sciendo

DOI: 10.2478/awutm-2025-0011

Annals of West University of Timisoara
Mathematics and Computer Science
61 (2025), 157–173

Analysis of RLC multi-term fractional boundary value problems

Ahlem Adoui, Assia Guezane-Lakoud, and Rabah Khaldi

Abstract This paper addresses the existence of solutions for a boundary value problem characterized by a fractional differential equation involving a multi-term formulation of the Riemann-Liouville-Caputo derivative. The multi-term approach allows for more accurate modelling of complex systems with diverse memory and hereditary properties. To explore the existence and uniqueness of solutions, fixed point theorems are employed. Additionally, two numerical examples are provided to illustrate the theoretical results.

AMS Subject Classification (2020) 34B15; 26A33
Keywords Riemann-Liouville-Caputo derivative; multi-term differential equation; boundary value problem

1 Introduction

Multi-term fractional differential equations (FDEs) are important for studying complex systems with memory and hereditary effects. Unlike single-term models, they use multiple fractional orders to provide a more detailed view of processes with different levels of memory and relaxation. This makes them better suited for accurately modelling real-world systems in areas like viscoelasticity, finance, and control theory. For example, in materials science, viscoelastic materials such as polymers or biological tissues have both elastic and viscous properties that change over time. Single-term models might give a basic understanding, but they often miss the full complexity of these materials, making multi-term fractional derivatives necessary in mathematical models. The numerical investigation of multi-term FDEs is a significantly active strand of research, since they are very hard to deal with analytically. In effect, the presence of both the perturbation and the convection terms adds much complexity to the problems analysis. Due to this degree of complexity, they are hardly addressed in the literature.

The Riemann-Liouville and Caputo fractional derivatives are the most commonly used fractional derivatives. Many papers have studied boundary value problems (BVPs) involving these derivatives with various boundary conditions. However, recent applications suggest that the new conservative Caputo derivative, also known as the Riemann-Liouville-Caputo (RLC) fractional derivative, might be more suitable in certain situations, see [5] and the references therein. Although the literature on the RLC derivative is still growing, numerical methods have primarily been used to study BVPs involving this specific fractional derivative, see [5–8, 14, 27]. For instance, in [5], the authors used a finite difference scheme to study the following BVP:

$$\begin{aligned}
&-D^{c}D^{\alpha-1}u(t)+b(t)u'(t)+c(t)u(t)=f(t),\ t\in(0,L),\quad 1<\alpha<2,\\
&{}^{c}D^{\alpha-1}u(0)=0,\\
&u(L)+\beta_1u'(L)=\gamma_1.
\end{aligned}$$

Many authors have studied FDEs with a convection term, see [1–6,8,9,13,18,23–25,28]. In [1, 3–8, 10, 13, 20, 24–26] the focus has been on FDEs with a perturbation term, which are challenging to solve. Nonetheless, recent studies have shown that it is possible to handle the challenges posed by the perturbation terms, see [4, 10, 26] for more details. The issue with FDEs with a convection term or a perturbation term is that they are overly specific, addressing particular physical situations. Hence, their applicability is limited. Their implementation in real-world phenomena is restricted to a smaller number of specific settings. However, multi-term FDEs have the advantage of covering a broader number of scenarios; thus, they represent a larger class of FDEs. Additionally, fractional boundary conditions provide an interesting generalisation of classical ones, but they add complexity to the problem by making the solution more difficult to find, see [11,12,14–16, 19, 22, 27, 29]. Forasmuch as using fractional operators in the differential equation gives better results, it is similarly more convenient to use fractional order derivatives in the boundary conditions too. This choice of the boundary condition corresponds to a more general situation, allowing for more freedom in the choice of the derivative; hence, they provide an advantageous accuracy when modelling numerous physical settings compared to the classical integer order boundary conditions.

In light of the considerations mentioned above, this work investigates the following fractional boundary value problem:

$$\begin{aligned}
&-D^{c}D^{\alpha}u(t)+a\,{}^{c}D^{\beta}u(t)+ku(t)=f(t,u(t)),t\in[0,1],\ 0<\alpha<\beta<1,\\
&{}^{c}D^{\alpha}u(0)=0,\\
&u(1)=\eta\,{}^{c}D^{\alpha}u(1).
\end{aligned}\tag{1.1}$$

The mixed derivative D^cD^α is The Riemann-Liouville-Caputo (RLC) fractional derivative, which is known as the conservative Caputo derivative. D is the classical derivative. a, k, η are positive constants. In some particular cases, this equation represents the advection diffusion reaction equation. u represents the concentration of the transported quantity. a is the advection velocity. k is the reaction rate.

To the best of our knowledge, no existing paper has addressed fractional multi-term boundary value problems using the RLC fractional derivative with separated Caputo fractional boundary conditions. This paper contributes to the theory and applications of fractional boundary value problems by exploring the aforementioned problem. It encapsulates a multiplicity of novel contributions. Primarily, most multi-term problems discussed in related literature involve an integer order convection term, while this paper considers a differential equation containing a fractional order convection term, thereby elevating the mathematical significance of our research question. Regarding the applicability of the problem, this consideration is more compelling as it models a wider range of physical problems, especially ones that have been largely overlooked in existing literature, for instance, the fractional Lienards equation and the Van der Pol Oscillator equation. Further, the boundary conditions in [13] and similar studies are predominantly of the Dirichlet type, rather than the ones considered herein, which are more flexible, resilient, and general than often. Thus, our work complements and advances existing literature, given the aforementioned matters.

Nevertheless, multi-term Fractional differential equations induce many challenges in the study of this type of problem, as they result in loading the solutions form with numerous parameters. Moreover, including fractional boundary conditions elevates the aforesaid hurdle by complicating the form of the solution, making it difficult to handle, especially when attempting to prove the existence of positive solutions. While the positivity of solutions to these problems is a fascinating avenue of research, it is hardly addressed in this case, due to its notable complexity; thus, we do not undertake this pursuit here. Nonetheless, we strive to surpass the aforementioned challenges encountered in the proof techniques, and we obtain the desired results by employing appropriate Green's functions and accurate computations.

To solve the problem at hand, we transform it into a Volterra equation. We overcome the difficulty arising from the intricate form of the solution through the proper use of corresponding Green's functions. We then use fixed point theory techniques, namely, the Banach contraction principle, the Leray-Schauder nonlinear alternative, and the Krasnoselskii fixed point theorem, to establish existence results under various conditions imposed on the nonlinear term f.

