sciendo
DOI: 10.2478/awutm-2025-0008

Annals of West University of Timisoara
Mathematics and Computer Science
61 (2025), 89–119

# New integral formulas with applications to integral inequalities

Christophe Chesneau

**Abstract** In this article, we derive new integral formulas involving a ratio function, a maximum function, and three adjustable parameters. Two of these parameters control the maximum function in different ways. The arctangent function plays a central role in the resulting expressions. These formulas are then used to construct new and varied types of integral inequalities. In particular, we present weighted Hölder-type integral inequalities, as well as new Hardy-Hilbert-type integral inequalities. Their novelty lies mainly in the inclusion of the maximum function and the two parameters governing it. Detailed proofs are given.

**AMS Subject Classification (2020)** 26D15; 33E20
**Keywords** maximum function; integral formulas; weighted Hölder-type integral inequalities; Hardy-Hilbert-type integral inequalities

## 1 Introduction

### 1.1 Context

Integral formulas and inequalities are fundamental to analysis. They are crucial for solving problems in both pure and applied mathematics. A comprehensive collection of the most important integral formulas can be found in the book [6], while important integral inequalities are presented in the books [2, 3, 5, 7, 15]. Among these inequalities, classical results, such as the Hölder and Hardy-Hilbert integral inequalities have inspired many generalizations. These often involve additional parameters, weight functions, extended dimensions, or modified functional forms. For a broader overview, see the survey [4] and the books [16, 17]. Recently, there has been a growing interest in integral inequalities with sophisticated functional structures, such as ratio or maximum functions. In particular, maximum functions with parameters enable finer control over the behavior of the integrand, making them useful for capturing extremal cases, modelling localized phenomena, or introducing adjustable complexity in various analytical settings. The results in this article contribute to this ongoing development.

### 1.2 Overview

For our mathematical background, a brief overview of Hardy-Hilbert integral inequalities involving ratio and maximum functions is necessary. The first result of this kind was obtained in [7], as formally mentioned below. Let $p > 1$, $q = p/(p-1)$ be the Hölder conjugate of $p$ and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} f^p(x)dx < +\infty$ and $\int_0^{+\infty} g^q(y)dy < +\infty$. Then the following inequality holds:

$$\iint_0^{+\infty} \frac{1}{\max(x,y)} f(x)g(y)dxdy \leq pq \left[\int_0^{+\infty} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} g^q(y)dy\right]^{1/q}, \tag{1.1}$$

where, for the sake of concision, we have set $\iint_0^{+\infty} = \int_0^{+\infty}\int_0^{+\infty}$. The upper bound is thus defined with a sharp constant factor, i.e., $pq = p^2/(p-1)$, and the unweighted integral norms of $f$ and $g$. A modification of this result was established in [14], with the addition of a parameter $\eta > 0$, as stated below. Assuming that $\int_0^{+\infty} x^{p-1-\eta} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q-1-\eta} g^q(y)dy < +\infty$, the following inequality holds:

$$\iint_0^{+\infty} \frac{1}{\max(x^\eta, y^\eta)} f(x)g(y)dxdy \leq \frac{pq}{\eta} \left[\int_0^{+\infty} x^{p-1-\eta} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q-1-\eta} g^q(y)dy\right]^{1/q}.$$

The upper bound is thus defined with a sharp constant factor, i.e., $pq/\eta = p^2/[\eta(p-1)]$, and the weighted integral norms of $f$ and $g$, with the weight functions $x^{p-1-\eta}$ and $y^{q-1-\eta}$, respectively. A more original variant, still defined with ratio and maximum functions, was introduced in [8] for the case $p = 2$. Formally, assuming that $\int_0^{+\infty} f^2(x)dx < +\infty$ and $\int_0^{+\infty} g^2(y)dy < +\infty$, the following inequality applies:

$$\iint_0^{+\infty} \frac{1}{x+y+\max(x,y)} f(x)g(y)dxdy$$
$$\leq \sqrt{2}\left\{\pi - 2\arctan[\sqrt{2}]\right\} \left[\int_0^{+\infty} f^2(x)dx\right]^{1/2} \left[\int_0^{+\infty} g^2(y)dy\right]^{1/2},$$

with the following numerical approximation fact: $\sqrt{2}\left\{\pi - 2\arctan[\sqrt{2}]\right\} \approx 1.74083$. This result was further generalized in [11] with the consideration of a norm parameter $p > 1$. Assuming that $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$, the following inequality holds:

$$\iint_0^{+\infty} \frac{1}{x+y+\max(x,y)} f(x)g(y)dxdy$$
$$\leq \sqrt{2}\left\{\pi - 2\arctan[\sqrt{2}]\right\} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}. \quad (1.2)$$

Still in [11], a one-power-parameter extension of this result was investigated, as mentioned below. For any $\tau > 1$, assuming that $\int_0^{+\infty} x^{1-\tau} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{1-\tau} g^q(y)dy < +\infty$, the following inequality applies:

$$\iint_0^{+\infty} \frac{1}{[x+y+\max(x,y)]^\tau} f(x)g(y)dxdy$$
$$\leq \frac{1}{2(\tau-1)}\left(\frac{1}{2^{\tau-2}} - \frac{1}{3^{\tau-1}}\right) \left[\int_0^{+\infty} x^{1-\tau} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{1-\tau} g^q(y)dy\right]^{1/q}.$$

A functional generalization was also examined in [11]. We complete this brief overview by mentioning other Hardy-Hilbert type integral inequalities defined with ratio and maximum functions in [1, 7–10, 12, 13]. These results illustrate the richness and versatility of this class of inequalities. They provide a foundation for the new contributions presented in this article.

### 1.3 Contributions

In this article, we aim to extend the current literature in two main directions. First, we establish new integral formulas that involve ratio and maximum functions. These formulas also have the property of depending on three adjustable parameters. Two of these parameters control the maximum function in different ways, of the form $\alpha\max(x,\gamma)$ or $\alpha\max(\gamma x, 1)$, where $\alpha$ and $\gamma$ denote the two parameters. Combined with the ratio function, this leads to original integral formulas in which the arctangent function appears naturally. To illustrate this claim, one of our results (the first) can be formulated as follows:

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha\max(x,\gamma)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\},$$

where $\alpha$, $\beta$ and $\gamma$ are the parameters, under some assumptions on them and on the integrability of $f$ and $g$.

Second, we use these formulas to construct new integral inequalities. These include weighted Hölder-type and Hardy-Hilbert-type integral inequalities, following the spirit of the results presented in the overview of the previous subsection. The main innovation is the way in which the maximum function and its governing parameters are incorporated. This structure leads to a variety of new and interesting inequality forms. To illustrate this claim, one of our results (the first) can be formulated as follows:

$$\iint_{0}^{+\infty} \frac{1}{1+\beta xy+\alpha\max(xy,\gamma)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}$$
$$\times\left[\int_{0}^{+\infty} x^{p/2-1}f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} y^{q/2-1}g^{q}(y)dy\right]^{1/q},$$

where $\alpha$, $\beta$ and $\gamma$ are the parameters, under some assumptions on them and on the integrability of $f$ and $g$. All of our results are clearly stated and accompanied by full proofs.

### 1.4 Organization

The article is organized as follows: Section 2 presents the main integral formulas, highlighting the role of each parameter. Section 3 applies these results to derive new integral inequalities. Detailed proofs are given throughout the text. We conclude with remarks on possible extensions and applications in Section 4.

## 2 New integral formulas

In this section, we present two sets of new integral formulas, where all formulas in each set share a common mathematical basis. As previously motivated, these formulas have the characteristic of involving a ratio functional structure, three adjustable parameters, and a maximum function controlled by two parameters.

### 2.1 First set of integral formulas

The first formula in this set is presented below.

