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On the partial parallelizable manifolds and associated invariants

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Abstract Our idea is to use the global diffeomorphism group $G(M)$ to study certain properties of differentiable manifolds endowed with global vector fields. We define a partial parallelizable manifold (p.p.m.) as a couple (M, ρ_p), where ρ_p is a global partial frame (g.p.f.) of the tangent bundle TM of M. In this text there are studied properties of the p.p.m. Certain specific invariants for these manifolds, relative to the group $G(M)$ of the global diffeomorphisms of M, are constructed. We also give some applications and examples. We obtain that a $G(M)$ -invariant $[M \times \mathbb{R}^p]$ of partial parallelizable manifold (M, ρ_p) is an equivalence class defined by action group $G(M)$ on M.

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1 Introduction

The work belongs to the geometry of vector fields. It is inspired from [1–5] and the notion of parallelizable manifold. Some interesting results on the geometry of vector fields are in [6] and [15–17]. Our method is based on the notion of vector bundle (see [8–14]). The geometrical objects considered in this paper are of C^∞ class and the morphisms are of constant rank. The existence results from this paper are realized by constructive method. The concept of a "partial parallelizable manifold" is fundamental in this paper and the manifolds are paracompact.

In [17] is studied the existence of vector fields on a differentiable manifold and fields of tangent planes. The homotopy-theoretic method is used. In [15] the existence of vector fields on Grassman manifold is studied.

In this paper we are interested in finding some invariants under the action of the group $G(M)$ of global diffeomorphisms of the manifold M.

We prove that the existence of a partial trivial structure in TM is reduced to the existence of a lift (g_0, g_1) of Gauss map g_{TM} in the commutative diagram (Figure 1) (and we construct this lift). The product bundle $\theta_1^s = (G_p(\mathbb{R}^s) \times \mathbb{R}^s, p_1, G_p(\mathbb{R}^s))$ over $G_p(\mathbb{R}^s)$ induces the partial trivial structure in TM. We say that a differential manifold M is partial parallelizable if the tangent vector bundle TM of M has a global partial frame $\rho_p = (X_1, X_2, \ldots, X_p)$, $1 \leq p < m$. Here the vectors $(X_1(x), X_2(x), \ldots, X_p(x))$ are independent vectors for all $x \in M$.

Also we show that partial parallelizable manifold (M, ρ_p) admits some $G(M)$-invariants $[M \times \mathbb{R}^p]$ and the structure of TM is given by the isomorphism $TM \cong (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$ (Theorem 4.2 and Theorem 4.3).

2 Preliminaries

Let M be a differentiable paracompact manifold of dimension m, TM its tangent vector bundle, and a fixed natural number p, $1 \leq p < m$. The essential property of M is the existence of global partial frame (g.p.f.) $\rho_p = (X_1, \ldots, X_p)$ of TM (i.e. M is a partial parallelizable manifold - p.p.m.). We suppose that the topology of M is induced by its differentiable structure.

Lemma 2.1. *Let (M, ρ_p) be a paracompact p.p.m. Then there exists a fibre bundle isomorphism $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$, where $E(\rho_p)$ denotes the vector subbundle of TM spanned by $\rho_p = (X_1, \ldots, X_p)$. Conversely, if TM admits such a structure, where $E(\rho_p) \simeq M \times \mathbb{R}^p$, then M is a p.p.m.*

Proof. Consider the vector bundle $\xi(\rho_p) = (E(\rho_p), \pi^p, M)$ over M, where $E(\rho_p) = \bigcup_{x \in M} E_x(\rho_p)$ and $E_x(\rho_p)$ is the vector space spanned by $\rho_p(x) = (X_1(x), \ldots, X_p(x))$ for all $x \in M$. Since M is paracompact the exact sequence $0 \longrightarrow E(\rho_p) \longrightarrow TM \longrightarrow TM/E(\rho_p) \longrightarrow 0$ gives the isomorphisms $TM \simeq E(\rho_p) \oplus TM/E(\rho_p)$, $E(\rho_p) \simeq M \times \mathbb{R}^p$ ([12], p. 37).