The paper is organised into four sections. Section 2 introduces essential definitions and results from fractional calculus. Section 3 discusses the solvability of the problem and provides conditions for existence and uniqueness. Finally, Section 4 presents two numerical examples to confirm our results.

2 Preliminaries

In this section, we review the Caputo fractional derivative and its key properties that will be used later. We also provide the solution form for the corresponding linear problem.

Definition 2.1. *[17] Let $\alpha \geq 0$, $n = [\alpha] + 1$. If $f \in L^1(a,b)$, then the left-sided Riemann-Liouville fractional integral is defined by*

$$I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ t > a.$$

If $f \in AC^n[a,b]$, the left sided Caputo fractional derivative of f exists almost everywhere on $[a,b]$ and is given by

$$ {}^cD^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(s)(t-s)^{n-\alpha-1}dt. $$

Lemma 2.1. *[17, 21] Let $\alpha, \beta > 0$, $n = [\alpha]+1$. Then*

$$ I^{\alpha}_{a+}(1) = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}, \tag{2.1} $$

$$ {}^cD^{\alpha}_{a+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}, \ \beta > [\alpha]+1. \tag{2.2} $$

If $f \in L^1(a,b)$, then

$$ I^{\alpha}_{a+}I^{\beta}_{a+}f(t) = I^{\alpha+\beta}_{a+}f(t), \text{ a.e. on } [a,b]. \tag{2.3} $$

If $f \in AC^n[a,b]$ or $f \in C^n[a,b]$, then

$$ I^{\alpha}_{a+}({}^cD^{\alpha}_{a+})f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} f^{(k)}(a), \ \text{ a.e. on } [a,b], \tag{2.4} $$

$$ {}^cD^{\alpha}_{a+}I^{\alpha}_{a+}f(t) = f(t), \ \text{ a.e. on } [a,b], \tag{2.5} $$

$$ {}^cD^{\alpha}_{a+}I^{\beta}_{a+}f(t) = I^{\beta-\alpha}_{a+}f(t), \ \alpha < \beta, \ \text{ a.e. on } [a,b]. \tag{2.6} $$

In what follows, we denote I^{α}_{0+} by I^{α}, ${}^cD^{\alpha}_{0+}$ by ${}^cD^{\alpha}$ and $C([0,1]\times\mathbb{R})$ by $\mathcal{C}$.

Lemma 2.2. *Let $h \in \mathcal{C}$. The corresponding linear BVP*

$$ \begin{aligned} &-D^cD^{\alpha}u(t) + a^cD^{\beta}u(t) + ku(t) = h(t), t \in (0,1), \\ &{}^cD^{\alpha}(0) = 0, \\ &u(1) = \eta^cD^{\alpha}u(1) \end{aligned} \tag{2.7} $$

is equivalent to the BVP

$$ \begin{aligned} &-^c D^{\alpha+1}u(t) + a^cD^{\beta}u(t) + ku(t) = h(t), t \in (0,1), \\ &u'(0) = 0, \\ &u(1) = \eta^cD^{\alpha}u(1). \end{aligned} \tag{2.8} $$

Moreover, if $\frac{a}{\Gamma(\alpha-\beta+2)} - \frac{\eta a}{\Gamma(2-\beta)} - 1 \neq 0$, then every solution $u \in \mathcal{C}$ to (2.8) satisfies the integral equation

$$ u(t) = \int_0^1 \mathcal{G}_1(t,s)u(s)ds - \int_0^1 \mathcal{G}_2(t,s)h(s)ds, $$

where

$$ \mathcal{G}_1(t,s) = \begin{cases} \frac{a\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{\eta(1-s)^{-\beta}}{\Gamma(1-\beta)}\right) + \frac{k\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} - \eta\right), & 0 \le t \le s \le 1, \\ \frac{a\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} - \frac{\eta(1-s)^{-\beta}}{\Gamma(1-\beta)}\right) + \frac{k\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)} - \eta\right) + \frac{a(t-s)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{k(t-s)^{\alpha}}{\Gamma(\alpha+1)}, & 0 \le s \le t \le 1, \end{cases} \tag{2.9} $$

and

$$\mathcal{G}_2(t,s)=\begin{cases}\frac{\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\right), & 0\le t\le s\le 1,\\ \frac{\phi(t)}{\xi}\left(\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}-\eta\right)+\frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)}, & 0\le s\le t\le 1,\end{cases}\tag{2.10}$$

with $\xi=\frac{a\eta}{\Gamma(2-\beta)}-\frac{a}{\Gamma(\alpha-\beta+2)}+1$ *and* $\phi(t)=\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1$. *Moreover,* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are uniformly bounded i.e., there exist* $\kappa_1,\kappa_2>0$ *such that*

$$\left|\int_0^1\mathcal{G}_1(t,s)ds\right|\le\kappa_1,\quad \left|\int_0^1\mathcal{G}_2(t,s)ds\right|\le\kappa_2,\quad \textit{for every } t\in[0,1].$$

Proof. For the first equivalence, see [5]. Next, let $u\in\mathcal{C}$ be a solution to (2.8), applying the operator $I^{\alpha+1}$ to the fractional differential equation

$$-{}^cD^{\alpha+1}u(t)+a{}^cD^{\beta}u(t)+ku(t)=h(t),$$

then using (2.3) and (2.4), we get

$$-(u(t)+c_0+c_1t)+aI^{\alpha+1-\beta}(u(t)+c_0)+kI^{\alpha+1}u(t)=I^{\alpha+1}h(t),$$

applying (2.1),

$$-u(t)-c_0-c_1t+aI^{\alpha-\beta+1}u(t)+\frac{ac_0t^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}+kI^{\alpha+1}u(t)=I^{\alpha+1}h(t).$$

Therefore,

$$u(t)=c_0\left(\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}-1\right)-c_1t+aI^{\alpha-\beta+1}u(t)+kI^{\alpha+1}u(t)-I^{\alpha+1}h(t).\tag{2.11}$$

Employing the boundary condition $u'(0)=0$, we obtain $c_1=0$. Now, applying the operator ${}^cD^{\alpha}$ to (2.11) besides using (2.2), (2.5) and (2.6), we retrieve

$${}^cD^{\alpha}u(t)=\frac{c_0at^{1-\beta}}{\Gamma(2-\beta)}+aI^{1-\beta}u(t)+kI^1u(t)-I^1h(t).$$