**Proposition 2.1.** *For any $\alpha,\beta,\gamma\in\mathbb{R}$ such that $(1+\alpha\gamma)\beta>0$, $\beta\gamma/(1+\alpha\gamma)\geq 0$, $\beta+\alpha>0$ and $(\beta+\alpha)\gamma>0$, we have*

$$\int_{0}^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(x,\gamma)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}.$$

*The case $\beta \to 0$ yields*

$$\int_{0}^{+\infty} \frac{x^{-1/2}}{1+\alpha \max(x,\gamma)}dx = 2\left\{\frac{\sqrt{\gamma}}{1+\alpha\gamma}+\frac{1}{\sqrt{\alpha}}\arctan\left[\frac{1}{\sqrt{\alpha\gamma}}\right]\right\}.$$

*The case $\gamma \to 0$ yields*

$$\int_{0}^{+\infty} \frac{x^{-1/2}}{1+(\beta+\alpha)x}dx = \frac{\pi}{\sqrt{\beta+\alpha}}.$$

*Proof.* Making the change of variables $x = \gamma y$, we find that

$$\int_{0}^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(x,\gamma)}dx = \int_{0}^{+\infty} \frac{(\gamma y)^{-1/2}}{1+\beta\gamma y+\alpha \max(\gamma y,\gamma)}(\gamma dy)$$
$$= \sqrt{\gamma}\int_{0}^{+\infty} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \max(y,1)}dy.$$

Decomposing this integral by the Chasles integral formula at the threshold value $y = 1$, taking into account the term $\max(y, 1)$, and using standard arctangent primitives and formulas, we get

$$\sqrt{\gamma}\int_{0}^{+\infty} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \max(y,1)}dy$$
$$= \sqrt{\gamma}\left\{\int_{0}^{1} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \max(y,1)}dy + \int_{1}^{+\infty} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \max(y,1)}dy\right\}$$
$$= \sqrt{\gamma}\left\{\int_{0}^{1} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \times 1}dy + \int_{1}^{+\infty} \frac{y^{-1/2}}{1+\beta\gamma y+\alpha\gamma \times y}dy\right\}$$
$$= \sqrt{\gamma}\left\{\frac{1}{1+\alpha\gamma}\int_{0}^{1} \frac{y^{-1/2}}{1+[\sqrt{\beta\gamma y/(1+\alpha\gamma)}]^2}dy + \int_{1}^{+\infty} \frac{y^{-1/2}}{1+[\sqrt{(\beta+\alpha)\gamma y}]^2}dy\right\}$$

$$
\begin{aligned}
&= \sqrt{\gamma}\left\{\frac{1}{1+\alpha\gamma}\left[2\sqrt{\frac{1+\alpha\gamma}{\beta\gamma}}\arctan\left[\sqrt{\frac{\beta\gamma y}{1+\alpha\gamma}}\right]\right]_{y=0}^{y=1}\right. \\
&\quad \left. + \left[\frac{2}{\sqrt{(\beta+\alpha)\gamma}}\arctan[\sqrt{(\beta+\alpha)\gamma y}]\right]_{y=1}^{y\to+\infty}\right\} \\
&= \sqrt{\gamma}\left\{\frac{2}{\sqrt{(1+\alpha\gamma)\beta\gamma}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{2}{\sqrt{(\beta+\alpha)\gamma}}\left\{\frac{\pi}{2} - \arctan[\sqrt{(\beta+\alpha)\gamma}]\right\}\right\} \\
&= \sqrt{\gamma}\left\{\frac{2}{\sqrt{(1+\alpha\gamma)\beta\gamma}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{2}{\sqrt{(\beta+\alpha)\gamma}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\} \\
&= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha\max(x,\gamma)}dx \\
&= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}.
\end{aligned}
$$

For the case $\beta \to 0$, using the equivalence result $\arctan(t) \sim t$ when $t \to 0$, we obtain

$$
\begin{aligned}
&\int_0^{+\infty} \frac{x^{-1/2}}{1+\alpha\max(x,\gamma)}dx \\
&= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \times \sqrt{\frac{\beta\gamma}{1+\alpha\gamma}} + \frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}\Bigg|_{\beta\to 0} \\
&= 2\left\{\frac{\sqrt{\gamma}}{1+\alpha\gamma} + \frac{1}{\sqrt{\alpha}}\arctan\left[\frac{1}{\sqrt{\alpha\gamma}}\right]\right\}.
\end{aligned}
$$

For the case $\gamma \to 0$, using the limit result $\arctan(t) \to \pi/2$ when $t \to +\infty$, we get

$$
\begin{aligned}
&\int_0^{+\infty} \frac{x^{-1/2}}{1+(\beta+\alpha)x}dx \\
&= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}\Bigg|_{\gamma\to 0} \\
&= \frac{\pi}{\sqrt{\beta+\alpha}}.
\end{aligned}
$$

This ends the proof of Proposition 2.1. □

The second formula for this set is given below. Compared to the previous formula, the parameter $\gamma$ controls another aspect of the maximum function.

**Proposition 2.2.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta + \alpha\gamma > 0$ and $\gamma/(\beta+\alpha\gamma) \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(\gamma x, 1)} dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

*The case $\beta \to 0$ yields*

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\alpha \max(\gamma x, 1)} dx = \frac{2}{\sqrt{\gamma}}\left\{\frac{1}{1+\alpha} + \frac{1}{\sqrt{\alpha}} \arctan\left[\sqrt{\frac{1}{\alpha}}\right]\right\}.$$

*The case $\gamma \to 0$ yields*

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha} dx = \frac{\pi}{\sqrt{(1+\alpha)\beta}}.$$

*Proof.* We can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(\gamma x, 1)} dx = \int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha\gamma \max(x, 1/\gamma)} dx.$$

Applying Proposition 2.1 with "$\alpha\gamma$" instead of "$\alpha$" and "$1/\gamma$" instead of "$\gamma$", we get

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha\gamma \max(x, 1/\gamma)} dx$$
$$= 2\left\{\frac{1}{\sqrt{[1+(\alpha\gamma)/\gamma]\beta}} \arctan\left[\sqrt{\frac{\beta/\gamma}{1+(\alpha\gamma)/\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha\gamma)/\gamma}}\right]\right\}$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

So we have

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(\gamma x, 1)} dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

For the case $\beta \to 0$, using the equivalence result $\arctan(t) \sim t$ when $t \to 0$, we obtain

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\alpha \max(\gamma x, 1)} dx$$
$$= 2\left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \times \sqrt{\frac{\beta}{(1+\alpha)\gamma}} + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\}\Bigg|_{\beta\to 0}$$
$$= \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{1+\alpha} + \frac{1}{\sqrt{\alpha}} \arctan\left[\sqrt{\frac{1}{\alpha}}\right] \right\}.$$

For the case $\gamma \to 0$, noticing that $\max(\gamma x, 1) = 1$ and using the limit result $\arctan(t) \to \pi/2$ when $t \to +\infty$, we get

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha} dx$$
$$= 2\left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\}\Bigg|_{\gamma\to 0}$$
$$= \frac{\pi}{\sqrt{(1+\alpha)\beta}}.$$

This ends the proof of Proposition 2.2. □

The third formula for this set is given below. Compared to the previous two results, $\beta$ has a different effect on the ratio function.

**Proposition 2.3.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $(1+\alpha\gamma)\beta > 0$, $\beta\gamma/(1+\alpha\gamma) \geq 0$, $\beta+\alpha > 0$ and $(\beta+\alpha)\gamma > 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha \max(\gamma x, 1)} dx$$
$$= 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right] \right\}.$$

*The case $\beta \to 0$ yields*

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\alpha \max(\gamma x, 1)} dx = 2\left\{ \frac{\sqrt{\gamma}}{1+\alpha\gamma} + \frac{1}{\sqrt{\alpha}} \arctan\left[\frac{1}{\sqrt{\alpha\gamma}}\right] \right\}.$$

*Proof.* It follows from Proposition 2.1 that

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(x, \gamma)} dx$$
$$= 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right] \right\}.$$

Making the change of variables $x = 1/y$, the integral can be expressed as

$$\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha \max(x,\gamma)}dx = \int_{+\infty}^{0} \frac{(1/y)^{-1/2}}{1+\beta(1/y)+\alpha\max(1/y,\gamma)}\left(-\frac{1}{y^2}dy\right)$$
$$= \int_0^{+\infty} \frac{y^{-1/2}}{y+\beta+\alpha y\max(1/y,\gamma)}dy = \int_0^{+\infty} \frac{y^{-1/2}}{y+\beta+\alpha\max(1,\gamma y)}dy$$
$$= \int_0^{+\infty} \frac{y^{-1/2}}{y+\beta+\alpha\max(\gamma y,1)}dy.$$

Uniformizing the notation, we therefore have

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\max(\gamma x,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}.$$

For the case $\beta \to 0$, using the equivalence result $\arctan(t) \sim t$ when $t \to 0$, we obtain

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\alpha\max(\gamma x,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\times\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}\Bigg|_{\beta\to 0}$$
$$= 2\left\{\frac{\sqrt{\gamma}}{1+\alpha\gamma}+\frac{1}{\sqrt{\alpha}}\arctan\left[\frac{1}{\sqrt{\alpha\gamma}}\right]\right\}.$$

This concludes the proof of Proposition 2.3. □

The fourth (and final) formula for this set is given below.

