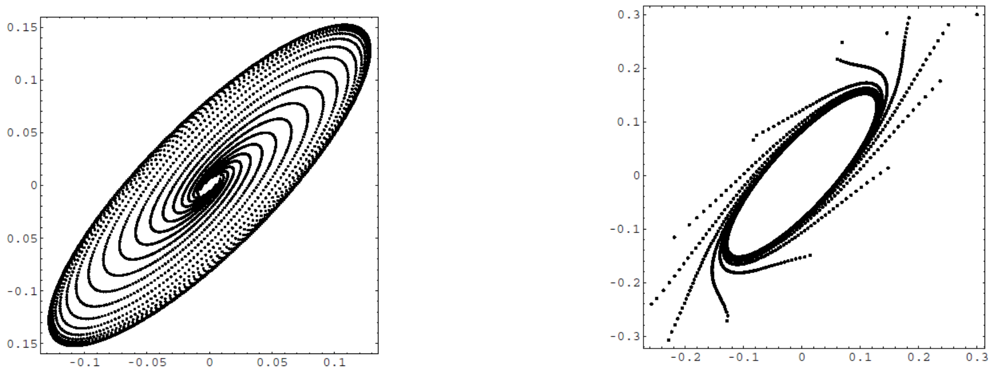
Fig. 9. Orbits after bifurcations.

The orbits presented in Figure 8 are not diffeomorphic with those presented in Figure 9. Therefore, for the bifurcation value of aggregated parameter the neural network (12) is structurally instable. The orbits at bifurcation point only exist theoretically, but they have no counterpart in the neural system.

In the paper [4] the classical discrete-time Hopfield neural network of two neurons with two delays and no self-connections defined by:

$$x_{p+1} = 1/4 \cdot x_p + \tanh(y_{p-2}) \quad , \quad y_{p+1} = 3/4 \cdot y_p + b \cdot \sin(x_{p-1}) \tag{13}$$

is analyzed. For different values of the aggregated parameter b some results of the above dynamics are represented in the following figures:

For b = −0.35, the null solution is asymptotically stable, and the trajectory converges to the origin Fig.10.

For b = −0.36, an asymptotically stable cycle (1-torus, drift ring) is present, and the trajectory converges to this cycle Fig.11.

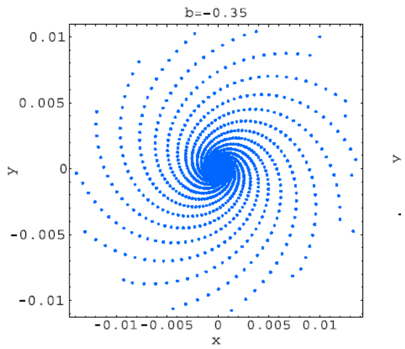

Fig. 10. Null solution is asymptotically stable

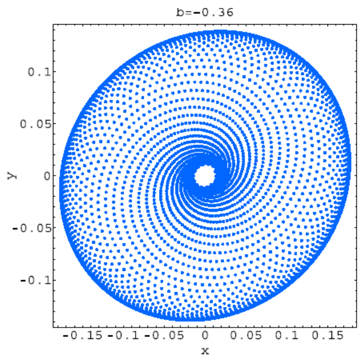

Fig. 11. Asymptotically stable cycle

The orbits presented in Fig.10. is not diffeomorphic with the orbits presented in Fig.11. Therefore, for the bifurcation value b=-0.35635 of aggregated parameter the neural network (13) is structurally unstable. The orbits at bifurcation point b=-0.35635 only exist theoretically, but they have no counterpart in the neuronal system.

At around b = −1.265, a bifurcation phenomenon takes place, which determines the appearance of two stable limit cycles (1-tori) close to the origin. For b = −1.26, there is only one asymptotically stable limit cycle (1-torus), symmetrical to the origin. For b = −1.27, there are two stable limit cycles, not symmetrical to the origin. These theoretical results are presented in Fig.12 and Fig.13.

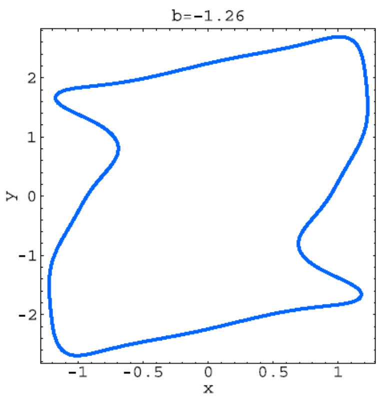

Fig.12. Limit cycle for b=-1.26

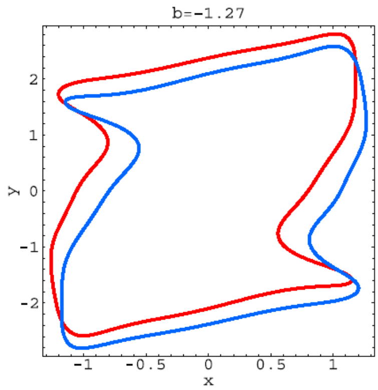

Fig.13. Two limit cycles for b=-1.27

The orbits presented in Fig.12. is not diffeomorphic with the orbits presented in Fig.13. Therefore, for the bifurcation value b=-1.265 of the aggregated parameter the neural network (13) is structurally unstable. The orbits at bifurcation point b=-1.265 only exist theoretically, but they have no counterpart in the neural system. For more detail regarding neural network (13) see paper [4].

Conclusion

Unstable equilibriums and bifurcation orbits exist only theoretically, but they have no counterpart in the neural system.

References

[1] St. Balint, L. Braescu and E. Kaslik. *Regions of attraction and applications to control theory*. Cambridge Scientific Publishers Ltd. 2008. Edited by S. Sivasundaram.

[2] M. Hirsh, S. Smale. *Differential equations, Dynamical systems and linear algebra*. Academic Press.1974.

[3] E. Kaslik, St. Balint. *Bifurcation analysis for a two-dimensional delayed discrete-time Hopfield neural network*. Chaos, Solitons and Fractals 34 (2007) 1245–1253.

[4] E. Kaslik ,St. Balint. *Chaotic Dynamics of a Delayed Discrete-Time Hopfield Network of Two Nonidentical Neurons with no Self-Connections*. J Nonlinear Sci. DOI 10.1007/s00332-007-9015-5. 2007.