Conversely, if (e_i) is the canonical base of $\mathbb{R}^p$, the global frame $r = \{(x, (e_i)); x \in M\}$ of $\theta^p = (M \times \mathbb{R}^p \longrightarrow M)$ defines a g.p.f. of TM. The result follows. □

We shall express this result also in this way: TM admits a partial trivial structure (p.t.s.) ([5]). The quotient vector bundle $TM/E(\rho_p)$ obstructs the trivialization of TM.

Lemma 2.2. *Let $G(M)$ be the group of global diffeomorphisms of M, and $\rho_p = (X_1, \ldots, X_p)$ a g.p.f. of TM. Then each $\varphi \in G(M)$ determines a g.p.f. $(\varphi, \acute{\varphi})(\rho_p)$ of TM, where $\acute{\varphi}$ denotes the differential of φ.*

Proof. The action of $\varphi \in G(M)$ on ρ_p is expressed by the relations:

$$\begin{aligned}(\acute{\varphi}(\rho_p))(\varphi(x)) &= (\varphi, \acute{\varphi})(\rho_p(x)) = (\varphi, \acute{\varphi})(X_1(x), \ldots, X_p(x)) \\ &= (\varphi(x); \acute{\varphi_x}(X_1(x)), \ldots, \acute{\varphi_x}(X_p(x))) \text{ for all } x \in M.\end{aligned}$$

Here $\acute{\varphi_x}$ denotes the differential of φ at x. Therefore $(\varphi, \acute{\varphi})(\rho_p)$ is a g.p.f., $(\acute{\varphi})^{-1} = (\varphi^{-1})\acute{}$. □

3 Constructions of partial trivial structures for TM

The following examples ensure the consistency of the notion of $G(M)$-invariant for manifolds with global vector fields.

3.1 General constructions

Case A.3.1. Let E be a subbundle of TM. There exits a infinity of partial trivial vector bundles over M. Particular case: $E = TM$.

Proof. Consider the product bundle $\theta^s = (M \times \mathbb{R}^s \longrightarrow M)$. Then $\eta^s = \theta^s \oplus E$, $s = 1, 2, \ldots$, is a partial trivial vector bundle over M. If $E = TM$ then $\widetilde{\eta}^s = \theta^s \oplus TM$ is a partial trivial vector bundle for all s. □

Case A.3.2. a) Let $G_p(\mathbb{R}^s)$ be the Grassmann manifold of p-dimensional subspaces of $\mathbb{R}^s$, $p \leq s \leq +\infty$ and M a paracompact manifold. For each map $f : M \to G_p(\mathbb{R}^s)$ there exists a partial trivial vector bundle (p.t.v.b.) over M.

b) Let G_p be the Grassmann variety of p-dimensional subspaces of $\mathbb{R}^\infty$ (similar meaning for G_{n-p}). If the classifying map of TM, $\gamma : M \to G_p \times G_{n-p}$ has the form $\gamma = (\gamma_0, \gamma_1)$, where γ_0 is a constant map, then TM is a p.t.v.b.

Proof. a) There exists a canonical vector bundle $\gamma_p^s = \left(E\left(\gamma_p^s\right), p^*, G_p(\mathbb{R}^s)\right)$ over $G_p(\mathbb{R}^s)$, for all $s \leq +\infty$ ([12], p.29), where the total space of γ_p^s is $E\left(\gamma_p^s\right) = \{(V, x) \in G_p(\mathbb{R}^s) \times \mathbb{R}^s;\ x \in V\}$. Consider also the product bundle $\theta_1^s = \{G_p(\mathbb{R}^s) \times \mathbb{R}^s, p_1, G_p(\mathbb{R}^s)\}$ over $G_p(\mathbb{R}^s)$. Then $f^*\left(\theta_1^s \oplus \gamma_p^s\right)$ is a p.t.v.b. over M.

b) The induced vector bundle under γ_0 is trivial: $\gamma_0^*\left(\gamma_p^s\right) \simeq \theta^s$. □

