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DOI: 10.2478/awutm-2025-0002

Annals of West University of Timisoara
Mathematics and Computer Science
61 (2025), 14–30

Study of some new integral inequalities involving four adaptable functions

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Abstract In this article, we establish new and flexible integral inequalities that have the property of involving four adaptable functions. Some of them generalize existing results in the literature. They can also be reformulated in the context of probability theory. Several special examples are discussed in detail. Additional integral inequalities based on the notion of convexity are also proposed.

AMS Subject Classification (2020) 26D15; 33E20
Keywords integral inequalities; expectation inequalities; convexity

1 Introduction

Inequalities play an important role in mathematics. In particular, many classical inequalities of the integral type, known as integral inequalities, have been fundamental tools in establishing crucial results. Among the best known integral inequalities are the Cauchy-Schwarz integral inequality, the Hölder integral inequality, the Minkowski integral inequality, the Hermite-Hadamard integral inequality and the Jensen integral inequality. A complete survey of the subject can be found in [1–3, 11, 12].

Recent research has focused on simple and original integral inequalities involving several adaptable functions and parameters. One notable result of this type is [4, Theorem 2.4]. For the purposes of this article, we will briefly describe it below. Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $p, q : [a, b] \mapsto [0, +\infty)$ be two functions such that p is integrable and q is non-decreasing, differentiable and integrable, and the following integral inequality holds:

$$\int_x^b p(t)dt \geq \int_x^b q(t)dt,$$

for any $x \in [a, b]$. Then, for any $(\alpha, \beta) \in [0, +\infty)^2$ such that $\alpha + \beta \geq 1$, we have

$$\int_a^b p^{\alpha+\beta}(t)dt \geq \int_a^b p^{\alpha}(t)q^{\beta}(t)dt.$$

This result extends those shown in [5,6,13], which are mainly based on specific values of β and the function $q(t) = t$. It has also been revisited with different integral techniques, and also applied in various mathematical scenarios, in [7–10]. In particular, the proofs in [10] show how the constraint $\alpha + \beta \geq 1$ can be relaxed under certain assumptions on p and q.

In this article, we propose a generalization of [4, Theorem 2.4] using four adaptable functions, say f, g, h and k. Under some assumptions on them, we establish comprehensive inequalities of the following form:

$$\text{"} \int_a^b f(t)g(t)dt \ (\leq \ \text{or} \ \geq) \int_a^b h(t)k[g(t)]dt.\text{"}$$

Compared to the existing works, the two main novelties are (i) the presence of k connecting the function g involved in the two integrals, and (ii) the fact that we can choose any type of functions for f, g, h and k, beyond the power type functions. Several examples are highlighted to illustrate possible applications of the established inequalities in mathematical analysis, mainly when g is chosen as a power, logarithmic or exponential function. We also discuss how some of these inequalities can be reformulated in the context of probability theory. In particular, by treating the integral as a probabilistic or expected value, we derive expectation inequalities of random variables involving f, g, h and k. Finally, some additional results are given, also based on convexity assumptions on k.

The structure of the article is as follows: Section 2 presents the main integral inequalities, including examples and some probabilistic formulations. Section 3 is devoted to the additional results. A conclusion is given in Section 4.

2 New integral inequalities

2.1 Two types of results

For the purposes of this article, we distinguish two types of results based on the assumptions made about the main functions f, g, h and k.

- We call results of type I those that use an assumption of the following form:

$$\int_x^b f(t)dt \ (\leq \ \text{or} \ \geq) \ k'[g(x)] \int_x^b h(t)dt.$$

- We call results of type II those that use an assumption of the following form:

$$\int_a^x f(t)dt \ (\leq \ \text{or} \ \geq) \ k'[g(x)] \int_a^x h(t)dt.$$

With these definitions, the main results of type I and type II of this article are the subject of the next two subsections.

2.2 Main results of type I

Our first main result of type I is shown in the theorem below. In this statement, and throughout the article, it is assumed that the integrals introduced exist (which is not guaranteed a priori, especially if $a \to -\infty$ or $b \to +\infty$).