**Proposition 2.4.** *For any* $\alpha, \beta, \gamma \in \mathbb{R}$ *such that* $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta+\alpha\gamma > 0$ *and* $\gamma/(\beta+\alpha\gamma) \geq 0$, *we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\max(x,\gamma)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

*Proof.* We can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\max(x,\gamma)}dx = \int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\gamma\max(x/\gamma,1)}dx.$$

Applying Proposition 2.3 with "$\alpha\gamma$" instead of "$\alpha$" and "$1/\gamma$" instead of "$\gamma$", we find that

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\gamma\max(x/\gamma,1)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+(\alpha\gamma)/\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta/\gamma}{1+(\alpha\gamma)/\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha\gamma)/\gamma}}\right]\right\}$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

We therefore have

$$\int_0^{+\infty} \frac{x^{-1/2}}{x+\beta+\alpha\max(x,\gamma)}dx$$
$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}.$$

This ends the proof of Proposition 2.4. □

The result is identical to that obtained in Proposition 2.2. This can be proved directly by making the change of variables $y = 1/x$ in the main integral.

### 2.2 Second set of integral formulas

The second set of integral formulas considers a slight modification in the functional structure of the ratio function. Either the term $x$ is removed, making the old constant term, which was set to 1, more flexible, or this constant is removed and the term $x$ is activated.

The first formula in this set is presented below.

**Proposition 2.5.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma > 0$, $\beta + \alpha\gamma \neq 0$, $\alpha\beta > 0$ and $\beta/(\alpha\gamma) \geq 0$, we obtain*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta+\alpha\max(x,\gamma)}dx = 2\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}.$$

*The case $\beta \to 0$ yields*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\max(x,\gamma)}dx = \frac{4}{\sqrt{\gamma}}.$$

*Proof.* Making the change of variables $x = \gamma y$, we get

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta+\alpha\max(x,\gamma)}dx = \int_0^{+\infty} \frac{(\gamma y)^{-1/2}}{\beta+\alpha\max(\gamma y,\gamma)}(\gamma dy) = \sqrt{\gamma}\int_0^{+\infty} \frac{y^{-1/2}}{\beta+\alpha\gamma\max(y,1)}dy.$$

Decomposing the integral by the Chasles integral formula at the threshold value $y = 1$, taking into account the term $\max(y,1)$, and using standard power and arctangent primitives and formulas, we establish that

$$\begin{aligned}
&\sqrt{\gamma}\int_0^{+\infty}\frac{y^{-1/2}}{\beta+\alpha\gamma\max(y,1)}dy\\
&=\sqrt{\gamma}\left\{\int_0^{1}\frac{y^{-1/2}}{\beta+\alpha\gamma\max(y,1)}dy+\int_1^{+\infty}\frac{y^{-1/2}}{\beta+\alpha\gamma\max(y,1)}dy\right\}\\
&=\sqrt{\gamma}\left\{\int_0^{1}\frac{y^{-1/2}}{\beta+\alpha\gamma\times 1}dy+\int_1^{+\infty}\frac{y^{-1/2}}{\beta+\alpha\gamma\times y}dy\right\}\\
&=\sqrt{\gamma}\left\{\frac{1}{\beta+\alpha\gamma}\int_0^{1}y^{-1/2}dy+\frac{1}{\beta}\int_1^{+\infty}\frac{y^{-1/2}}{1+[\sqrt{\alpha\gamma y/\beta}]^2}dy\right\}\\
&=\sqrt{\gamma}\left\{\frac{1}{\beta+\alpha\gamma}\left[2\sqrt{y}\right]_{y=0}^{y=1}+\frac{1}{\beta}\left[2\sqrt{\frac{\beta}{\alpha\gamma}}\arctan\left[\sqrt{\frac{\alpha\gamma}{\beta}y}\right]\right]_{y=1}^{y\to+\infty}\right\}\\
&=\sqrt{\gamma}\left\{\frac{2}{\beta+\alpha\gamma}+\frac{2}{\sqrt{\alpha\beta\gamma}}\left\{\frac{\pi}{2}-\arctan\left[\sqrt{\frac{\alpha\gamma}{\beta}}\right]\right\}\right\}\\
&=\sqrt{\gamma}\left\{\frac{2}{\beta+\alpha\gamma}+\frac{2}{\sqrt{\alpha\beta\gamma}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}\\
&=2\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}.
\end{aligned}$$

So we have

$$\int_0^{+\infty}\frac{x^{-1/2}}{\beta+\alpha\max(x,\gamma)}dx=2\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}.$$

For the case $\beta\to 0$, using the equivalence result $\arctan(t)\sim t$ when $t\to 0$, we obtain

$$\int_0^{+\infty}\frac{x^{-1/2}}{\max(x,\gamma)}dx=\alpha\int_0^{+\infty}\frac{x^{-1/2}}{\alpha\max(x,\gamma)}dx=2\alpha\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\times\sqrt{\frac{\beta}{\alpha\gamma}}\right\}\Bigg|_{\beta\to 0}=\frac{4}{\sqrt{\gamma}}.$$

This concludes the proof of Proposition 2.5. □

The second formula for this set is given below. Compared to the previous formula, the parameter $\gamma$ affects the maximum function in another way.

**Proposition 2.6.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma > 0$, $\beta + \alpha \neq 0$, $\alpha\beta > 0$ and $\beta/\alpha \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(\gamma x, 1)} dx = \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}.$$

*The case $\beta \to 0$ yields*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\max(\gamma x, 1)} dx = \frac{4}{\sqrt{\gamma}}.$$

*Proof.* We can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(\gamma x, 1)} dx = \int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha\gamma \max(x, 1/\gamma)} dx.$$

Applying Proposition 2.5 with "$\alpha\gamma$" instead of "$\alpha$" and "$1/\gamma$" instead of "$\gamma$", we obtain

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha\gamma \max(x, 1/\gamma)} dx = 2 \left\{ \frac{\sqrt{1/\gamma}}{\beta + (\alpha\gamma)/\gamma} + \frac{1}{\sqrt{\alpha\gamma\beta}} \arctan\left[\sqrt{\frac{\beta}{(\alpha\gamma)/\gamma}}\right] \right\}$$
$$= \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}.$$

So we have

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(\gamma x, 1)} dx = \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}.$$

For the case $\beta \to 0$, using the equivalence result $\arctan(t) \sim t$ when $t \to 0$, we obtain

$$\int_0^{+\infty} \frac{x^{-1/2}}{\max(\gamma x, 1)} dx = \alpha \int_0^{+\infty} \frac{x^{-1/2}}{\alpha \max(\gamma x, 1)} dx = \alpha \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \times \sqrt{\frac{\beta}{\alpha}} \right\} \Bigg|_{\beta \to 0} = \frac{4}{\sqrt{\gamma}}.$$

This ends the proof of Proposition 2.6. □

The third formula for this set is presented below.