3.2 Special constructions

Case B.3.1. Let TM be the tangent bundle of a paracompact manifold M. There exists a Gauss map $\widetilde{g}_{TM} : TM \to \mathbb{R}^\infty$ of TM ([12], p.30) and a vector bundle morphism $(u, g_{TM}) : TM \to \gamma_m^s$, $m \leq s \leq +\infty$, $\widetilde{g}_{TM} = qu$, $q : TM \to \mathbb{R}^s$, $q(Vx) = x$, $x \in V$. The morphism (u, g_{TM}) is a isomorphism when restricted to T_xM, for all $x \in M$, and $g_{TM} : TM \to G_m(\mathbb{R}^s)$, $g_{TM}(x) = \widetilde{g}_{TM}(\pi^{-1}(x))$, π being the projection of TM. In addition $u(y) = (g_{TM}(\pi(y)), \widetilde{g}_{TM}(y))$, $y \in TM$. Consider the map $K : G_p \times G_{m-p} \to G_m$ induced by the direct sum $G_p(\mathbb{R}^s) \times G_{m-p}\left(\mathbb{R}^{\acute{s}}\right) \to G_m\left(\mathbb{R}^{s+\acute{s}}\right)$ sending $(\alpha, \beta) \to\to \alpha \oplus \beta$. If TM admits a lift $g = (g_0, g_1)$ in the commutative diagram (see Figure 1), where $g_0 : M \to G_p$ is a constant map, then we have the induced bundle $g^*\left(\gamma_p^s \times \gamma_{m-p}^{\acute{s}}\right) = g_0^*\left(\gamma_p^s\right) \oplus g_1^*\left(\gamma_{m-p}^{\acute{s}}\right)$, $m \leq s, \acute{s} \leq +\infty$, (case A.3.2).

The morphism $u = f_{\gamma_m} v$, where $v : TM \to g^*\left(\gamma_p^s \times \gamma_{m-p}^{\acute{s}}\right)$ is an isomorphism, and $f_{\gamma_m} : g^*\left(\gamma_p^s \times \gamma_{m-p}^{\acute{s}}\right) \to \gamma_p^s \times \gamma_{m-p}^{\acute{s}}$, $f_{\gamma_m}(x, y) = y$ is the canonical morphism. Hence we have the isomorphism $TM \simeq g_0^*\left(\gamma_p^s\right) \oplus g_1^*\left(\gamma_{m-p}^{\acute{s}}\right)$ and therefore TM is a p.t.v.b. Conversely, let TM be a p.t.v.b. There exists the isomorphism $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$, $E(\rho_p) \simeq \theta^p$, via Lemma 2.1. Let $\acute{g_0} : M \to G_p(\mathbb{R}^s)$ be a classifying map of θ^p and $g_0 : M \to G_p(\mathbb{R}^s)$ be a constant map. From the classic result ([9]) $Vect(M) \simeq [M, G_p]$ (the set of homotopy classes of maps of M into G_p) and $g_0^*(\gamma_p) \simeq \theta^p$, we obtain

$$\begin{tikzcd}
 & G_p \times G_{m-p} \arrow[d, "K"] \\
M \arrow[ur, dashed, "(g_0, g_1)"] \arrow[r, "g_{TM}"'] & G_m
\end{tikzcd}$$

Figure 1: Lift for Gauss map g_{TM}

a bijection $[g_0] \longrightarrow [\theta^p] \longrightarrow [g_0']$ for homotopy classes. If $g_1 : M \to G_{m-p}$ classifies $TM/E(\rho_p)$, then (g_0, g_1) classifies TM.