Theorem 2.1. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is integrable, g is differentiable and non-decreasing, h is integrable, k is differentiable and monotone, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_x^b f(t)dt \geq k'[g(x)] \int_x^b h(t)dt, \tag{2.1}$$

for any $x \in [a, b]$, and

$$g(a) \int_a^b f(t)dt \geq k[g(a)] \int_a^b h(t)dt. \tag{2.2}$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

Proof. Since g is differentiable, we can write

$$g(t) = g(t) - g(a) + g(a) = \int_a^t g'(u)du + g(a).$$

Using this and a change in the order of integration, which is valid thanks to the Fubini-Tonelli theorem (since the integrand has a constant sign), we obtain

$$\begin{aligned} \int_a^b f(t)g(t)dt &= \int_a^b f(t) \left[\int_a^t g'(u)du + g(a)\right] dt \\ &= \int_a^b \int_a^t f(t)g'(u)dudt + g(a) \int_a^b f(t)dt \\ &= \int_a^b \int_u^b f(t)g'(u)dtdu + g(a) \int_a^b f(t)dt \\ &= \int_a^b g'(u) \left[\int_u^b f(t)dt\right] du + g(a) \int_a^b f(t)dt. \end{aligned} \tag{2.3}$$

It follows from $g'(u) \geq 0$ for any $u \in [a, b]$ (since g is differentiable and non-decreasing) and Equation (2.1) that

$$\begin{aligned} &\int_a^b g'(u) \left[\int_u^b f(t)dt\right] du + g(a) \int_a^b f(t)dt \geq \\ &\geq \int_a^b g'(u) \left\{k'[g(u)] \int_u^b h(t)dt\right\} du + g(a) \int_a^b f(t)dt \\ &= \int_a^b \int_u^b g'(u)k'[g(u)]h(t)dtdu + g(a) \int_a^b f(t)dt. \end{aligned} \tag{2.4}$$

Doing again a change of order of integration by the Fubini-Tonelli theorem (still possible because the integrand has a constant sign, partly because of the monotone assumption on k), we find that

$$\begin{aligned}
&\int_a^b \int_u^b g'(u)k'[g(u)]h(t)dtdu + g(a)\int_a^b f(t)dt = \\
&\qquad = \int_a^b \int_a^t g'(u)k'[g(u)]h(t)dudt + g(a)\int_a^b f(t)dt \\
&\qquad = \int_a^b h(t)\left\{\int_a^t g'(u)k'[g(u)]du\right\}dt + g(a)\int_a^b f(t)dt \\
&\qquad = \int_a^b h(t)\left\{k[g(t)] - k[g(a)]\right\}dt + g(a)\int_a^b f(t)dt \\
&\qquad = \int_a^b h(t)k[g(t)]dt + \left\{g(a)\int_a^b f(t)dt - k[g(a)]\int_a^b h(t)dt\right\}.
\end{aligned} \tag{2.5}$$

It follows from Equation (2.2) that

$$\int_a^b h(t)k[g(t)]dt + \left\{g(a)\int_a^b f(t)dt - k[g(a)]\int_a^b h(t)dt\right\} \geq \int_a^b h(t)k[g(t)]dt. \tag{2.6}$$

Putting Equations (2.3), (2.4), (2.5) and (2.6) together, we obtain

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

The desired result is obtained. □

Note that if k is non-increasing, the assumption in Equation (2.1) is directly satisfied, since f and h are positive. Furthermore, after analyzing the proof of Theorem 2.1, we can avoid the monotone assumption on k by assuming that $\int_a^b \int_a^b g'(u)|k'[g(u)]|h(t)dtdu$ exists, i.e., $\int_a^b g'(u)|k'[g(u)]|du$ exists since h is integrable, and use the Fubini theorem instead of the Fubini-Tonelli theorem for the change of order of integration in Equation (2.5) (the sign of the integrated term may not be constant, justifying this technical change). We may also mention that Theorem 2.1 can be extended to functions $f, g, h : [a, b] \mapsto \mathbb{R}$ and $k : [0, +\infty) \mapsto \mathbb{R}$, but under additional integration assumptions to activate the Fubini theorem in Equations (2.3) and (2.5).