**Proposition 2.7.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma > 0$, $\beta + \alpha\gamma \neq 0$, $\alpha\beta > 0$ and $\beta/(\alpha\gamma) \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha \max(\gamma x, 1)} dx = 2 \left\{ \frac{\sqrt{\gamma}}{\beta + \alpha\gamma} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right] \right\}.$$

*Proof.* It follows from Proposition 2.5 that

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(x, \gamma)} dx = 2 \left\{ \frac{\sqrt{\gamma}}{\beta + \alpha\gamma} + \frac{1}{\sqrt{\alpha\beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha\gamma}} \right] \right\}.$$

Making the change of variables $x = 1/y$, the integral can be expressed as

$$\begin{aligned}
\int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(x, \gamma)} dx &= \int_{+\infty}^{0} \frac{(1/y)^{-1/2}}{\beta + \alpha \max(1/y, \gamma)} \left( -\frac{1}{y^2} dy \right) \\
&= \int_0^{+\infty} \frac{y^{-1/2}}{\beta y + \alpha y \max(1/y, \gamma)} dy = \int_0^{+\infty} \frac{y^{-1/2}}{\beta y + \alpha \max(1, \gamma y)} dy \\
&= \int_0^{+\infty} \frac{y^{-1/2}}{\beta y + \alpha \max(\gamma y, 1)} dy.
\end{aligned}$$

Uniformizing the notation, we therefore have

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha \max(\gamma x, 1)} dx = 2 \left\{ \frac{\sqrt{\gamma}}{\beta + \alpha\gamma} + \frac{1}{\sqrt{\alpha\beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha\gamma}} \right] \right\}.$$

This concludes the proof of Proposition 2.7. □

The fourth (and final) formula for this set is given below.

**Proposition 2.8.** *For any $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\gamma > 0$, $\beta + \alpha > 0$, $\alpha\beta \neq 0$ and $\beta/\alpha \geq 0$, we have*

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha \max(x, \gamma)} dx = \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha}} \right] \right\}.$$

*Proof.* We can write

$$\int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha \max(x, \gamma)} dx = \int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha\gamma \max(x/\gamma, 1)} dx.$$

Applying Proposition 2.7 with "$\alpha\gamma$" instead of "$\alpha$" and "$1/\gamma$" instead of "$\gamma$", we get

$$\begin{aligned}
\int_0^{+\infty} \frac{x^{-1/2}}{\beta x + \alpha\gamma \max(x/\gamma, 1)} dx &= 2 \left\{ \frac{\sqrt{1/\gamma}}{\beta + (\alpha\gamma)/\gamma} + \frac{1}{\sqrt{(\alpha\gamma)\beta}} \arctan \left[ \sqrt{\frac{\beta}{(\alpha\gamma)/\gamma}} \right] \right\} \\
&= \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha}} \right] \right\}.
\end{aligned}$$

This ends the proof of Proposition 2.8. □

The result is identical to that obtained in Proposition 2.6. We can prove that the two integrals involved are equal using the change of variables $y = 1/x$.

The integral formulas presented in these propositions are new to the best of our knowledge. In particular, they are not referenced in [6]. They can be used to derive new integral inequalities involving maximum and ratio functions. This claim is supported by the contributions in the next section.

# 3 Applications to integral inequalities

In this section, we establish weighted Hölder-type and Hardy-Hilbert-type integral inequalities based on the new integral formulas.

## 3.1 Some new weighted Hölder-type integral inequalities

A set of new weighted Hölder-type integral inequalities is given in the proposition below. The corresponding integrands have the property of involving ratio and maximum functions, as well as three adjustable parameters.

**Proposition 3.1.** *Let $p > 1$, $q = p/(p-1)$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a function such that $\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx < +\infty$. Then the inequalities below hold.*

**Inequality 1:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha\gamma)\beta > 0$, $\beta\gamma/(1+\alpha\gamma) \geq 0$, $\beta + \alpha > 0$ and $(\beta+\alpha)\gamma > 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{[1 + \beta x + \alpha \max(x, \gamma)]^{1/p}} f(x)dx \\
&\leq 2^{1/p} \left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right] \right\}^{1/p} \\
&\quad \times \left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

**Inequality 2:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta + \alpha\gamma > 0$ and $\gamma/(\beta + \alpha\gamma) \geq 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{[1+\beta x+\alpha \max(\gamma x, 1)]^{1/p}} f(x)dx \\
&\leq 2^{1/p}\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}^{1/p} \\
&\quad \times \left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

**Inequality 3:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha\gamma)\beta > 0$, $\beta\gamma/(1+\alpha\gamma) \geq 0$, $\beta+\alpha > 0$ and $(\beta+\alpha)\gamma > 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{[x+\beta+\alpha \max(\gamma x, 1)]^{1/p}} f(x)dx \\
&\leq 2^{1/p}\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}^{1/p} \\
&\quad \times \left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

**Inequality 4:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta+\alpha\gamma > 0$ and $\gamma/(\beta+\alpha\gamma) \geq 0$, we have*

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{[x+\beta+\alpha \max(x, \gamma)]^{1/p}} f(x)dx \\
&\leq 2^{1/p}\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}^{1/p} \\
&\quad \times \left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

**Inequality 5:** *For any $\alpha, \beta, \gamma \geq 0$ such that $\gamma > 0$, $\beta+\alpha\gamma > 0$, $\alpha\beta > 0$ and $\beta/(\alpha\gamma) \geq 0$, we obtain*

$$\begin{aligned}
&\int_0^{+\infty} \frac{1}{[\beta+\alpha \max(x, \gamma)]^{1/p}} f(x)dx \\
&\leq 2^{1/p}\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}^{1/p}\left[\int_0^{+\infty} x^{(q-1)/2} f^q(x)dx\right]^{1/q}.
\end{aligned}$$

**Inequality 6:** *For any $\alpha, \beta, \gamma \geq 0$ such that $\gamma > 0$, $\beta + \alpha > 0$, $\alpha\beta > 0$ and $\beta/\alpha \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[\beta + \alpha \max(\gamma x, 1)]^{1/p}} f(x)dx$$

$$\leq \frac{2^{1/p}}{\gamma^{1/(2p)}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}^{1/p} \left[ \int_0^{+\infty} x^{(q-1)/2} f^q(x)dx \right]^{1/q}.$$

**Inequality 7:** *For any $\alpha, \beta, \gamma \geq 0$ such that $\gamma > 0$, $\beta + \alpha\gamma > 0$, $\alpha\beta > 0$ and $\beta/(\alpha\gamma) \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[\beta x + \alpha \max(\gamma x, 1)]^{1/p}} f(x)dx$$

$$\leq 2^{1/p} \left\{ \frac{\sqrt{\gamma}}{\beta + \alpha\gamma} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right] \right\}^{1/p} \left[ \int_0^{+\infty} x^{(q-1)/2} f^q(x)dx \right]^{1/q}.$$

**Inequality 8:** *For any $\alpha, \beta, \gamma \geq 0$ such that $\gamma > 0$, $\beta + \alpha > 0$, $\alpha\beta > 0$ and $\beta/\alpha \geq 0$, we have*

$$\int_0^{+\infty} \frac{1}{[\beta x + \alpha \max(x, \gamma)]^{1/p}} f(x)dx$$

$$\leq \frac{2^{1/p}}{\gamma^{1/(2p)}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}^{1/p} \left[ \int_0^{+\infty} x^{(q-1)/2} f^q(x)dx \right]^{1/q}.$$

*For each of these inequalities, we can apply the limit $\beta \to 0$ or $\gamma \to 0$ if the convergence holds.*

*Proof.* We will only give a detailed proof of the first inequality. The other inequalities are based on the same mathematical principle, each using one of the propositions established in the previous section.

**Inequality 1:** A suitable product decomposition of the main integrand using the identity $1 = x^{-1/(2p)} x^{1/(2p)}$, followed by the application of the Hölder integral inequality at $p$ and $q$, gives

$$\int_0^{+\infty} \frac{1}{[1 + \beta x + \alpha \max(x, \gamma)]^{1/p}} f(x)dx = \int_0^{+\infty} \frac{x^{-1/(2p)}}{[1 + \beta x + \alpha \max(x, \gamma)]^{1/p}} x^{1/(2p)} f(x)dx$$

$$\leq \left[ \int_0^{+\infty} \frac{x^{-1/2}}{1 + \beta x + \alpha \max(x, \gamma)} dx \right]^{1/p} \left[ \int_0^{+\infty} x^{q/(2p)} f^q(x)dx \right]^{1/q}.$$

Using Proposition 2.1 and $p = q/(q-1)$, we can express the upper bound as

$$\left[\int_0^{+\infty} \frac{x^{-1/2}}{1+\beta x+\alpha\max(x,\gamma)}dx\right]^{1/p}\left[\int_0^{+\infty} x^{q/(2p)}f^q(x)dx\right]^{1/q}$$
$$= 2^{1/p}\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}^{1/p}$$
$$\times\left[\int_0^{+\infty} x^{(q-1)/2}f^q(x)dx\right]^{1/q}.$$

We therefore have

$$\int_0^{+\infty} \frac{1}{[1+\beta x+\alpha\max(x,\gamma)]^{1/p}}f(x)dx$$
$$\leq 2^{1/p}\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}}\arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha}}\arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}^{1/p}$$
$$\times\left[\int_0^{+\infty} x^{(q-1)/2}f^q(x)dx\right]^{1/q}.$$

The desired result is established.