Owing to the boundary condition $u(1)=\eta{}^cD^{\alpha}u(1)$, we find

$$\begin{aligned}c_0\left(\frac{a}{\Gamma(\alpha-\beta+2)}-1\right)+aI^{\alpha-\beta+1}u(1)+kI^{\alpha+1}u(1)-I^{\alpha+1}h(1)&\\ =\frac{\eta c_0a}{\Gamma(2-\beta)}+a\eta I^{1-\beta}u(1)+k\eta I^1u(1)-\eta I^1h(1).&\end{aligned}$$

Hence,

$$\begin{aligned}&c_0\left(\frac{a\eta}{\Gamma(2-\beta)}-\frac{a}{\Gamma(\alpha-\beta+2)}+1\right)\\ &\quad=a\Big(I^{\alpha-\beta+1}u(1)-\eta I^{1-\beta}u(1)\Big)+k\Big(I^{\alpha+1}u(1)-\eta I^1u(1)\Big)-\Big(I^{\alpha+1}h(1)-\eta I^1h(1)\Big).\end{aligned}$$

Setting $\xi = \frac{a\eta}{\Gamma(2-\beta)} - \frac{a}{\Gamma(\alpha-\beta+2)} + 1$, $\phi(t) = \frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} - 1$ and substituting c_0 in (2.11) gives

$$\begin{aligned}
u(t) =& \frac{\phi(t)}{\xi}\Big(a(I^{\alpha-\beta+1}u(1) - \eta I^{1-\beta}u(1)) + k(I^{\alpha+1}u(1) - \eta I^{1}u(1)) - (I^{\alpha+1}h(1) - \eta I^{1}h(1))\Big) \\
&+ aI^{\alpha-\beta+1}u(t) + kI^{\alpha+1}u(t) - I^{\alpha+1}h(t) \\
=& \int_0^1 \mathcal{G}_1(t,s)u(s)ds - \int_0^1 \mathcal{G}_2(t,s)h(s)ds.
\end{aligned}$$

□

Now, we shall prove the uniform boundedness of $\mathcal{G}_1$ and $\mathcal{G}_2$.

$$\begin{aligned}
\left|\int_0^1 \mathcal{G}_1(t,s)ds\right| \leq& a\left(\left|\frac{\phi(t)}{\xi}\right|\left(\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^1 (1-s)^{\alpha-\beta}ds + \frac{\eta}{\Gamma(1-\beta)}\int_0^1 (1-s)^{-\beta}ds\right)\right. \\
&\left.+\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^t (t-s)^{\alpha-\beta}ds\right) \\
&+ k\left(\left|\frac{\phi(t)}{\xi}\right|\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}ds + \eta\right) + \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha}ds\right) \\
\leq& a\left(\frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}\right) + \frac{1}{\Gamma(\alpha-\beta+2)}\right) \\
&+ k\left(\frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) + \frac{1}{\Gamma(\alpha+2)}\right) := \kappa_1,
\end{aligned}$$

where $\nu = \max_{0\leq t\leq 1}\left|\frac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} - 1\right| = \frac{a}{\Gamma(\alpha-\beta+2)} - 1$, if $\frac{a}{\Gamma(\alpha-\beta+2)} > 1$ and $\nu = 1$ if $\frac{a}{\Gamma(\alpha-\beta+2)} \leq 1$. Similarly,

$$\begin{aligned}
\left|\int_0^1 \mathcal{G}_2(t,s)ds\right| \leq& \left|\frac{\phi(t)}{\xi}\right|\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}ds + \eta\right) + \frac{1}{\Gamma(\alpha+1)}\int_0^1 (t-s)^{\alpha}ds \\
\leq& \frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) + \frac{1}{\Gamma(\alpha+2)} := \kappa_2.
\end{aligned}$$

3 Main Results

In this section, we present three results on existence and uniqueness. The proofs are provided using fixed point theorems.

Theorem 3.1. *Assume $f\colon [0,1]\times\mathcal{C}\to\mathbb{R}$ satisfies the Lipschitz condition with respect to the second variable, i.e., there exists a constant $L>0$ such that*

$$|f(t,u) - f(t,v)| \leq L\|u-v\|, \quad t\in[0,b]\ u,v\in\mathcal{C},$$

and $\kappa_1 + L\kappa_2 < 1$. Then the BVP (1.1) has a unique solution.

Proof. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be the operator defined by

$$\mathcal{T}u(t) = \int_0^1 \mathcal{G}_1(t,s)u(s)ds - \int_0^1 \mathcal{G}_2(t,s)f(s,u(s))ds.$$

$\mathcal{T}$ is a contraction mapping. Indeed, let $u, v \in \mathcal{C}$, then

$$\begin{aligned}|\mathcal{T}u(t) - \mathcal{T}v(t)| &\leq \int_0^1 |\mathcal{G}_1(t,s)||u(s) - v(s)|ds + \int_0^1 |\mathcal{G}_2(t,s)||f(s,u(s)) - f(s,v(s))|ds \\ &\leq \int_0^1 |\mathcal{G}_1(t,s)||u(s) - v(s)|ds + L\int_0^1 |\mathcal{G}_2(t,s)||u(s) - v(s)|ds \\ &\leq (\kappa_1 + L\kappa_2)\|u - v\|.\end{aligned}$$

Hence, the existence of a unique fixed point follows by means of the Banach contraction principle. □

Theorem 3.2. *If $\kappa_1 < 1$, then the BVP (1.1) has at least one solution, provided that there exists $\zeta \in \mathcal{C}$ so that $|f(t,u)| \leq \zeta(t)$, for every $u \in \mathcal{C},\ t \in [0,1]$.*

Proof. Let $\mathcal{A}, \mathcal{B} : \mathcal{C} \to \mathcal{C}$ be two operators defined by $\mathcal{A}u(t) = \int_0^1 \mathcal{G}_1(t,s)u(s)ds$ and $\mathcal{B}u(t) = -\int_0^1 \mathcal{G}_2(t,s)f(s,u(s))ds$. We easily verify that $\mathcal{A}$ and $\mathcal{B}$ are continuous. Next, we prove that for every $u, v \in B_r$, we have $\mathcal{A}u + \mathcal{B}v \in B_r$.
By assumption, $\kappa_1 < 1$. Fix a positive number r such that $r > \dfrac{\kappa_2\|\zeta\|}{1-\kappa_1}$. Set $B_r = \{u \in \mathcal{C}, \|u\| \leq r\}$. Then for every $u, v \in B_r$ we have

$$\begin{aligned}|\mathcal{A}u(t) + \mathcal{B}v(t)| &\leq |\mathcal{A}u(t)| + |\mathcal{B}v(t)| \\ &\leq r\int_0^1 |\mathcal{G}_1(t,s)|ds + \int_0^1 |\mathcal{G}_2(t,s)||f(s,v(s))|ds \\ &\leq r\kappa_1 + \|\zeta\|\kappa_2 < r.\end{aligned}$$