Theorem 2.1 generalizes some existing results in the literature. In particular, taking $k(t) = t$, $f(t) = p^{\alpha}(t)$, with $p : [a, b] \mapsto [0, +\infty)$ and $\alpha > 0$, and $g(t) = q^{\beta}(t)$, with $q : [a, b] \mapsto [0, +\infty)$ and $\beta > 0$, Theorem 2.1 becomes [10, Lemma 2.1 under the assumption in Equation (2.1)]. Furthermore, still with $k(t) = t$, with $f(t) = \int_a^t p^{\alpha}(x)dx$, with $p : [a, b] \mapsto [0, +\infty)$ and $\alpha > 0$, and $g(t) = \int_a^t q^{\beta}(x)dx$, with $q : [a, b] \mapsto [0, +\infty)$ and $\beta > 0$, Theorem 2.1 becomes [8, Equation (2.3) in the proof of Theorem 2.1].

Some examples are given below, adopting the framework of Theorem 2.1, with $a = 0$ and $b = 1$, and the choice of several functions k.

- Considering $k(t) = \sqrt{t}$, with $t \in [0, +\infty)$, the assumption in Equation (2.1) becomes

$$\int_x^1 f(t)dt \geq \frac{1}{2\sqrt{g(x)}} \int_x^1 h(t)dt,$$

for any $x \in [0, 1]$, the assumption in Equation (2.2) becomes

$$g(0) \int_0^1 f(t)dt \geq \sqrt{g(0)} \int_0^1 h(t)dt,$$

(which is obviously true if $g(0) = 0$, for instance, otherwise it is reduced to $\sqrt{g(0)} \int_0^1 f(t)dt \geq \int_0^1 h(t)dt$), and the result in Theorem 2.1 gives

$$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t)\sqrt{g(t)}dt.$$

- Considering $k(t) = \log(1 + t)$, with $t \in [0, +\infty)$, the assumption in Equation (2.1) becomes

$$\int_x^1 f(t)dt \geq \frac{1}{1 + g(x)} \int_x^1 h(t)dt,$$

for any $x \in [0, 1]$, the assumption in Equation (2.2) becomes

$$g(0) \int_0^1 f(t)dt \geq \log[1 + g(0)] \int_0^1 h(t)dt,$$

and the result in Theorem 2.1 gives

$$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t) \log[1 + g(t)]dt.$$

- Considering $k(t) = 1 - e^{-ct}$, with $c > 0$ and $t \in [0, +\infty)$, the assumption in Equation (2.1) becomes

$$\int_x^1 f(t)dt \geq ce^{-cg(x)} \int_x^1 h(t)dt,$$

for any $x \in [0, 1]$, the assumption in Equation (2.2) becomes

$$g(0) \int_0^1 f(t)dt \geq [1 - e^{-cg(0)}] \int_0^1 h(t)dt,$$

and the result in Theorem 2.1 gives

$$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t)[1 - e^{-cg(t)}]dt = \int_0^1 h(t)dt - \int_0^1 h(t)e^{-cg(t)}dt.$$

The result below completes Theorem 2.1 by considering the counterpart of the assumptions in Equations (2.1) and (2.2).

Theorem 2.2. *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a,b] \mapsto [0,+\infty)$ and $k : [0,+\infty) \mapsto [0,+\infty)$ be four functions, such that f is integrable, g is differentiable and non-decreasing, h is integrable, k is differentiable and non-decreasing, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_x^b f(t)dt \leq k'[g(x)] \int_x^b h(t)dt,$$

for any $x \in [a,b]$, and

$$g(a) \int_a^b f(t)dt \leq k[g(a)] \int_a^b h(t)dt.$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \leq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is strictly similar to that of Theorem 2.1; we only need to replace "$\geq$" by "$\leq$" in Equations (2.4) and (2.6) (and in the final inequality). The details are therefore omitted. □

Another result of type I is given below by assuming that g is non-increasing instead of non-decreasing as in Theorem 2.1.