**Inequalities 2-8:** For proving Inequalities 2-8, we proceed as the proof in Inequality 1 but using Propositions 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8, instead of Proposition 2.1, respectively.

This concludes the proof of Proposition 3.1. □

These weighted Hölder-type integral inequalities can be used to study integral operators involving ratio and maximum functions. In particular, they can be instrumental in analyzing the continuity of such operators with respect to the integral norms under consideration.

## 3.2 Some new Hardy-Hilbert-type integral inequalities

A set of new Hardy-Hilbert-type integral inequalities is given in the proposition below. They mainly deal with integrands involving ratio and maximum functions, three adjustable parameters, and the product variable $xy$.

**Proposition 3.2.** *Let $p > 1$, $q = p/(p-1)$ and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1}f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1}g^q(y)dy < +\infty$. Then the inequalities below hold.*

**Inequality 1:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha\gamma)\beta > 0$, $\beta\gamma/(1+\alpha\gamma) \geq 0$, $\beta+\alpha > 0$ and $(\beta+\alpha)\gamma > 0$, we have*

$$\iint_0^{+\infty} \frac{1}{1+\beta xy+\alpha \max(xy,\gamma)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}$$
$$\times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

**Inequality 2:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta+\alpha\gamma > 0$ and $\gamma/(\beta+\alpha\gamma) \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{1+\beta xy+\alpha \max(\gamma xy,1)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}$$
$$\times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

**Inequality 3:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha\gamma)\beta > 0$, $\beta\gamma/(1+\alpha\gamma) \geq 0$, $\beta+\alpha > 0$ and $(\beta+\alpha)\gamma > 0$, we have*

$$\iint_0^{+\infty} \frac{1}{xy+\beta+\alpha \max(\gamma xy,1)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\}$$
$$\times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

**Inequality 4:** *For any $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta+\alpha\gamma > 0$ and $\gamma/(\beta+\alpha\gamma) \geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{xy+\beta+\alpha\max(xy,\gamma)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}$$
$$\times\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

**Inequality 5:** *For any $\alpha,\beta,\gamma\geq 0$ such that $\gamma>0$, $\beta+\alpha\gamma>0$, $\alpha\beta>0$ and $\beta/(\alpha\gamma)\geq 0$, we obtain*

$$\iint_0^{+\infty} \frac{1}{\beta+\alpha\max(xy,\gamma)} f(x)g(y)dxdy$$
$$\leq 2\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

**Inequality 6:** *For any $\alpha,\beta,\gamma\geq 0$ such that $\gamma>0$, $\beta+\alpha>0$, $\alpha\beta>0$ and $\beta/\alpha\geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{\beta+\alpha\max(\gamma xy,1)} f(x)g(y)dxdy$$
$$\leq \frac{2}{\sqrt{\gamma}}\left\{\frac{1}{\beta+\alpha}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha}}\right]\right\}\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

**Inequality 7:** *For any $\alpha,\beta,\gamma\geq 0$ such that $\gamma>0$, $\beta+\alpha\gamma>0$, $\alpha\beta>0$ and $\beta/(\alpha\gamma)\geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{\beta xy+\alpha\max(\gamma xy,1)} f(x)g(y)dxdy \leq 2\left\{\frac{\sqrt{\gamma}}{\beta+\alpha\gamma}+\frac{1}{\sqrt{\alpha\beta}}\arctan\left[\sqrt{\frac{\beta}{\alpha\gamma}}\right]\right\}$$
$$\times\left[\int_0^{+\infty} x^{p/2-1}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1}g^q(y)dy\right]^{1/q}.$$

**Inequality 8:** *For any $\alpha,\beta,\gamma\geq 0$ such that $\gamma>0$, $\beta+\alpha>0$, $\alpha\beta>0$ and $\beta/\alpha\geq 0$, we have*

$$\iint_0^{+\infty} \frac{1}{\beta xy + \alpha \max(xy, \gamma)} f(x)g(y)dxdy \leq \frac{2}{\sqrt{\gamma}} \left\{ \frac{1}{\beta + \alpha} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[\sqrt{\frac{\beta}{\alpha}}\right] \right\}$$
$$\times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

*For each of these inequalities, we can apply the limit $\beta \to 0$ or $\gamma \to 0$ if the convergence holds.*

*Proof.* We will only give a detailed proof of the first inequality. The other inequalities are based on the same mathematical principle, each using one of the propositions established in the previous section.

**Inequality 1:** A suitable product decomposition of the main integrand considering the identity $1 = x^{1/(2q)} y^{-1/(2p)} x^{-1/(2q)} y^{1/(2p)}$ and the equality $1/p + 1/q = 1$, followed by the application of the Hölder integral inequality at $p$ and $q$, gives

$$\begin{aligned} &\iint_0^{+\infty} \frac{1}{1 + \beta xy + \alpha \max(xy, \gamma)} f(x)g(y)dxdy \\ &= \iint_0^{+\infty} \frac{x^{1/(2q)} y^{-1/(2p)}}{[1 + \beta xy + \alpha \max(xy, \gamma)]^{1/p}} f(x) \frac{x^{-1/(2q)} y^{1/(2p)}}{[1 + \beta xy + \alpha \max(xy, \gamma)]^{1/q}} g(y)dxdy \\ &\leq \mathfrak{A}^{1/p} \mathfrak{B}^{1/q}, \end{aligned} \tag{3.1}$$

where

$$\mathfrak{A} = \iint_0^{+\infty} \frac{x^{p/(2q)} y^{-1/2}}{1 + \beta xy + \alpha \max(xy, \gamma)} f^p(x)dxdy$$

and

$$\mathfrak{B} = \iint_0^{+\infty} \frac{x^{-1/2} y^{q/(2p)}}{1 + \beta xy + \alpha \max(xy, \gamma)} g^q(y)dxdy.$$

Let us determine the expressions of $\mathfrak{A}$ and $\mathfrak{B}$, one after the other.

Since the integrand associated with $\mathfrak{A}$ is clearly non-negative, we can apply the Fubini-Tonelli integral theorem that ensures the exchange of the order of integration. This, followed by the change of variables $u = xy$ with respect to $y$, the equality $q = p/(p-1)$ and the application of Proposition 2.1, gives

$$
\begin{aligned}
\mathfrak{A} &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[ \int_0^{+\infty} \frac{(xy)^{-1/2}}{1+\beta xy+\alpha \max(xy,\gamma)} x dy \right] dx \\
&= \int_0^{+\infty} x^{p/2-1} f^p(x) \left[ \int_0^{+\infty} \frac{u^{-1/2}}{1+\beta u+\alpha \max(u,\gamma)} du \right] dx \\
&= \int_0^{+\infty} x^{p/2-1} f^p(x) \\
&\quad \times \left[ 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[ \sqrt{\frac{\beta\gamma}{1+\alpha\gamma}} \right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[ \frac{1}{\sqrt{(\beta+\alpha)\gamma}} \right] \right\} \right] dx \\
&= 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[ \sqrt{\frac{\beta\gamma}{1+\alpha\gamma}} \right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[ \frac{1}{\sqrt{(\beta+\alpha)\gamma}} \right] \right\} \\
&\quad \times \int_0^{+\infty} x^{p/2-1} f^p(x) dx.
\end{aligned}
\tag{3.2}
$$