This proves the following theorem:

Theorem 3.1. *Let M be a partial parallelizable (paracompact) manifold and TM its tangent vector bundle. The construction of partial trivial structure of TM is equivalent to give a lift (g_0, g_1) of Gauss map g_{TM} in the diagram from Figure 1.*

Case B.3.2. Let $\theta_1^p = (G_p(\mathbb{R}^s) \times \mathbb{R}^s \to G_p(\mathbb{R}^s))$, $p \le s \le +\infty$ be a product bundle over $G_p(\mathbb{R}^s)$. Consider the tangent bundle TM of a paracompact manifold M, and g_{TM} a classifying map of TM. Denote by $g = (g_0, g_1)$ a lift of g_{TM} in the commutative diagram from Figure 1, where g_0, g_1 are not constant maps. Then the bundle TM and $g^*\left(\gamma_p^s \times \gamma_{m-p}^{s'}\right) = g_0^*(\theta_1^p) \oplus g_1^*\left(\gamma_{m-p}^{s'}\right)$ are isomorphic. Hence TM is a p.t.v.b. This proves the following result:

Theorem 3.2. *Let TM the tangent vector bundle of a paracompact manifold M and let $[g] = [(g_0, g_1)]$ be a lift of $[g_{TM}]$ (homotopy classes), where (g_0, g_1) are or are not constant maps. The trivial structure of θ_1^p induces a partial trivial structure in TM.*

Case B.3.3. Let M be a paracompact manifold, TM its tangent bundle and g_{TM} the classifying map of TM. If g_{TM} admits a lift $[g] = [(g_0, g_1)]$, where g_0, g_1 are constant, then there exists the isomorphisms $TM \simeq g_0^*(\gamma_p) \oplus g_1^*(\gamma_{m-p}) \simeq M \times \mathbb{R}^m$. Therefore TM is trivial.

The intrinsic meaning of this construction and its relation with certain invariants will be discussed in Section 4.

4 Invariants of the partial parallelizable manifolds

In the sequel we prove that the constructive method and the action of the group $G(M)$ on (M, ρ_p) guarantees that a p.p.m. has some invariants $[M \times \mathbb{R}^i]$, $i = 1, 2, \ldots, p$.

With the preceding notations we can formulate the following:

Theorem 4.1. *Any partial parallelizable manifold (M, ρ_p) admits some unique invariants relative to the action of $G(M)$, $[M \times \mathbb{R}^i]$, $i = 1, 2, \ldots, p < m$ and if M is parallelizable then M has also the invariant $[M \times \mathbb{R}^m]$. The structure of TM is given, respectively, by the isomorphisms: $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$, $TM \simeq M \times \mathbb{R}^m$.*

Proof. These invariants can be constructed as follows. We discern certain situations.

I. Let $\rho_p = (X_1, X_2, \ldots, X_p)$, be a g.p.f. of TM. On construct the vector subbundle of TM spanned by ρ_p (the section 1): $\xi(\rho_p) =(E(\rho_p), \pi^p, M)$. For $\varphi \in G(M)$ we obtain a g.p.f. $\widetilde{\rho}_p = (\varphi, \varphi')\rho_p$ and the vector subbundle spanned by $\widetilde{\rho}_p$, $\xi(\widetilde{\rho}_p) = (E(\widetilde{\rho}_p), \widetilde{\pi}^p, M)$ of TM. $\xi(\widetilde{\rho}_p)$ is isomorphic with θ^p (Lemma 2.2). Consider the equivalence class of isomorphic subbundles $[M \times \mathbb{R}^p] = \{E((\varphi, \varphi')(\rho_p)/\rho_p = (X_1, \ldots, X_p),\ \varphi \in G(M)\}$. We remark that the class $[M \times \mathbb{R}^p]$ is invariable to the action of $G(M)$. Via Lemma 2.1 we have $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$.