Theorem 2.3. *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a,b] \mapsto [0,+\infty)$ and $k : [0,+\infty) \mapsto [0,+\infty)$ be four functions, such that f is integrable, g is differentiable and non-increasing, h is integrable, k is differentiable and non-increasing, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_x^b f(t)dt \leq k'[g(x)] \int_x^b h(t)dt, \tag{2.7}$$

for any $x \in [a,b]$, and

$$g(a) \int_a^b f(t)dt \geq k[g(a)] \int_a^b h(t)dt. \tag{2.8}$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is identical to that of Theorem 2.1; we only need to note that $g'(u) \leq 0$ for any $u \in [a,b]$ combined with the assumption in Equation (2.7) gives the same two lines in Equation (2.4), i.e.,

$$\int_a^b g'(u) \left[\int_u^b f(t)dt\right] du + g(a) \int_a^b f(t)dt \geq$$
$$\geq \int_a^b g'(u) \left\{k'[g(u)] \int_u^b h(t)dt\right\} du + g(a) \int_a^b f(t)dt.$$

The details are therefore omitted. □

The result below completes Theorem 2.3 by considering the counterpart of the assumptions in Equations (2.7) and (2.8).

Theorem 2.4. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is integrable, g is differentiable and non-increasing, h is integrable, k is differentiable and monotone, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_x^b f(t)dt \geq k'[g(x)] \int_x^b h(t)dt, \tag{2.9}$$

for any $x \in [a, b]$, and

$$g(a) \int_a^b f(t)dt \leq k[g(a)] \int_a^b h(t)dt.$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \leq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is identical to that of Theorem 2.2. We just mention that, using $g'(u) \leq 0$ for any $u \in [a, b]$ and the assumption in Equation (2.9), we get

$$\int_a^b g'(u) \left[\int_u^b f(t)dt\right] du + g(a) \int_a^b f(t)dt \leq$$
$$\leq \int_a^b g'(u) \left\{k'[g(u)] \int_u^b h(t)dt\right\} du + g(a) \int_a^b f(t)dt.$$

The details are therefore omitted. □

We end this subsection with a remark. Integral inequalities often have analogues in probability theory, especially when integrals represent expectations. Their formulation can be somewhat simpler thanks to the notion of a probability density function. As an example, the result below is the probability formulation of our main theorem: Theorem 2.1.

Theorem 2.5. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is a probability density function, g is differentiable and non-decreasing, h is a probability density function, and k is differentiable and non-decreasing.*

Let X be a random variable with probability density function f, Y be a random variable with probability density function h, $\mathbb{P}$ be the probability operator and $\mathbb{E}$ be the expectation operator.

We suppose that

$$\mathbb{P}(X \geq x) \geq k'[g(x)]\mathbb{P}(Y \geq x), \tag{2.10}$$

for any $x \in [a, b]$, *and*

$$g(a) \geq k[g(a)].$$

Then the following expectation inequality holds:

$$\mathbb{E}\left[g(X)\right] \geq \mathbb{E}\left\{k[g(Y)]\right\}.$$

Proof. This theorem is a probability formulation of Theorem 2.1, taking into account that

$$\mathbb{P}(X \geq x) = \int_x^b f(t)dt, \quad \mathbb{P}(Y \geq x) = \int_x^b h(t)dt,$$

$\int_a^b f(t)dt = 1$, $\int_a^b h(t)dt = 1$, and, by the law of the unconscious statistician,

$$\mathbb{E}\left[g(X)\right] = \int_a^b f(t)g(t)dt, \quad \mathbb{E}\left\{k[g(Y)]\right\} = \int_a^b h(t)k[g(t)]dt.$$

This ends this proof. □

In a similar probabilistic framework, the other theorems in this subsection can be reformulated.

2.3 Main results of type II

The result below shows that Theorem 2.1 holds for a different set of assumptions; it is the first main result of type II.

Theorem 2.6. *Let* $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ *with* $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ *and* $k : [0, +\infty) \mapsto [0, +\infty)$ *be four functions, such that* f *is integrable,* g *is differentiable and non-decreasing,* h *is integrable,* k *is differentiable and non-decreasing, and* f, g, h *and* k *satisfy the following two integral inequalities:*

$$\int_a^x f(t)dt \leq k'[g(x)] \int_a^x h(t)dt, \tag{2.11}$$

for any $x \in [a, b]$, *and*

$$g(b) \int_a^b f(t)dt \geq k[g(b)] \int_a^b h(t)dt. \tag{2.12}$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