Now, we shall show that $\mathcal{A}$ is a contraction mapping.

$$\begin{aligned}|\mathcal{A}u(t) - \mathcal{A}v(t)| &\leq \int_0^1 |\mathcal{G}_1(t,s)||u(s) - v(s)|ds \\ &\leq \|u - v\|\int_0^1 |\mathcal{G}_1(t,s)|ds \leq \kappa_1\|u - v\|.\end{aligned}$$

Since $\kappa_1 < 1$ by assumption, $\mathcal{A}$ is indeed a contraction mapping.
To this end, we prove that $\mathcal{B}$ is compact. First, $\mathcal{B}$ is uniformly bounded on B_r, since

$$\begin{aligned}|\mathcal{B}u(t)| &\leq \int_0^1 |\mathcal{G}_2(t,s)f(s,u(s))|ds \\ &\leq \|\zeta\|\int_0^1 |\mathcal{G}_2(t,s)|ds \leq \|\zeta\|\kappa_2.\end{aligned}$$

Second, we prove that $\mathcal{B}$ is equicontinuous. Let $t_1, t_2 \in [0,1]$ with $0 \le t_1 < t_2 \le 1$ and $u \in B_r$. We have

$$\begin{aligned}
|\mathcal{B}u(t_2) - \mathcal{B}u(t_1)| &= \left| \int_0^{t_2} \frac{(t_2-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds - \frac{1}{\Gamma(\alpha+1)} \int_0^{t_1} \frac{(t_1-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds \right| \\
&\quad + \frac{|\phi(t_1)-\phi(t_2)|}{\xi} \left| \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} f(s,u)ds - \eta \int_0^1 f(s,u)ds \right| \\
&\le \frac{1}{\Gamma(\alpha+1)} \left| \int_0^{t_1} \big((t_2-s)^\alpha - (t_1-s)^\alpha\big) f(s,u)ds - \int_{t_1}^{t_2} (t_2-s)^\alpha f(s,u)ds \right| \\
&\quad + \frac{a|t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1}|}{|\xi|} \left(\int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s,u)|ds + \eta \int_0^1 |f(s,u)|ds \right) \\
&\le \frac{\|\zeta\|}{\Gamma(\alpha+1)} \left(\int_0^{t_1} |(t_2-s)^\alpha - (t_1-s)^\alpha| ds + \int_{t_1}^{t_2} |(t_2-s)^\alpha| ds \right) \\
&\quad + \frac{a\|\zeta\|}{|\xi|\Gamma(\alpha-\beta+2)} |t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}| \left(\frac{1}{\Gamma(\alpha+2)} + \eta \right) \\
&\le \frac{\|\zeta\|}{\Gamma(\alpha+2)} (t_2^{\alpha+1} - t_1^{\alpha+1}) + \frac{a\|\zeta\|(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{|\xi|\Gamma(\alpha-\beta+2)} \left(\frac{1}{\Gamma(\alpha+2)} + \eta \right).
\end{aligned}$$

It is clear that the right-hand side of the inequality approaches zero as $t_1 \to t_2$. By the Arzelà-Ascoli theorem, we conclude that $\mathcal{B} : \mathcal{C} \to \mathcal{C}$ is completely continuous. Hence, by means of the Krasnoselskii's fixed point theorem, $\mathcal{T}$ has a fixed point $u \in B_r$. This means the boundary value problem has at least one solution. □

Theorem 3.3. *Under the following assumptions*

1. *f is a continuous function,*
2. *there exist a function $g \in C([0,1], \mathbb{R}_+)$, and a nondecreasing function $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ verifying $|f(t,u)| \le g(t)\psi(\|u\|)$, for every $t \in [0,1], u \in \mathcal{C}$, and*

$$\kappa_1 + \kappa_2 \|g\| \limsup_{r \to +\infty} \frac{\psi(r)}{r} < 1,$$

the BVP (1.1) has at least one solution.

Proof. Let $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be defined as

$$\mathcal{T}u(t) = \int_0^1 \mathcal{G}_1(t,s)u(s)ds - \int_0^1 \mathcal{G}_2(t,s)f(s,u(s))ds.$$

Using the Leray-Schauder nonlinear alternative, we will demonstrate that $\mathcal{T}$ is both continuous and completely continuous.

Step 1: $\mathcal{T}$ is continuous.

The proof of the continuity of $\mathcal{T}$ is straightforward and is therefore omitted.

Step 2: $\mathcal{T}$ maps bounded sets into bounded sets in $\mathcal{C}$.
Let $u \in B_r := \{u \in \mathcal{C}, \|u\| \leq r\}$. Since f satisfies the growth condition,

$$\begin{aligned}|\mathcal{T}u(t)| &\leq \int_0^1 |\mathcal{G}_1(t,s)||u(s)|ds + \int_0^1 |\mathcal{G}_2(t,s)||f(s,u(s))|ds \\ &\leq \kappa_1\|u\| + \kappa_2\|g\|\psi(\|u\|) \leq \kappa_1 r + \kappa_2\|g\|\psi(r) := l.\end{aligned}$$

Hence $\|\mathcal{T}u\| \leq l$.
Step 3: $\mathcal{T}$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.
Let $t_1,\ t_2 \in [0,1],\ t_1 < t_2$ and let $u \in B_r := \{u \in \mathcal{C}, \|u\| \leq r\}$ be a bounded set of $\mathcal{C}$. Then

$$\begin{aligned}&|\mathcal{T}u(t_2) - \mathcal{T}u(t_1)| \\ &\quad \leq \left|\int_0^1 (\mathcal{G}_1(t_2,s) - \mathcal{G}_1(t_1,s))u(s)ds\right| + \left|\int_0^1 (\mathcal{G}_2(t_2,s) - \mathcal{G}_2(t_1,s))f(s,u(s))ds\right| := I_1 + I_2.\end{aligned}$$