II. Let $\rho_{p-1} = (X_{i_1}, \ldots, X_{i_{p-1}}) \subset \rho_p$ be a global partial frame of TM and $\xi(\rho_{p-1}) = (E(\rho_{p-1}), \pi^{p-1}, M)$ the vector subbundle corresponding to ρ_{p-1}. We construct the vector bundle $\xi(\widetilde{\rho}_{p-1}) = (E(\widetilde{\rho}_{p-1}), \widetilde{\pi}^{p-1}, M)$ corresponding to an arbitrary frame $\widetilde{\rho}_{p-1} = (\varphi, \varphi')\rho_{p-1}, \varphi \in G(M)$. The vector bundles $\xi(\rho_{p-1})$, $\xi(\widetilde{\rho}_{p-1}))$ are isomorphic and they define an equivalence class $[M \times \mathbb{R}^{p-1}] = \{E((\varphi, \varphi')\rho_{p-1})/\rho_{p-1} = (X_{i_1}, \ldots, X_{i_{p-1}}), \varphi \in G(M)\}$. The class $[M \times \mathbb{R}^{p-1}]$ is independent of the selection of ρ_{p-1} and $\varphi \in G(M)$. In this case the structure of TM is $TM \simeq (M \times \mathbb{R}^{p-1}) \oplus TM/E(\rho_{p-1})$.

III. Repeating this process step by step, we arrive at $\rho_1 = \{X\} \subset \rho_2 \subset \ldots \subset \rho_p$ and $\xi(\rho_1) = (E(\rho_1), \widetilde{\pi}^1, M)$. Thus is obtained the class $[M \times \mathbb{R}^1] = \{E((\rho, \rho')\rho_1)/\rho_1 = \{X\}, \varphi \in G(M)\}$. Now TM has the structure $TM \simeq (M \times \mathbb{R}) \oplus TM/E(\rho_1)$.

IV. Analysis of excepted case $p = m$. In this situation M is a parallelizable manifold and $[M \times \mathbb{R}^m] = \{E((\varphi, \varphi')\rho_m)/\rho_m = (X_1, \ldots, X_m),\ \varphi \in G(M)\} = [TM]$. Moreover, M also admits the invariants $[M \times \mathbb{R}^i] = \{E((\varphi, \varphi')\rho_i)/\rho_i = (X_1, \ldots, X_i),\ \varphi \in G(M)\}$, $i = 1, 2, \ldots, m-1$. Every invariant $[M \times \mathbb{R}^i]$ can be written uniquely as an equivalence class. □

Definition 4.1. *The classes $[M \times \mathbb{R}^k]$, $k = 1, 2, \ldots, p$ are invariants of partial parallelizable manifold (M, ρ_p).*

The next results mark the connection between TM and $[M \times \mathbb{R}^p]$:

Theorem 4.2. *Let M be a paracompact C^∞-differentiable manifold and TM be its tangent vector bundle. The following statements are equivalent:*

1) TM admits a global partial frame $\rho_p = (X_1, \ldots, X_p)$;

2) TM is endowed with a trivial subbundle $E(\rho_p)$.

Proof. Let $R(\rho_p)$ be the principal bundle of the frames of $E(\rho_p)$. The bundles $R(\rho_p)$ and $E(\rho_p)$ are associated. Then the transition functions of $R(\rho_p)$ and $E(\rho_p)$ coincide. It follows that $R(\rho_p)$ and $E(\rho_p)$ are simultaneously trivial. Therefore 1)$\Longleftrightarrow$2). □

For each fixed p the existence of a $G(M)$-invariant $[M \times \mathbb{R}^p]$ on M describes the structure of TM. This special structure of TM is given in the following theorem:

Theorem 4.3. *Let M be a paracompact C^∞-differentiable manifold and TM be its tangent vector bundle. The following statements are equivalent:*

1) The structure of TM is given by the isomorphism $TM \simeq (M \times \mathbb{R}^p) \oplus TM/E(\rho_p)$.

2) M is endowed with an invariant $[M \times \mathbb{R}^p]$, where p is fixed, $1 \leq p < m$.

Moreover, M admits the invariants $[M \times \mathbb{R}^i]$, $i = 1, 2, \dots, p-1$.

If M is a parallelizable manifold, then M admits the invariant $[M \times \mathbb{R}^m]$, too.

Proof. The structure of TM given by condition 1) defines a global partial frame $\theta^p = (M \times \mathbb{R}^p \longrightarrow M)$. This means that θ^p has a global section generated by p vector fields $X_1, \dots, X_p$. Using the definition of the class $[M \times \mathbb{R}^p]$, it results that 1)$\Longleftrightarrow$2).