Proof. We can write

$$g(t) = g(b) - [g(b) - g(t)] = g(b) - \int_t^b g'(u)du.$$

With this and a change in the order of integration, which is valid thanks to the Fubini-Tonelli theorem (since the integrand has a constant sign, that is, positive), we obtain

$$\begin{aligned}\int_a^b f(t)g(t)dt &= \int_a^b f(t)\left[g(b) - \int_t^b g'(u)du\right]dt \\ &= g(b)\int_a^b f(t)dt - \int_a^b \int_t^b f(t)g'(u)dudt \\ &= g(b)\int_a^b f(t)dt - \int_a^b \int_a^u f(t)g'(u)dtdu \\ &= g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[\int_a^u f(t)dt\right]du.\end{aligned} \tag{2.13}$$

It follows from $g'(u) \geq 0$ for any $u \in [a,b]$ (since g is differentiable and non-decreasing) and Equation (2.11) that

$$\begin{aligned}&g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[\int_a^u f(t)dt\right]du \geq \\ &\geq g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[k'[g(u)]\int_a^u h(t)dt\right]du \\ &= g(b)\int_a^b f(t)dt - \int_a^b \int_a^u g'(u)k'[g(u)]h(t)dtdu.\end{aligned} \tag{2.14}$$

Doing again a change of order of integration by the Fubini-Tonelli theorem, we find that

$$\begin{aligned}&g(b)\int_a^b f(t)dt - \int_a^b \int_a^u g'(u)k'[g(u)]h(t)dtdu = \\ &= g(b)\int_a^b f(t)dt - \int_a^b \int_t^b g'(u)k'[g(u)]h(t)dudt \\ &= g(b)\int_a^b f(t)dt - \int_a^b h(t)\left[\int_t^b g'(u)k'[g(u)]du\right]dt \\ &= g(b)\int_a^b f(t)dt - \int_a^b h(t)\left\{k[g(b)] - k[g(t)]\right\}dt \\ &= \left\{g(b)\int_a^b f(t)dt - k[g(b)]\int_a^b h(t)dt\right\} + \int_a^b h(t)k[g(t)]dt.\end{aligned} \tag{2.15}$$

It follows from Equation (2.12) that

$$\left\{g(b)\int_a^b f(t)dt - k[g(b)]\int_a^b h(t)dt\right\} + \int_a^b h(t)k[g(t)]dt \geq \int_a^b h(t)k[g(t)]dt. \tag{2.16}$$

Putting Equations (2.13), (2.14), (2.15) and (2.16) together, we obtain

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

The desired result is obtained. □

Theorem 2.6 generalizes some existing results in the literature. In particular, taking $k(t) = t$, $f(t) = p^{\alpha}(t)$, with $p : [a, b] \mapsto [0, +\infty)$ and $\alpha > 0$, and $g(t) = q^{\beta}(t)$, with $q : [a, b] \mapsto [0, +\infty)$ and $\beta > 0$, Theorem 2.1 becomes [10, Lemma 2.1 under the assumption in Equation (2.2)]. Furthermore, still in the case $k(t) = t$, with $f(t) = \int_t^b p^{\alpha}(x)dx$, with $p : [a, b] \mapsto [0, +\infty)$ and $\alpha > 0$, and $g(t) = \int_t^b q^{\beta}(x)dx$, with $q : [a, b] \mapsto [0, +\infty)$ and $\beta > 0$, Theorem 2.1 becomes [8, Equation (2.17) in the proof of Theorem 2.5].

Some examples are given below, adopting the framework of Theorem 2.6, with $a = 0$ and $b = 1$, and the choice of several functions k.

- Considering $k(t) = \sqrt{t}$, with $t \in [0, +\infty)$, the assumption in Equation (2.11) becomes

  $$\int_0^x f(t)dt \leq \frac{1}{2\sqrt{g(x)}} \int_0^x h(t)dt,$$

  for any $x \in [0, 1]$, the assumption in Equation (2.12) becomes

  $$g(1) \int_0^1 f(t)dt \geq \sqrt{g(1)} \int_0^1 h(t)dt,$$

  so that, for $g(1) \neq 0$, $\sqrt{g(1)} \int_0^1 f(t)dt \geq \int_0^1 h(t)dt$, and the result in Theorem 2.6 gives

  $$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t)\sqrt{g(t)}dt.$$

- Considering $k(t) = \log(1 + t)$, with $t \in [0, +\infty)$, the assumption in Equation (2.11) becomes