On the one hand,

$$\begin{aligned}I_1 =& \left|\int_0^1 (\mathcal{G}_1(t_2,s) - \mathcal{G}_1(t_1,s))u(s)ds\right| \\ \leq& |\phi(t_2) - \phi(t_1)|\Big(\frac{a}{|\xi|}\Big|\frac{1}{\Gamma(\alpha-\beta+1)}\int_0^1 (1-s)^{\alpha-\beta}u(s)ds - \frac{\eta}{\Gamma(1-\beta)}\int_0^1 (1-s)^{-\beta}u(s)ds\Big| \\ &+ \frac{k}{|\xi|}\Big|\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-s)^{\alpha}ds - \eta\int_0^1 u(s)ds\Big|\Big) \\ &+ \frac{a}{\Gamma(\alpha-\beta+1)}\Big|\int_0^{t_2} (t_2-s)^{\alpha-\beta}u(s)ds - \int_0^{t_1} (t_1-s)^{\alpha-\beta}u(s)ds\Big| \\ &+ k\frac{1}{\Gamma(\alpha+1)}\Big|\int_0^{t_2} (t_2-s)^{\alpha}u(s)ds - \int_0^{t_1} (t_1-s)^{\alpha}u(s)ds\Big| \\ \leq& r\frac{a(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{\Gamma(\alpha-\beta+2)}\left(\frac{a}{|\xi|}\left(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}\right) + \frac{k}{|\xi|}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right)\right) \\ &+ r\frac{2a(t_2-t_1)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} + r\frac{k(t_2^{\alpha+1} - t_1^{\alpha+1})}{\Gamma(\alpha+2)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}I_2 =& \left|\int_0^1 (\mathcal{G}_2(t_2,s) - \mathcal{G}_2(t_1,s))f(s,u(s))ds\right| \\ \leq& \frac{\|g\|\psi(\|u\|)}{\Gamma(\alpha+1)}\left(\int_0^{t_1} |(t_2-s)^{\alpha} - (t_1-s)^{\alpha}|ds + \int_{t_1}^{t_2} |(t_2-s)^{\alpha}|ds\right) \\ &+ \frac{a\|g\|\psi(\|u\|)}{|\xi|\Gamma(\alpha-\beta+2)}|t_1^{\alpha-\beta+1} - t_2^{\alpha-\beta+1}|\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) \\ \leq& \|g\|\psi(r)\left(\frac{a(t_2^{\alpha-\beta+1} - t_1^{\alpha-\beta+1})}{|\xi|\Gamma(\alpha-\beta+2)}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) + \frac{1}{\Gamma(\alpha+2)}(t_2^{\alpha+1} - t_1^{\alpha+1})\right).\end{aligned}$$

Both I_1 and I_2 tend to zero as t_1 tends to t_2. This completes the proof of equicontinuity. Given Steps $1-3$ and the Arzel-Ascoli theorem, we conclude that $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Step 4: A priori bounds

By assumption, we have that there exists $M > 0$ such that $\kappa_1 M + \kappa_2 \|g\|\psi(M) < M$. Let $\Omega = \{u \in \mathcal{C},\ \|u\| < M\}$. According to the previous steps $\mathcal{T} : \bar{\Omega} \to \bar{\Omega}$ is completely continuous. If there exists $u \in \partial\Omega,\ \lambda \in (0,1)$ such that $u = \lambda \mathcal{T}u$, then

$$\begin{aligned}|u(t)| \le |\lambda \mathcal{T}u(t)| \le |\mathcal{T}u(t)| &\le \int_0^1 |\mathcal{G}_1(t,s)||u(s)|ds + \int_0^1 |\mathcal{G}_2(t,s)||f(s,u(s))|ds\\ &\le \kappa_1\|u\| + \kappa_2\|g\|\psi(\|u\|) \le \kappa_1 M + \kappa_2\|g\|\psi(M) < M,\end{aligned}$$

which contradicts the fact that $u \in \partial\Omega$. Therefore, by the Leray-Schauder nonlinear alternative, we conclude that $\mathcal{T}$ has a fixed point in $\bar{\Omega}$. □

4 Examples

In this section, we provide three numerical examples to support the results obtained.

Example 4.1. Let us take $\alpha = 0.5,\ \beta = 0.6,\ a = 0.1,\ k = 0.2,\ \eta = 0.5$, and $f(t,u) = \frac{t+1}{20}\left(u - \frac{e^t}{t^2+1}\right)$. Then

$$\begin{aligned}\Gamma(\alpha-\beta+2) &= \Gamma(1.9) = 0.9618,\\ \Gamma(2-\beta) &= \Gamma(1.4) = 0.8873,\\ \Gamma(\alpha+2) &= \Gamma(2.5) = 1.3293.\end{aligned}$$

Then $\nu = \max\limits_{0\le t\le 1}\left|\dfrac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} - 1\right| = 1$. Hence,

$$\begin{aligned}&a\left(\frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}\right) + \frac{1}{\Gamma(\alpha-\beta+2)}\right) = 0.2723,\\ &\kappa_2 = \frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) + \frac{1}{\Gamma(\alpha+2)} = 2.0671,\\ &\kappa_1 = 0.2723 + 0.2(2.0671) = 0.6857.\end{aligned}$$

Moreover,

$$|f(t,u) - f(t,v)| \le \frac{t+1}{20}|u-v| \le \frac{1}{10}|u-v|.$$

Then $\kappa_1 + L\kappa_2 = 0.6857 + 0.1(2.0671) = 0.8924 < 1$. Therefore, by Theorem 3.1, the BVP (1.1) has a unique solution. Additionally, it is evident that the conditions of Theorem 3.2 are satisfied.

Furthermore, we employ the $L1$ and $L2$ discretisations of the Caputo fractional derivative given in [5] to obtain the numerical scheme by the finite difference method. We provide in Figure 1 the approximate solutions for Example 4.1 with different orders α and β.