We remark that $\rho_i = (X_1, \dots, X_i) \subset \rho_p = (X_1, \dots, X_p)$, $i = 1, 2, \dots, p-1$ and ρ_i generates the class $[M \times \mathbb{R}^i]$.

If M is parallelizable then $\rho_m = (X_1, \dots, X_m)$ generates $[M \times \mathbb{R}^m]$. □

Now, we can formulate the following.

Definition 4.2. *A differentiable manifold in which one, hence all of these conditions holds, is called a partial parallelizable manifold.*

Evident properties of the p.p.m. (M, ρ_p) are:

1. There exist the following bijections: $\{\rho_i\} \longmapsto \{[M \times \mathbb{R}^i]\} \longmapsto \xi(\rho_i)$, $i = 1, 2, \dots, m$.
2. The p.p.m. (M, ρ_p) is a parallelizable manifold iff $[M \times \mathbb{R}^p] = [M \times \mathbb{R}^m]$;
3. The definition of $[M \times \mathbb{R}^i]$ indicates the way by which the submanifolds $E(\rho_i), \dots, E(\widetilde{\rho}_i), \dots$ are glued together;
4. There is possibility to define (using the direct sum of subbundles) an associative and commutative operation $*$ in a natural way: $[M \times \mathbb{R}^i] * [M \times \mathbb{R}^j] = [M \times \mathbb{R}^{i+j}]$, $1 < i + j \leq m$. The null vector bundle $(M, 1_M, M)$ defines the unit element for $*$. $[M \times \mathbb{R}^i] = [M \times \mathbb{R}^j] \Longleftrightarrow i = j$.
5. If $G(M) = \{1_M\}$, then $[M \times \mathbb{R}^i] = \{E(\rho_i), i = 1, 2, \dots, m-1\}$; $[M \times \mathbb{R}^m] = \{E(\rho_m)\} = [TM]$.

5 Applications to integrable partial trivial structures

Now we present some applications to integrable partial trivial structures.

Definition 5.1. *A p.t.s. ρ_p is integrable if the corresponding tangent subbundle $E(\rho_p)$ is integrable.*

Lemma 5.1. *Let M, M' be differentiable manifolds, $\varphi : M \to M'$ is a diffeomorphism, and $\rho_p = (X_i)$, $\widetilde{\xi}_p(\widetilde{X}_i)$, $i = 1, 2, \dots, p$, g.p.f. of TM and TM', respectively. Then, $E(\rho_p)$ is integrable iff $E(\widetilde{\rho}_p)$ is integrable.*

Proof. The diffeomorphism φ induces the following relations: $[\varphi' X_i, \varphi' X_j] = \varphi'[X_i, X_j] = [\widetilde{X}_i, \widetilde{X}_j]$ where $[X_i, X_j] = C_{ij}^k X_k$, $[\widetilde{X}_i, \widetilde{X}_j] = \widetilde{C}_{ij}^k \widetilde{X}_k$ and C_{ij}^k are some functions, $C_{ij}^k = \widetilde{C}_{ij}^k \varphi$. Obviously, φ' is linear on the fibres. Then the result follows from the definitions of $E(\rho_p)$ and $E(\widetilde{\rho}_p)$. □

Theorem 5.2. *1) If $E(\rho_p), E(\widetilde{\rho}_p)$ are integrable subbundle of TM then Lie algebras of tangent vector fields to the corresponding foliations, are isomorphic.*

2) The subbundles $E(\rho_p), E(\widetilde{\rho}_p)$ of $[M \times \mathbb{R}^p]$ are simultaneous integrable.

Proof. 1) Through Lemma 5.1, the diffeomorphism $\varphi : M \to M'$ defines an isomorphism φ' between those Lie algebras.