  $$\int_0^x f(t)dt \leq \frac{1}{1 + g(x)} \int_0^x h(t)dt,$$

  for any $x \in [0, 1]$, the assumption in Equation (2.12) becomes

  $$g(1) \int_0^1 f(t)dt \geq \log[1 + g(1)] \int_0^1 h(t)dt,$$

  and the result in Theorem 2.6 gives

  $$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t) \log[1 + g(t)]dt.$$

- Considering $k(t) = 1-e^{-ct}$, with $c > 0$ and $t \in [0, +\infty)$, the assumption in Equation (2.11) becomes

$$\int_0^x f(t)dt \leq ce^{-cg(x)} \int_0^x h(t)dt,$$

for any $x \in [0, 1]$, the assumption in Equation (2.12) becomes

$$g(1) \int_0^1 f(t)dt \geq [1 - e^{-cg(1)}] \int_0^1 h(t)dt,$$

and the result in Theorem 2.6 gives

$$\int_0^1 f(t)g(t)dt \geq \int_0^1 h(t)[1 - e^{-cg(t)}]dt = \int_0^1 h(t)dt - \int_0^1 h(t)e^{-cg(t)}dt.$$

The result below completes Theorem 2.6 by considering the counterparts of the assumptions in Equations (2.11) and (2.12).

Theorem 2.7. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is integrable, g is differentiable and non-decreasing, h is integrable, k is differentiable and monotone, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_a^x f(t)dt \geq k'[g(x)] \int_a^x h(t)dt, \tag{2.17}$$

for any $x \in [a, b]$, and

$$g(b) \int_a^b f(t)dt \leq k[g(b)] \int_a^b h(t)dt.$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \leq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is strictly similar to that of Theorem 2.6; we only need to replace "$\geq$" by "$\leq$" in Equations (2.14) and (2.16) (and in the final inequality). The details are therefore omitted. □

Note that, if k is non-increasing, the assumption in Equation (2.17) is directly satisfied since f and h are positive. Moreover, we can avoid the monotone assumption on k by assuming that $\int_a^b \int_a^b g'(u)|k'[g(u)]|h(t)dtdu$ exists, i.e., $\int_a^b g'(u)|k'[g(u)]|du$ exists since h is integrable.

Another result of type II is given below by assuming that g is non-increasing instead of non-decreasing as in Theorem 2.1.

Theorem 2.8. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is integrable, g is differentiable and non-increasing, h is integrable, k is differentiable and non-decreasing, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_a^x f(t)dt \geq k'[g(x)] \int_a^x h(t)dt, \tag{2.18}$$

for any $x \in [a, b]$, and

$$g(b) \int_a^b f(t)dt \geq k[g(b)] \int_a^b h(t)dt. \tag{2.19}$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is identical to that of Theorem 2.6; we only need to note that $g'(u) \leq 0$ for any $u \in [a, b]$ combined with the assumption in Equation (2.18) gives the same two lines in Equation (2.14), i.e.,

$$\begin{aligned} g(b) \int_a^b f(t)dt - \int_a^b g'(u) \left[\int_a^u f(t)dt\right] du \geq \\ \geq g(b) \int_a^b f(t)dt - \int_a^b g'(u) \left[k'[g(u)] \int_a^u h(t)dt\right] du. \end{aligned}$$

The details are therefore omitted. □

The result below completes Theorem 2.8 by considering the counterparts of the assumptions in Equations (2.18) and (2.19).

Theorem 2.9. *Let $(a, b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a, b] \mapsto [0, +\infty)$ and $k : [0, +\infty) \mapsto [0, +\infty)$ be four functions, such that f is integrable, g is differentiable and non-increasing, h is integrable, k is differentiable and non-decreasing, and f, g, h and k satisfy the following two integral inequalities:*

$$\int_a^x f(t)dt \leq k'[g(x)] \int_a^x h(t)dt, \tag{2.20}$$

for any $x \in [a, b]$, and

$$g(b) \int_a^b f(t)dt \leq k[g(b)] \int_a^b h(t)dt.$$

Then the following integral inequality holds:

$$\int_a^b f(t)g(t)dt \leq \int_a^b h(t)k[g(t)]dt.$$

Proof. The proof is identical to that of Theorem 2.7. We just mention that, using $g'(u) \leq 0$ for any $u \in [a,b]$ and the assumption in Equation (2.20), we get

$$g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[\int_a^u f(t)dt\right]du \leq$$
$$\leq g(b)\int_a^b f(t)dt - \int_a^b g'(u)\left[k'[g(u)]\int_a^u h(t)dt\right]du.$$

The details are therefore omitted. □

We end this section with a probabilistic formulation of our main theorem: Theorem 2.6.