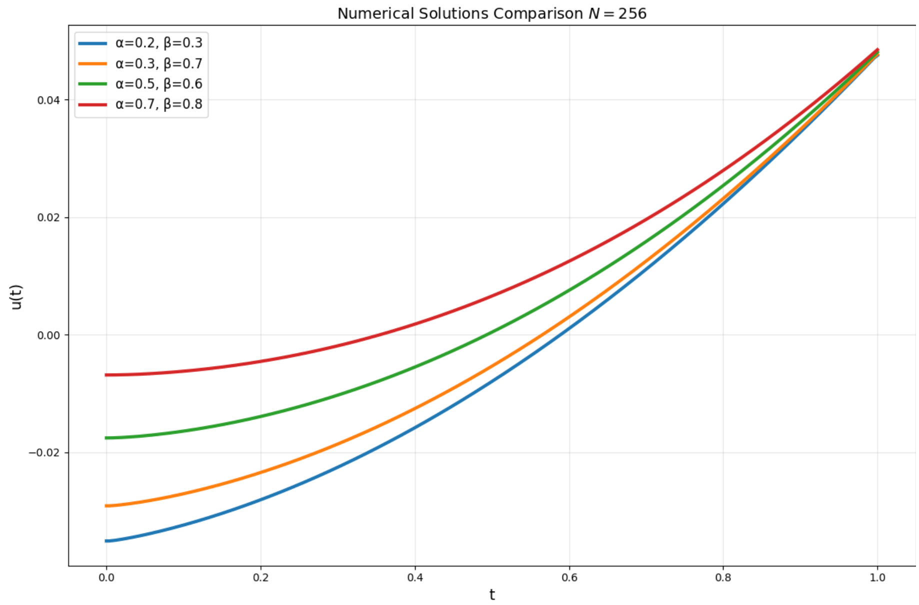

Figure 1: Numerical solutions for Example 4.1 with different orders α and β.

Table 1: Example 4.1 Errors and orders of convergence

	1/N	L_2 Error	L_2 Order	L_∞ Error	L_∞ Order
$\alpha = 0.2, \beta = 0.3$	1/16	1.6827e-03		6.6310e-03	
$\kappa_1 + L\kappa_2 = 0.987$	1/32	8.7757e-04	0.939	3.2391e-03	1.034
	1/64	4.6839e-04	0.906	1.5208e-03	1.091
	1/128	2.2699e-04	1.045	6.5448e-04	1.216
	1/256	8.3259e-05	1.447	2.1880e-04	1.581
$\alpha = 0.3, \beta = 0.7$	1/16	2.2420e-03		6.7160e-03	
$\kappa_1 + L\kappa_2 = 0.978$	1/32	1.2627e-03	0.828	3.2762e-03	1.036
	1/64	6.7057e-04	0.913	1.5361e-03	1.093
	1/128	3.1637e-04	1.084	6.6032e-04	1.218
	1/256	1.1291e-04	1.486	2.2055e-04	1.582
$\alpha = 0.5, \beta = 0.6$	1/16	3.2154e-03		6.7814e-03	
$\kappa_1 + L\kappa_2 = 0.892$	1/32	1.7546e-03	0.874	3.3074e-03	1.036
	1/64	8.8745e-04	0.983	1.5503e-03	1.093
	1/128	4.0087e-04	1.147	6.6613e-04	1.219
	1/256	1.3823e-04	1.536	2.2239e-04	1.583
$\alpha = 0.7, \beta = 0.8$	1/16	3.9474e-03		6.7658e-03	
$\kappa_1 + L\kappa_2 = 0.827$	1/32	2.0596e-03	0.939	3.2927e-03	1.039
	1/64	1.0022e-03	1.039	1.5408e-03	1.096
	1/128	4.3931e-04	1.190	6.6123e-04	1.220
	1/256	1.4817e-04	1.568	2.2055e-04	1.584

Example 4.2. Let us take $\alpha = 0.5,\ \beta = 0.6,\ a = 0.1,\ k = 0.2,\ \eta = 0.5$, and $f(t,u) = \frac{t^3}{2}\left(1 + \frac{|u|}{1+|u|}\right)$. Clearly

$$|f(t,u)| \leq g(t)\psi(\|u\|),$$

where $g(t) = \frac{t^3}{2}$ and $\psi(|u|) = 1 + \frac{|u|}{1+|u|}$. Then

$$\begin{aligned}
&\Gamma(\alpha-\beta+2) = \Gamma(1.9) = 0.9618,\\
&\Gamma(2-\beta) = \Gamma(1.4) = 0.8873,\\
&\Gamma(\alpha+2) = \Gamma(2.5) = 1.3293.
\end{aligned}$$

Then $\nu = \max\limits_{0\leq t\leq 1}\left|\dfrac{at^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)} - 1\right| = 1$. Hence,

$$\begin{aligned}
&a\left(\frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha-\beta+2)} + \frac{\eta}{\Gamma(2-\beta)}\right) + \frac{1}{\Gamma(\alpha-\beta+2)}\right) = 0.2723,\\
&\kappa_2 = \frac{\nu}{|\xi|}\left(\frac{1}{\Gamma(\alpha+2)} + \eta\right) + \frac{1}{\Gamma(\alpha+2)} = 2.0671,\\
&\kappa_1 = 0.2723 + 0.2(2.0671) = 0.6857.
\end{aligned}$$

Moreover,

$$|f(t,u) - f(t,v)| \leq \frac{1}{2}|u-v|.$$

Then $\kappa_1 + L\kappa_2 = 0.6857 + 0.5(2.0671) = 1.7193$. We lost the uniqueness of the solution. However, it is clear that $\kappa_1 + \kappa_2\|g\|\limsup\limits_{r\to+\infty}\dfrac{\psi(r)}{r} < 1$, so Theorem 3.3 applies. This guarantees the existence of at least one solution.

We provide in Figure 2 below the approximate solutions for Example 4.2 with different orders α and β.

Example 4.3. Let $u(t) = t^2$, then using (2.2) we have ${}^cD^{\alpha+1}u(t) = \frac{2}{\Gamma(2-\alpha)}t^{1-\alpha}$, ${}^cD^{\beta}u(t) = \frac{2}{\Gamma(3-\beta)}t^{2-\beta}$. Hence,

$$f(t,u) = -{}^cD^{\alpha+1}u(t) + a{}^cD^{\beta}u(t) + ku(t) = -\frac{2}{\Gamma(2-\alpha)}t^{1-\alpha} + a\frac{2}{\Gamma(3-\beta)}t^{2-\beta} + kt^2.$$

From the boundary condition $u(1) = \eta{}^cD^{\alpha}u(1)$ we choose $\eta = \frac{\Gamma(3-\alpha)}{2}$. Setting $\alpha = 0.7, \beta = 0.8, a = 0.1$, and $k = 0.1$ we find $\kappa_1 = 0.4715 < 1$. Clearly, the nonlinearity f satisfies the Lipschitz condition since $L = 0$. Therefore, by Theorem 3.1, the BVP (1.1) has a unique solution. We provide in Figure 3 the approximate solutions for $N = 128, 256, 512$ along with the exact solution.