2) The result follows from Lemma 5.1 and the definition of $[M \times \mathbb{R}^p]$. □

Remark 5.1. Let $E(\rho_p)$ be integrable subbundle. Then:

a) The transverse bundle $\nu_p = TM/E(\rho_p)$ is $E(\rho_p)$-flat, i.e. its transition functions are constant along the leaves of $\widetilde{E}(\rho_p) =$ the foliation defined by $E(\rho_p)$ ([4]).

b) $G(M)$ contains the subgroups: $G_1(M) = \{\varphi \in G(M) | \varphi' \text{ conserves } \widetilde{E}(\rho_p)\}$, $G_2(M) = \{\varphi \in G_1(M) | \varphi \text{ conserves every leaf of } \widetilde{E}(\rho_p)\}$. Indeed, $\varphi \in G(M)$ conserves $E(\rho_p)$ iff φ' conserves $\widetilde{E}(\rho_p)$, i.e. $(\varphi, \varphi')(E(\rho_p)) = E(\rho_p) \iff \varphi(\widetilde{E}(\rho_p)) = \widetilde{E}(\rho_p)$.

Remark 5.2. A subbundle $E \subset TM$ is global integrable if there exists an integrable subbundle E' of TM, isomorphic with E ([8]).

The following result is immediate. Obviously, for the subbundles of $[M \times \mathbb{R}^p]$ the global integrability and usual integrability are equivalent.

Now, we suppose that each leaf $L \in \widetilde{E}(\rho_p)$ is Hausdorff and proper imbedded.

Remark 5.3. Let $\widetilde{M} = \cup\{L | L \in \widetilde{E}(\rho_p)\}$ be the maximal integrable manifold of $E(\rho_p)$. We will identify $\widetilde{M}$ with its image $i(\widetilde{M}) \subset M$, i being the inclusion. Let $G(\widetilde{M})$ be the group of global diffeomorphisms of $\widetilde{M}$. Then $[\widetilde{M} \times \mathbb{R}^k] = \{(\varphi, \varphi')E(\rho_k) | \varphi \in G(\widetilde{M})\}$, $k = 1, 2, \ldots, p-1$, $T\widetilde{M} \simeq \widetilde{M} \times \mathbb{R}^p$, $p = dim\, \widetilde{M}$ are invariants under the action of $G(\widetilde{M})$.

The following results are immediate:

Proposition 5.3. *Let $i^*(TM|E(\rho_p))$ be the induced bundle of $TM/E(\rho_p)$ under the inclusion $i^*\widetilde{M} \subset M$. Consider the vector bundle over $\widetilde{M}$, $\tau(\widetilde{M}) = T\widetilde{M} \oplus i^*(TM/E(\rho_p))$. Then $\tau(\widetilde{M}) = T\widetilde{M} \simeq \widetilde{M} \times \mathbb{R}^p$ if and only if $i^*(TM|E(\rho_p)) = (\widetilde{M}, 1_{\widetilde{M}}, \widetilde{M}) =$ the null vector bundle of $\widetilde{M}$.*

Proposition 5.4. *Let $G_1(M)$ be the subgroup of $G(M)$ which conserves $\widetilde{M}$ (i.e. $\varphi(\widetilde{M}) = \widetilde{M}$, $\forall \varphi \in G_1(M)$), and $\rho_p = (X_1, \ldots, X_p)$. Then $(\varphi, \varphi')E(\rho_p) = E(\rho_p)$ iff $\varphi(\widetilde{M}) = \widetilde{M}$.*

Proposition 5.5. *If $G(\widetilde{M})$ denotes the group of global diffeomorphisms of $\widetilde{M}$, then $[\widetilde{M} \times \mathbb{R}^k] = \{E((\varphi, \varphi')\rho_k | \varphi \in G(\widetilde{M})\}$, $k = 1, 2, \ldots, p-1$ and $[\widetilde{M} \times \mathbb{R}^p] = [T\widetilde{M}]$.*