Theorem 2.10. *Let $(a,b) \in \mathbb{R}^2 \cup \{\pm\infty\}^2$ with $a < b$, $f, g, h : [a,b] \mapsto [0,+\infty)$ and $k : [0,+\infty) \mapsto [0,+\infty)$ be four functions, such that f is a probability density function, g is differentiable and non-decreasing, h is a probability density function, and k is differentiable and non-decreasing.*

Let X be a random variable with probability density function f, Y be a random variable with probability density function h, $\mathbb{P}$ be the probability operator and $\mathbb{E}$ be the expectation operator.

We suppose that

$$\mathbb{P}(X \leq x) \leq k'[g(x)]\mathbb{P}(Y \leq x), \tag{2.21}$$

for any $x \in [a,b]$, and

$$g(b) \geq k[g(b)].$$

Then the following expectation inequality holds:

$$\mathbb{E}\left[g(X)\right] \geq \mathbb{E}\left\{k[g(Y)]\right\}.$$

Proof. This theorem is a probability formulation of Theorem 2.6, taking into account that

$$\mathbb{P}(X \leq x) = \int_a^x f(t)dt, \quad \mathbb{P}(Y \leq x) = \int_a^x h(t)dt,$$

$\int_a^b f(t)dt = 1$, $\int_a^b h(t)dt = 1$, and, by the law of the unconscious statistician,

$$\mathbb{E}\left[g(X)\right] = \int_a^b f(t)g(t)dt, \quad \mathbb{E}\left\{k[g(Y)]\right\} = \int_a^b h(t)k[g(t)]dt.$$

This ends this proof. □

In a similar probabilistic framework, the other theorems in this subsection can be reformulated.

3 Additional results

In this section, some additional integral inequalities are presented based on the previous results.

The theorem below is a consequence of Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8, under an additional assumption on the function k.

Proposition 3.1. *We consider the framework of Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8, under their respective assumptions, and under the additional assumption that k is convex. Then we have*

$$\int_a^b f(t)g(t)dt \geq \left[\int_a^b h(t)dt\right] k\left\{\frac{1}{\int_a^b h(t)dt}\int_a^b h(t)g(t)dt\right\}.$$

Proof. Under their respective assumptions, it follows from Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8 that

$$\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt = \left[\int_a^b h(t)dt\right]\int_a^b j(t)k[g(t)]dt, \tag{3.1}$$

where

$$j(t) := \frac{1}{\int_a^b h(t)dt}h(t).$$

Since $j(t) \geq 0$ for any $t \in [a,b]$ and $\int_a^b j(t)dt = 1$ (so j is a probability density function), and k is convex, the Jensen integral inequality implies that

$$\int_a^b j(t)k[g(t)]dt \geq k\left[\int_a^b j(t)g(t)dt\right] = k\left\{\frac{1}{\int_a^b h(t)dt}\int_a^b h(t)g(t)dt\right\}. \tag{3.2}$$

Combining Equations (3.1) and (3.2), we get

$$\int_a^b f(t)g(t)dt \geq \left[\int_a^b h(t)dt\right] k\left\{\frac{1}{\int_a^b h(t)dt}\int_a^b h(t)g(t)dt\right\}.$$

This ends the proof. □

The result below is another consequence of Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8.

Proposition 3.2. *Let $\lambda > 0$. We consider the framework of Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8 under their respective assumptions, and a function $\ell : [a,b] \mapsto \mathbb{R}$ such that*

$$\ell(t) \geq (1+\lambda)f(t)g(t) - \lambda h(t)k[g(t)], \tag{3.3}$$

for any $t \in [a,b]$.