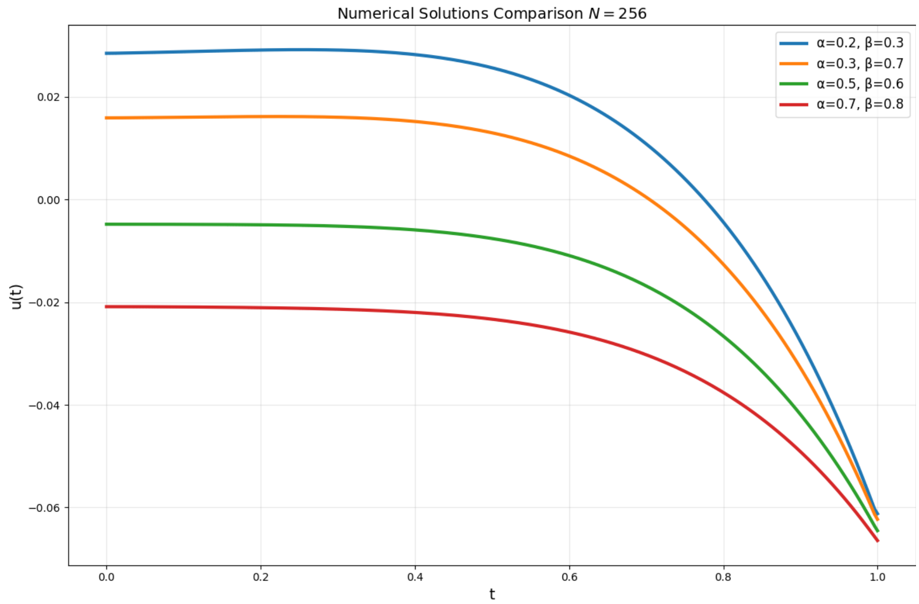

Figure 2: Numerical solutions for Example 4.2 with different orders α and β.

Table 2: Example 4.2 Errors and orders of convergence

	1/N	L_2 Error	L_2 Order	L_∞ Error	L_∞ Order
$\alpha = 0.2, \beta = 0.3$	$1/16$	4.6668e-03		2.0359e-02	
$\kappa_1 + L\kappa_2 = 1.943$	$1/32$	1.8775e-03	1.314	1.0710e-02	0.927
	$1/32$	1.8775e-03	1.314	1.0710e-02	0.927
	$1/64$	8.5120e-04	1.141	5.2309e-03	1.034
	$1/128$	4.1189e-04	1.047	2.3002e-03	1.185
	$1/256$	1.5910e-04	1.372	7.7829e-04	1.563
$\alpha = 0.3, \beta = 0.7$	$1/16$	4.7386e-03		2.0890e-02	
$\kappa_1 + L\kappa_2 = 1.896$	$1/32$	2.9624e-03	0.678	1.0996e-02	0.926
	$1/64$	1.7212e-03	0.783	5.3730e-03	1.033
	$1/128$	8.6158e-04	0.998	2.3621e-03	1.186
	$1/256$	3.1905e-04	1.433	7.9882e-04	1.564
$\alpha = 0.5, \beta = 0.6$	$1/16$	1.0461e-02		2.2040e-02	
$\kappa_1 + L\kappa_2 = 1.719$	$1/32$	6.1884e-03	0.757	1.1644e-02	0.921
	$1/64$	3.2629e-03	0.923	5.6922e-03	1.033
	$1/128$	1.5053e-03	1.116	2.5009e-03	1.187
	$1/256$	5.2457e-04	1.521	8.4483e-04	1.566
$\alpha = 0.7, \beta = 0.8$	$1/16$	1.5045e-02		2.2743e-02	
$\kappa_1 + L\kappa_2 = 1.569$	$1/32$	8.4398e-03	0.834	1.1975e-02	0.925
	$1/64$	4.2629e-03	0.985	5.8369e-03	1.037
	$1/128$	1.9039e-03	1.163	2.5577e-03	1.190
	$1/256$	6.4804e-04	1.555	8.6219e-04	1.569

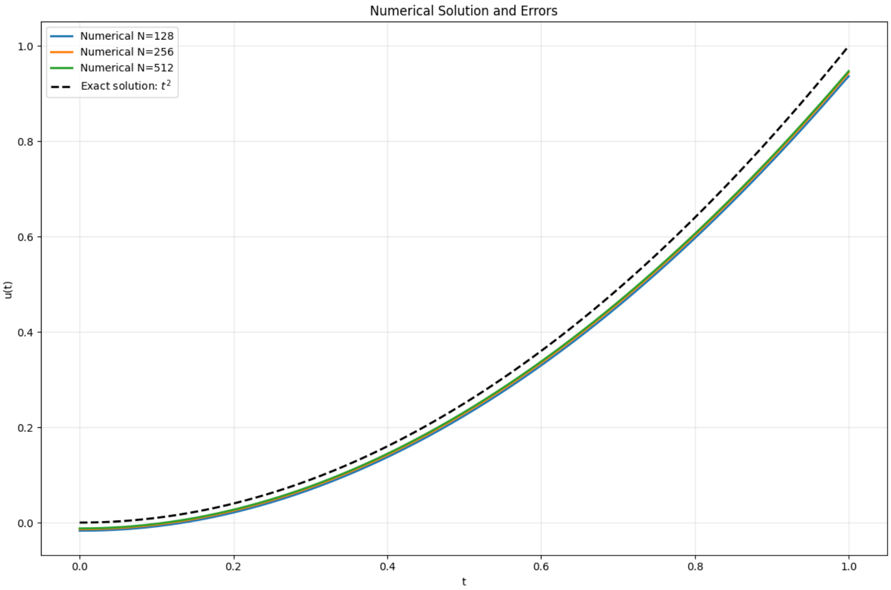

Figure 3: Exact and numerical solutions for Example 4.3 for different values of N.

Conclusion

In this paper, we explore a new setting involving fractional multi-term differential equations with the Riemann-Liouville-Caputo fractional derivative. We address the challenges posed by the three terms and the fractional boundary condition using Green's functions. By applying fixed point theorems, we obtain existence results under various conditions on the nonlinearity. Finally, we provide three numerical examples via the finite difference method to confirm our findings. Future work could explore other fractional derivatives and their impact on multi-term equations. It could also investigate higher-dimensional problems, system stability, and cases on infinite intervals.