Example 5.1. Let M, M' be some differentiable manifolds, $m = dim\, M$, $m' = dim\, M'$, where M' is a partial parallelizable (or parallelizable) manifold. Then $M \times M'$ is a partial parallelizable manifold. Models: $M \times S^n$, $M \times T^K$, $T(G_k(\mathbb{R}^s)) \oplus \theta^s$ where $\theta^s = (G(\mathbb{R}^n) \times \mathbb{R}^s \to G_k(\mathbb{R}^n))$, G a Lie group. $T(M \times M') \simeq ((M \times M') \times \mathbb{R}^s) \oplus T(M \times M')|E(\rho_s)$, where $E(\rho_s)$ denotes the subbundle spanned by ρ_s, $1 \leq s \leq m'$. The invariants are $[M \times \mathbb{R}^s] = \{E(\rho_s) \simeq E(\widetilde{\rho}_s) \simeq \ldots\}$.

Example 5.2. a) We associate to $G_p(\mathbb{R}^n)$ the set $A = \left\{\binom{2}{1}, \binom{4}{1}, \binom{4}{3}, \binom{8}{1}, \binom{8}{7}\right\}$. If $\binom{n}{p} \in A$ then $G_p(\mathbb{R}^n)$ is a parallelizable manifold ([7]) and $[G_p(\mathbb{R}^n) \times \mathbb{R}^{p(n-p)}] = [T(G_p(\mathbb{R}^n))]$, dim $G_p(\mathbb{R}^n) = p(n-p)$.

b) If $\binom{n}{p} \notin A$ then certain manifolds $G_p(\mathbb{R}^n)$ are parallelizable manifolds ([18]) (a necessary condition: $n \geq p(n-p)$). There exists the invariants $[G_p(\mathbb{R}^n) \times \mathbb{R}^i]$, $i = 1, 2, \ldots, \theta(n)$, where $\theta(n)$ denotes the maximal number of linearly independent tangent vector fields on $G_p(\mathbb{R}^n)$ ([18]).

Example 5.3. For parallelizable spheres the following properties hold: $[S^k \times \mathbb{R}^k] = [TS^k]$, $k = 1, 3, 7$. If $k \neq 1, 3, 7$, we have $[S^k \times \mathbb{R}^i] = \{E(\rho_i), \ldots\}$, $i \neq k$, i is fixed.

Example 5.4. Let G be a Lie group, $m = dim\ G$. Then $[G \times \mathbb{R}^m] = [TG]$, $[G \times \mathbb{R}^i] = \ldots$ according to Section 4, $i = 1, 2, \ldots, p-1$.

Example 5.5. Consider the differentiable manifold $\mathbb{R}^n$ (defined by $\mathbb{R}^n$) and its tangent bundle $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$. One can foliate $\mathbb{R}^n$ by parallel copies of $\mathbb{R}^i$. Then $[\mathbb{R}^n \times \mathbb{R}^n] = [T\mathbb{R}^n]$, $[\mathbb{R}^n \times \mathbb{R}^i] = \{E(\rho_i), \ldots\}$, $i = 1, 2, \ldots, n-1$.

Example 5.6. The case A.2.1 of construction of partial trivial structures shows that there exists an infinity of partial trivial bundles associated to TM.

Example 5.7. a) If there exists a regular vector field X on M, then $\rho_1 = \{X\}$ defines an integrable partial trivial structure of TM.

b) For a parallelizable manifold (M, ρ_m), $m = dim\, M$, the global frame ρ_m defines an integrable trivial structure in TM.

Example 5.8. Let G be a Lie transformations (with effective action) group on a C^∞-manifold M, $m = dim\, M$, $n = dim\, G$. Consider the Lie algebra $\overline{G}$ of G and $\chi(M) =$ the space of all C^∞-vector fields on M. Since G is a parallelizable manifold, it follows that there exists p.t.s. on G. It is well known that there is a homomorphism $h : \overline{G} \to \chi(M)$. Suppose $1 \leq dim\ (\operatorname{Im} h) \leq n < m$ and $\sigma_k = (Y_1, \ldots, Y_k)$ defines a p.t.s. on G. If hY_i and Y_i are h-corresponding, then $h\sigma_k = (hY_1, \ldots, hY_k)$ defines a p.t.s. on M (Theorem 4.2).

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