For the term $\mathfrak{B}$, we proceed in a similar way but with the change of variables $v = xy$ with respect to $x$, as follows:

$$
\begin{aligned}
\mathfrak{B} &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[ \int_0^{+\infty} \frac{(xy)^{-1/2}}{1+\beta xy+\alpha \max(xy,\gamma)} y dx \right] dy \\
&= \int_0^{+\infty} y^{q/2-1} g^q(y) \left[ \int_0^{+\infty} \frac{v^{-1/2}}{1+\beta v+\alpha \max(v,\gamma)} dv \right] dy \\
&= \int_0^{+\infty} y^{q/2-1} g^q(y) \\
&\quad \times \left[ 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[ \sqrt{\frac{\beta\gamma}{1+\alpha\gamma}} \right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[ \frac{1}{\sqrt{(\beta+\alpha)\gamma}} \right] \right\} \right] dy \\
&= 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[ \sqrt{\frac{\beta\gamma}{1+\alpha\gamma}} \right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[ \frac{1}{\sqrt{(\beta+\alpha)\gamma}} \right] \right\} \\
&\quad \times \int_0^{+\infty} y^{q/2-1} g^q(y) dy.
\end{aligned}
\tag{3.3}
$$

Substituting the expressions of $\mathfrak{A}$ and $\mathfrak{B}$ determined in Equations (3.2) and (3.3) into Equation (3.1), and using the equality $1/p + 1/q = 1$, we get

$$\begin{aligned}
&\iint_0^{+\infty} \frac{1}{1+\beta xy + \alpha \max(xy,\gamma)} f(x)g(y)dxdy \\
&\leq \left[ 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\} \right. \\
&\quad \left. \times \int_0^{+\infty} x^{p/2-1} f^p(x)dx \right]^{1/p} \\
&\quad \times \left[ 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\} \right. \\
&\quad \left. \times \int_0^{+\infty} y^{q/2-1} g^q(y)dy \right]^{1/q} \\
&= 2\left\{ \frac{1}{\sqrt{(1+\alpha\gamma)\beta}} \arctan\left[\sqrt{\frac{\beta\gamma}{1+\alpha\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha}} \arctan\left[\frac{1}{\sqrt{(\beta+\alpha)\gamma}}\right]\right\} \\
&\quad \times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

The desired inequality is established.

**Inequalities 2-8:** For proving Inequalities 2-8, we proceed as the proof in Inequality 1 but using Propositions 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 and 2.8, instead of Proposition 2.1, respectively.

This concludes the proof of Proposition 3.2. □

These results can be applied to the study of various integral operators defined using the proposed integrands and function space embeddings. They also highlight the importance of our integral formulas in establishing boundedness, continuity and other functional properties of such operators.

The proposition below can be viewed as a three-parameter extension of the result in [11], recalled in Equation (1.2).

**Proposition 3.3.** *Let $p > 1$, $q = p/(p-1)$, $\alpha, \beta, \gamma \geq 0$ such that $(1+\alpha)\beta > 0$, $\beta/[(1+\alpha)\gamma] \geq 0$, $\beta + \alpha\gamma > 0$ and $\gamma/(\beta+\alpha\gamma) \geq 0$, and $f, g : (0, +\infty) \mapsto (0, +\infty)$ be two functions such that $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ and $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$. Then we have*

$$\begin{aligned}
&\iint_0^{+\infty} \frac{1}{x+\beta y+\alpha \max(x,\gamma y)} f(x)g(y)dxdy \\
&\leq 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\} \\
&\quad \times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.
\end{aligned}$$

*Note that we can apply the limit $\beta \to 0$ or $\gamma \to 0$.*

*Proof.* A suitable product decomposition of the main integrand using the identity $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and the equality $1/p+1/q = 1$, followed by the application of the Hölder integral inequality at $p$ and $q$, gives

$$\begin{aligned}
&\iint_0^{+\infty} \frac{1}{x+\beta y+\alpha \max(x,\gamma y)} f(x)g(y)dxdy \\
&= \iint_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{[x+\beta y+\alpha \max(x,\gamma y)]^{1/p}} f(x) \frac{x^{-1/(2q)}y^{1/(2p)}}{[x+\beta y+\alpha \max(x,\gamma y)]^{1/q}} g(y)dxdy \\
&\leq \mathfrak{C}^{1/p}\mathfrak{D}^{1/q}, \qquad (3.4)
\end{aligned}$$

where

$$\mathfrak{C} = \iint_0^{+\infty} \frac{x^{p/(2q)}y^{-1/2}}{x+\beta y+\alpha \max(x,\gamma y)} f^p(x)dxdy$$

and

$$\mathfrak{D} = \iint_0^{+\infty} \frac{x^{-1/2}y^{q/(2p)}}{x+\beta y+\alpha \max(x,\gamma y)} g^q(y)dxdy.$$

Let us determine the expressions of $\mathfrak{C}$ and $\mathfrak{D}$, one after the other.

Since the integrand associated with $\mathfrak{C}$ is clearly non-negative, we can exchange the order of integration by the Fubini-Tonelli integral theorem. This, followed by the change of variables $u = y/x$ with respect to $y$, the use of $q = p/(p-1)$, and the application of Proposition 2.2, gives

$$\begin{aligned}
\mathfrak{C} &= \int_0^{+\infty} x^{p/(2q)-1/2} f^p(x) \left[\int_0^{+\infty} \frac{(y/x)^{-1/2}}{1+\beta(y/x)+\alpha \max(1,\gamma y/x)} \times \frac{1}{x}dy\right] dx \\
&= \int_0^{+\infty} x^{p/2-1} f^p(x) \left[\int_0^{+\infty} \frac{u^{-1/2}}{1+\beta u+\alpha \max(1,\gamma u)} du\right] dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{+\infty} x^{p/2-1} f^p(x) \\
&\quad \times \left\{ 2 \left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\} \right\} dx \\
&= 2 \left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\} \\
&\quad \times \int_0^{+\infty} x^{p/2-1} f^p(x) dx.
\end{aligned} \tag{3.5}
$$

For the term $\mathfrak{D}$, we proceed in a similar way but with the change of variables $v = x/y$ with respect to $x$, and Proposition 2.4, as follows:

$$
\begin{aligned}
\mathfrak{D} &= \int_0^{+\infty} y^{q/(2p)-1/2} g^q(y) \left[ \int_0^{+\infty} \frac{(x/y)^{-1/2}}{x/y+\beta+\alpha\max(x/y,\gamma)} \times \frac{1}{y} dx \right] dy \\
&= \int_0^{+\infty} y^{q/2-1} g^q(y) \left[ \int_0^{+\infty} \frac{v^{-1/2}}{v+\beta+\alpha\max(v,\gamma)} dv \right] dy \\
&= \int_0^{+\infty} y^{q/2-1} g^q(y) \\
&\quad \times \left\{ 2 \left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\} \right\} dy \\
&= 2 \left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\} \\
&\quad \times \int_0^{+\infty} y^{q/2-1} g^q(y) dy.
\end{aligned} \tag{3.6}
$$

Substituting the expressions of $\mathfrak{C}$ and $\mathfrak{D}$ obtained in Equations (3.5) and (3.6) into Equation (3.4), and using the equality $1/p + 1/q = 1$, we obtain

$$
\begin{aligned}
&\iint_0^{+\infty} \frac{1}{x+\beta y+\alpha\max(x,\gamma y)} f(x)g(y) dx dy \\
&\leq \left[ 2 \left\{ \frac{1}{\sqrt{(1+\alpha)\beta}} \arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right] + \frac{1}{\sqrt{\beta+\alpha\gamma}} \arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right] \right\} \right.
\end{aligned}
$$

$$\times \int_0^{+\infty} x^{p/2-1} f^p(x)dx\Bigg]^{1/p}$$

$$\times \left[2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}\right.$$

$$\left.\times \int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}$$

$$= 2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}$$

$$\times \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This ends the proof of Proposition 3.3. □

In particular, taking $\alpha = 1$, $\beta = 1$ and $\gamma = 1$, the constant factor becomes

$$2\left\{\frac{1}{\sqrt{(1+\alpha)\beta}}\arctan\left[\sqrt{\frac{\beta}{(1+\alpha)\gamma}}\right]+\frac{1}{\sqrt{\beta+\alpha\gamma}}\arctan\left[\sqrt{\frac{\gamma}{\beta+\alpha\gamma}}\right]\right\}$$
$$= 2\left\{\frac{1}{\sqrt{2}}\arctan\left[\frac{1}{\sqrt{2}}\right]+\frac{1}{\sqrt{2}}\arctan\left[\frac{1}{\sqrt{2}}\right]\right\} = 2\sqrt{2}\arctan\left[\frac{1}{\sqrt{2}}\right]$$
$$= \sqrt{2}\left\{\pi - 2\arctan[\sqrt{2}]\right\},$$

which corresponds to the setting of [11]. See also [8] for the special case $p = 2$. Proposition 3.3 thus significantly generalizes these results and adds a higher degree of flexibility thanks to the parameters $\alpha$, $\beta$ and $\gamma$.