Acknowledgement

The authors express their sincere gratitude to the anonymous referees for their valuable comments and thoughtful remarks, which improved this manuscript. The authors are thankful to Dr. A. Souahi, Department of Mathematics, Badji Mokhtar-Annaba University, Algeria for his efforts and insightful comments incorporated in the manuscript.

References

[1] M. Al-Refai, *Basic results on nonlinear eigenvalue problems of fractional order*, Electron. J. Differ. Equ. **2012** (2012), 1-12.

[2] M. Al-Refai, K. Pal, *A maximum principle for a fractional boundary value problem with convection term and applications*, Math. Model. Anal. **24** (2018), 62-71.

[3] O. Brandibur, E. Kaslik, *Stability properties of multi-term fractional-differential equations*, Fractal Fract. **7** (2023), 117.

[4] P. Eloe, J. Neugebauer, *Green's functions for a fractional boundary value problem with three terms*, Foundations **2** (2022), 885-897.

[5] J. L. Gracia, E. O'Riordan, M. Stynes, *Convergence analysis of a finite difference scheme for a two-point boundary value problem with a Riemann-Liouville-Caputo fractional derivative*, BIT Numer. Math. **60** (2020), 411-439.

[6] J. L. Gracia, M. Stynes, *Formal consistency versus actual convergence rates of difference schemes for fractional-derivative boundary value problems*, Fract. Calc. Appl. Anal. **18** (2015), 419-436.

[7] J. L. Gracia, M. Stynes, *A finite difference method for an initial-boundary value problem with a Riemann-Liouville-Caputo spatial fractional derivative*, J. Comput. Appl. Math. **381** (2021), 113020.

[8] J. L. Gracia, M. Stynes, *A collocation method for an RLC fractional derivative two-point boundary value problem with a singular solution*, Comput. Appl. Math. **43** (2024), 199.

[9] J. Graef, L. Kong, Q. Kong, M. Wang, *Positive solutions of nonlocal fractional boundary value problems*, Discrete Contin. Dyn. Syst. **2013** (2013), 282-290.

[10] J. Graef, L. Kong, Q. Kong, M. Wang., *On a fractional boundary value problem with a perturbation term*, J. Appl. Anal. Comput. **7** (2017), 57-66.

[11] A. Guezane-Lakoud, S. Bensebaa, *Solvability of a fractional boundary value problem with fractional derivative condition*, Arab. J. Math. **3** (2014), 39-48.

[12] A. Guezane-Lakoud, R. Khaldi, D. F. M. Torres, *On a fractional oscillator equation with natural boundary conditions*, Prog. Fract. Differ. Appl. **3** (2017), 1-7.

[13] B. Ibrahim, Q.-X. Dong, Z. Fan, *Existence for boundary value problems of two-term Caputo fractional differential equations*, J. Nonlinear Sci. Appl. **10** (2017), 511-520.

[14] L. Jia, H. Chen, V. Ervin, *Existence and regularity of solutions to 1-D fractional order diffusion equations*, preprint, arXiv: 1808.10555.

[15] R. Khaldi, A. Guezane-Lakoud, *Upper and lower solutions method for higher order boundary value problems*, Prog. Fract. Differ. Appl. **3** (2017), 53-57.

[16] R. Khaldi, A. Guezane-Lakoud, *On generalized nonlinear Euler-Bernoulli beam type equations*, Acta Univ. Sapientiae, Math. **10** (2018), 90-100.

[17] A. Kilbas, H. Srivastava, J.Trujillo, *Theory and applications of fractional differential equations*, Elsevier Amsterdam, 2006.

[18] X. Meng, M. Stynes, *The Green's function and a maximum principle for a Caputo two-point boundary value problem with a convection term*, J. Math. Anal. Appl. **461** (2018), 198-218.

[19] J. Rong, C. Bai, *Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions*, Adv. Difference Equ. **2015** (2015), 82.

[20] B. Sambandham, A. Vatsala, *Basic results for sequential Caputo fractional differential equations*, Mathematics **3** (2015), 76-91.

[21] S. Samko, A. Kilbas, O. Marichev, *Fractional integrals and derivatives*, Taylor & Francis, 1993.

[22] A. Souahi, A. Guezane-Lakoud, A. Hitta, *Positive solutions for higher-order nonlinear fractional differential equations*, Vietnam J. Math. **45** (2017), 441-450.

[23] S. Stanek, *Periodic problem for two-term fractional differential equations*, Fract. Calc. Appl. Anal. **20** (2017), 662-678.

[24] M. Stynes, J. L. Gracia, *A finite difference method for a two-point boundary value problem with a Caputo fractional derivative*, IMA J. Numer. Anal. **35** (2015), 698-721.

[25] A. Vatsala, G. Pageni, V. Vijesh, *Analysis of sequential Caputo fractional differential equations versus non-sequential Caputo fractional differential equations with applications*, Foundations **2** (2022), 1129-1142.

[26] O. K. Wanassi, F. Toumi, *Existence results for perturbed boundary value problem with fractional order*, Ric. Mat. **73** (2024), 1367-1383.

[27] H. Wang, D. Yang, *Wellposedness of Neumann boundary-value problems of space-fractional differential equations*, Fract. Calc. Appl. Anal. **20** (2017), 1356-1381.

[28] Y. Wei, Z. Bai, *Solvability of some fractional boundary value problems with a convection term*, Discrete Dyn. Nat. Soc. **2019** (2019), 1230502.

[29] C. Xie, S. Fang, *Finite difference scheme for timespace fractional diffusion equation with fractional boundary conditions*, Math. Methods Appl. Sci. **43** (2020), 3473-3487.

Ahlem Adoui
Laboratory of Systems and Advanced Materials
Faculty of Sciences, Badji Mokhtar-Annaba University
P.O Box.12, Annaba, 23000 Algeria
E-mail: `ahlem.adoui@univ-annaba.dz, ahlem.a.adoui@gmail.com`

Assia Guezane-Lakoud
Laboratory of Systems and Advanced Materials
Faculty of Sciences, Badji Mokhtar-Annaba University
P.O Box.12, Annaba, 23000 Algeria
E-mail: `a_guezane@yahoo.fr`

Rabah Khaldi
Laboratory of Systems and Advanced Materials
Faculty of Sciences, Badji Mokhtar-Annaba University
P.O Box.12, Annaba, 23000 Algeria
E-mail: `rkhadi@yahoo.fr`

Received: 7.09.2024
Revised: 25.05.2025, 19.07.2025
Accepted: 23.08.2025