The proposition below presents a new variant of the Hardy-Hilbert-type integral inequality, which can be seen as a three-parameter version of the one in Equation (1.1), distinguishing two types of integrability assumptions or weighted integral norms in the upper bounds. Note that the presence of the three parameters complicates the situation considerably and explains this distinction.

**Proposition 3.4.** *Let $p > 1$, $q = p/(p-1)$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $f, g : (0,+\infty) \mapsto (0,+\infty)$ be two functions. We study upper bounds for the following double integral:*

$$\iint_0^{+\infty} \frac{1}{\beta + \alpha \max(x, \gamma y)} f(x)g(y)dxdy,$$

*distinguishing two types of weighted integral norms for $f$ and $g$, denoted Types I and II.*

**Type I.** *Assuming that* $\int_0^{+\infty} x^{p/2-1} f^p(x)dx < +\infty$ *and* $\int_0^{+\infty} y^{q/2-1} g^q(y)dy < +\infty$, *we have*

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{\beta+\alpha\max(x,\gamma y)} f(x)g(y)dxdy\\ &\leq \frac{4}{\alpha\sqrt{\gamma}}\left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.\end{aligned}$$

**Type II.** *Assuming that* $\int_0^{+\infty} x^{(p-1)/2} f^p(x)dx < +\infty$ *and* $\int_0^{+\infty} y^{(q-1)/2} g^q(y)dy < +\infty$, *we have*

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{\beta+\alpha\max(x,\gamma y)} f(x)g(y)dxdy\\ &\leq \frac{1+\pi}{\sqrt{\alpha\beta}\gamma^{1/(2p)}}\left[\int_0^{+\infty} x^{(p-1)/2} f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(q-1)/2} g^q(y)dy\right]^{1/q}.\end{aligned}$$

*For each of these inequalities, we can apply the limit* $\beta \to 0$ *or* $\gamma \to 0$ *if the convergence holds.*

*Proof.* A suitable product decomposition of the main integrand using the identity $1 = x^{1/(2q)}y^{-1/(2p)}x^{-1/(2q)}y^{1/(2p)}$ and the equality $1/p+1/q=1$, followed by the application of the Hölder integral inequality at $p$ and $q$, gives

$$\begin{aligned}&\iint_0^{+\infty} \frac{1}{\beta+\alpha\max(x,\gamma y)} f(x)g(y)dxdy\\ &= \iint_0^{+\infty} \frac{x^{1/(2q)}y^{-1/(2p)}}{[\beta+\alpha\max(x,\gamma y)]^{1/p}} f(x)\frac{x^{-1/(2q)}y^{1/(2p)}}{[\beta+\alpha\max(x,\gamma y)]^{1/q}} g(y)dxdy\\ &\leq \mathfrak{E}^{1/p}\mathfrak{F}^{1/q}, \end{aligned}\tag{3.7}$$

where

$$\mathfrak{E} = \iint_0^{+\infty} \frac{x^{p/(2q)}y^{-1/2}}{\beta+\alpha\max(x,\gamma y)} f^p(x)dxdy$$

and

$$\mathfrak{F} = \iint_0^{+\infty} \frac{x^{-1/2}y^{q/(2p)}}{\beta+\alpha\max(x,\gamma y)} g^q(y)dxdy.$$

Let us determine the expressions of $\mathfrak{E}$ and $\mathfrak{F}$, one after the other.

Since the integrand associated with $\mathfrak{E}$ is clearly non-negative, we can apply the Fubini-Tonelli integral theorem that allows the exchange of the order of integration. This, followed by the use of $q = p/(p-1)$ and the application of Proposition 2.5 with "$\alpha\gamma$" instead of "$\alpha$" and "$x/\gamma$" instead of "$\gamma$", gives

$$\begin{aligned}
\mathfrak{E} &= \int_0^{+\infty} x^{p/(2q)} f^p(x) \left[ \int_0^{+\infty} \frac{y^{-1/2}}{\beta + \alpha \max(x, \gamma y)} dy \right] dx \\
&= \int_0^{+\infty} x^{(p-1)/2} f^p(x) \left[ \int_0^{+\infty} \frac{y^{-1/2}}{\beta + \alpha\gamma \max(x/\gamma, y)} dy \right] dx \\
&= \int_0^{+\infty} x^{(p-1)/2} f^p(x) \left[ 2 \left\{ \frac{\sqrt{x/\gamma}}{\beta + (\alpha\gamma)(x/\gamma)} + \frac{1}{\sqrt{\alpha\gamma\beta}} \arctan\left[ \sqrt{\frac{\beta}{\alpha\gamma(x/\gamma)}} \right] \right\} \right] dx \\
&= \frac{2}{\sqrt{\gamma}} \int_0^{+\infty} x^{(p-1)/2} f^p(x) \left\{ \frac{\sqrt{x}}{\beta + \alpha x} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[ \sqrt{\frac{\beta}{\alpha x}} \right] \right\} dx.
\end{aligned} \tag{3.8}$$

Let us now majorize $\mathfrak{F}$ by distinguishing between Types I and II, one after the other.

**Type I.** Using the inequality $\arctan(a) \leq a$ for any $a > 0$, with $a = \sqrt{\beta/(\alpha x)}$, and $\beta > 0$, we have

$$\frac{\sqrt{x}}{\beta + \alpha x} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[ \sqrt{\frac{\beta}{\alpha x}} \right] \leq \frac{\sqrt{x}}{\alpha x} + \frac{1}{\sqrt{\alpha\beta}} \times \sqrt{\frac{\beta}{\alpha x}} = \frac{2}{\alpha\sqrt{x}}.$$

This, and Equation (3.8), implies that

$$\mathfrak{E} \leq \frac{2}{\sqrt{\gamma}} \int_0^{+\infty} x^{(p-1)/2} f^p(x) \times \frac{2}{\alpha\sqrt{x}} dx = \frac{4}{\alpha\sqrt{\gamma}} \int_0^{+\infty} x^{p/2-1} f^p(x) dx. \tag{3.9}$$

**Type II.** For this type, using the formula $a^2 + b^2 \geq 2ab$ for any $a, b > 0$, with $a = \sqrt{\beta}$ and $b = \sqrt{\alpha x}$, and $\arctan(a) \leq \pi/2$ for any $a > 0$, we have

$$\frac{\sqrt{x}}{\beta + \alpha x} + \frac{1}{\sqrt{\alpha\beta}} \arctan\left[ \sqrt{\frac{\beta}{\alpha x}} \right] \leq \frac{\sqrt{x}}{2\sqrt{\alpha\beta x}} + \frac{1}{\sqrt{\alpha\beta}} \times \frac{\pi}{2} = \frac{1+\pi}{2\sqrt{\alpha\beta}}.$$

This, and Equation (3.8), implies that

$$\mathfrak{E} \leq \frac{2}{\sqrt{\gamma}} \int_0^{+\infty} x^{(p-1)/2} f^p(x) \times \frac{1+\pi}{2\sqrt{\alpha\beta}} dx = \frac{1+\pi}{\sqrt{\alpha\beta\gamma}} \int_0^{+\infty} x^{(p-1)/2} f^p(x) dx. \tag{3.10}$$

For the term $\mathfrak{F}$, we proceed in a similar way, but with the application of Proposition 2.5 with "$\gamma y$" instead of "$\gamma$". This yields

$$\begin{aligned}\mathfrak{F} &= \int_0^{+\infty} y^{q/(2p)} g^q(y) \left[ \int_0^{+\infty} \frac{x^{-1/2}}{\beta + \alpha \max(x, \gamma y)} dx \right] dy \\ &= \int_0^{+\infty} y^{(q-1)/2} g^q(y) \left[ 2 \left\{ \frac{\sqrt{\gamma y}}{\beta + \alpha \gamma y} + \frac{1}{\sqrt{\alpha \beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha \gamma y}} \right] \right\} \right] dy \qquad (3.11) \\ &= 2 \int_0^{+\infty} y^{(q-1)/2} g^q(y) \left\{ \frac{\sqrt{\gamma y}}{\beta + \alpha \gamma y} + \frac{1}{\sqrt{\alpha \beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha \gamma y}} \right] \right\} dy.\end{aligned}$$

Let us now majorize $\mathfrak{F}$ by distinguishing between Types I and II, one after the other.

**Type I.** Using the inequality $\arctan(a) \leq a$ for any $a > 0$, with $a = \sqrt{\beta/(\alpha\gamma y)}$, and $\beta > 0$, we have

$$\frac{\sqrt{\gamma y}}{\beta + \alpha \gamma y} + \frac{1}{\sqrt{\alpha \beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha \gamma y}} \right] \leq \frac{\sqrt{\gamma y}}{\alpha \gamma y} + \frac{1}{\sqrt{\alpha \beta}} \times \sqrt{\frac{\beta}{\alpha \gamma y}} = \frac{2}{\alpha \sqrt{\gamma y}}.$$

This, and Equation (3.11), implies that

$$\mathfrak{F} \leq 2 \int_0^{+\infty} y^{(q-1)/2} g^q(y) \times \frac{2}{\alpha \sqrt{\gamma y}} dy = \frac{4}{\alpha \sqrt{\gamma}} \int_0^{+\infty} y^{q/2-1} g^q(y) dy. \qquad (3.12)$$

**Type II.** For this type, using the formula $a^2 + b^2 \geq 2ab$ for any $a, b > 0$, with $a = \sqrt{\beta}$ and $b = \sqrt{\alpha\gamma y}$, and $\arctan(a) \leq \pi/2$ for any $a > 0$, we have

$$\frac{\sqrt{\gamma y}}{\beta + \alpha \gamma y} + \frac{1}{\sqrt{\alpha \beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha \gamma y}} \right] \leq \frac{\sqrt{\gamma y}}{2\sqrt{\alpha \beta \gamma y}} + \frac{1}{\sqrt{\alpha \beta}} \times \frac{\pi}{2} = \frac{1 + \pi}{2\sqrt{\alpha \beta}}.$$

This, and Equation (3.8), implies that

$$\mathfrak{F} \leq 2 \int_0^{+\infty} y^{(q-1)/2} g^q(y) \times \frac{1 + \pi}{2\sqrt{\alpha \beta}} dy = \frac{1 + \pi}{\sqrt{\alpha \beta}} \int_0^{+\infty} y^{(q-1)/2} g^q(y) dy. \qquad (3.13)$$

We are now in a position to conclude by combining the upper bounds obtained. Let us do this in detail by distinguishing between Types I and II, one after the other.

**Type I.** Putting the upper bounds of $\mathfrak{E}$ and $\mathfrak{F}$ obtained in Equations (3.9) and (3.12) into Equation (3.7), and using the equality $1/p + 1/q = 1$, we obtain

$$\iint_0^{+\infty} \frac{1}{\beta + \alpha \max(x, \gamma y)} f(x)g(y)dxdy$$
$$\leq \left[\frac{4}{\alpha\sqrt{\gamma}} \int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\frac{4}{\alpha\sqrt{\gamma}} \int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}$$
$$= \frac{4}{\alpha\sqrt{\gamma}} \left[\int_0^{+\infty} x^{p/2-1} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{q/2-1} g^q(y)dy\right]^{1/q}.$$

This is the desired result for Type I.

**Type II.** Putting the upper bounds of $\mathfrak{E}$ and $\mathfrak{F}$ obtained in Equations (3.10) and (3.13) into Equation (3.7), and using the equality $1/p + 1/q = 1$, we obtain

$$\iint_0^{+\infty} \frac{1}{\beta + \alpha \max(x, \gamma y)} f(x)g(y)dxdy$$
$$\leq \left[\frac{1+\pi}{\sqrt{\alpha\beta\gamma}} \int_0^{+\infty} x^{(p-1)/2} f^p(x)dx\right]^{1/p} \left[\frac{1+\pi}{\sqrt{\alpha\beta}} \int_0^{+\infty} y^{(q-1)/2} g^q(y)dy\right]^{1/q}$$
$$= \frac{1+\pi}{\sqrt{\alpha\beta}\gamma^{1/(2p)}} \left[\int_0^{+\infty} x^{(p-1)/2} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{(q-1)/2} g^q(y)dy\right]^{1/q}.$$

This is the desired result for Type II.

This ends the proof of Proposition 3.4. □

For the special case $\alpha = 1$, $\beta = 1$, $\gamma = 1$ and $p = 2$, the result obtained for Type I reduces to the inequality given in Equation (1.1). To the best of our knowledge, the result for other parameter values is new. Moreover, the introduction and analysis of Type II inequalities appear to be original contributions as well.

## 4 Conclusion

In conclusion, this article provides contributions to the field of integral calculus and inequalities. The integral formulas obtained, which feature ratio functions, maximum terms, and adjustable parameters, offer a flexible framework. The natural emergence of the arctangent function also points to meaningful connections with classical analysis and special functions. The new weighted Hölder-type and Hardy-Hilbert-type integral inequalities highlight the potential of our formulas in more technical contexts. The role of the maximum function, which is governed by two independent parameters, introduces additional complexity that may inspire further developments in related functional inequalities.

Potential extensions include generalization of our results to multidimensional domains, formulation of operator versions of inequalities, and identification of extremal functions or equality cases to better understand sharpness. In addition, possible areas of application include harmonic analysis, optimization, probability theory, and partial differential equations. We therefore believe that the methods and results presented here lay a versatile foundation for both theoretical advancement and practical application.

## Acknowledgement

The author would like to thank the reviewer for his constructive comments on the article.

## References

[1] L. E. Azar, *On some extensions of Hardy-Hilbert's inequality and applications*, J. Ineq. Appl. **2008** (2008), 1-14.

[2] D. Bainov, P. Simeonov, *Integral Inequalities and Applications*, Kluwer Academic, Dordrecht, 1992.

[3] E. F. Beckenbach, R. Bellman, *Inequalities*, Springer, Berlin, 1961.

[4] Q. Chen, B. C. Yang, *A survey on the study of Hilbert-type inequalities*, J. Inequal. Appl. **2015** (2015), 1-29.

[5] Z. Cvetkovski, *Inequalities: Theorems, Techniques and Selected Problems*, SpringerLink : Bücher, Springer Berlin Heidelberg, 2012.

[6] I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th Edition, Academic Press, 2007.

[7] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[8] Y. Li, J. Wu, B. He, *A new Hilbert-type integral inequality and the equivalent form*, Int. J. Math. Math. Sci. **8** (2006), 1-6.

[9] A. Saglam, H. Yildirim, M. Z. Sarikaya, *Generalization of Hardy-Hilbert's inequality and applications*, Kyungpook Math. J. **50** (2010), 131-152.

[10] M. Z. Sarikaya, M. S. Bingol, *Recent developments of integral inequalities of the Hardy-Hilbert type*, Turkish J. Ineq. **8** (2024), 43-54.

[11] W. T. Sulaiman, *New Hardy-Hilbert's-type integral inequalities*, Int. Math. Forum **3** (2008), 2139-2147.

[12] W. T. Sulaiman, *New kinds of Hardy-Hilbert's integral inequalities*, Appl. Math. Lett. **23** (2010), 361-365.

[13] W. T. Sulaiman, *An extension of Hardy-Hilbert's integral inequality*, Afr. Diaspora J. Math. **10** (2010), 66-71.

[14] B. Sun, *Best generalization of a Hilbert type inequality*, J. Ineq. Pure Appl. Math. **7** (2006), 1-7.

[15] W. Walter, *Differential and Integral Inequalities*, Springer, Berlin, 1970.

[16] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

[17] B. C. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.

Christophe Chesneau

Department of Mathematics, LMNO, University of Caen-Normandie, 14032, Caen, France.

E-mail: `christophe.chesneau@gmail.com`

Received: 5.05.2025

Revised: 20.06.2025

Accepted: 29.06.2025