Then we have

$$\int_a^b \ell(t)dt \geq \max\left\{\int_a^b f(t)g(t)dt, \int_a^b h(t)k[g(t)]dt\right\}.$$

Proof. Using Equation (3.3), $\lambda > 0$ and applying Theorem 2.1, Theorem 2.3, Theorem 2.6 or Theorem 2.8 under their respective assumptions, i.e., $\int_a^b f(t)g(t)dt \geq \int_a^b h(t)k[g(t)]dt$, we get

$$\begin{aligned}\int_a^b \ell(t)dt &\geq \int_a^b \{(1+\lambda)f(t)g(t) - \lambda h(t)k[g(t)]\}\, dt \\ &= (1+\lambda)\int_a^b f(t)g(t)dt - \lambda \int_a^b h(t)k[g(t)]dt \\ &\geq (1+\lambda)\int_a^b f(t)g(t)dt - \lambda \int_a^b f(t)g(t)dt \\ &= \int_a^b f(t)g(t)dt.\end{aligned} \tag{3.4}$$

With similar arguments, we have

$$\begin{aligned}\int_a^b \ell(t)dt &\geq \int_a^b \{(1+\lambda)f(t)g(t) - \lambda h(t)k[g(t)]\}\, dt \\ &= (1+\lambda)\int_a^b f(t)g(t)dt - \lambda \int_a^b h(t)k[g(t)]dt \\ &\geq (1+\lambda)\int_a^b h(t)k[g(t)]dt - \lambda \int_a^b h(t)k[g(t)]dt \\ &= \int_a^b h(t)k[g(t)]dt.\end{aligned} \tag{3.5}$$

It follows from Equations (3.4) and (3.5) that

$$\int_a^b \ell(t)dt \geq \max\left\{\int_a^b f(t)g(t)dt, \int_a^b h(t)k[g(t)]dt\right\}.$$

The desired result is obtained. □

Taking $k(t) = t$, $f(t) = p^{\alpha}(t)$, with $p : [a, b] \mapsto [0, +\infty)$ and $\alpha > 0$, and $g(t) = q^{\beta}(t)$, with $q : [a, b] \mapsto [0, +\infty)$ and $\beta > 0$, we get [10, Part of the proof of Theorem 2.3].

The result below is a consequence of Theorem 2.2, Theorem 2.4, Theorem 2.7 or Theorem 2.9, under an additional assumption on the function k.

Proposition 3.3. *We consider the framework of Theorem 2.2, Theorem 2.4, Theorem 2.7 or Theorem 2.9, under their respective assumptions, and under the additional assumption that k is concave. Then we have*

$$\int_a^b f(t)g(t)dt \leq \left[\int_a^b h(t)dt\right] k\left\{\frac{1}{\int_a^b h(t)dt}\int_a^b h(t)g(t)dt\right\}.$$

Proof. The proof is strictly similar to that of Proposition 3.1; we only need to replace "$\geq$" by "$\leq$" in Equations (3.1) and (3.2). The details are therefore omitted. □

The result below is another consequence of Theorem 2.2, Theorem 2.4, Theorem 2.7 or Theorem 2.9.

Proposition 3.4. *Let $\lambda > 0$. We consider the framework of Theorem 2.2, Theorem 2.4, Theorem 2.7 or Theorem 2.9, under their respective assumptions, and a function $\ell : [a, b] \mapsto \mathbb{R}$ such that*

$$\ell(t) \leq (1 + \lambda)f(t)g(t) - \lambda h(t)k[g(t)],$$

for any $t \in [a, b]$.

Then we have

$$\int_a^b \ell(t)dt \leq \min \left\{ \int_a^b f(t)g(t)dt, \int_a^b h(t)k[g(t)]dt \right\}.$$

Proof. The proof is strictly similar to that of Proposition 3.2; we only need to replace "$\geq$" by "$\leq$" in Equations (3.4) and (3.5) (and "min" instead of "max" in the final inequality). The details are therefore omitted. □

4 Conclusion

In conclusion, this article presents new integral inequalities involving four adaptable functions "f, g, h and k", some of which generalize existing results in the literature. These inequalities can be reformulated in the context of probability theory. Several precise examples are given. Thanks to their flexibility, these inequalities open up new applications to integrals. Another possible perspective is the extension of the results to the multidimensional case. These lines require further investigation, which we leave to future research.

Acknowledgement

The author would like to thank the reviewer for his constructive comments on the article.

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Received: 21.11.2024
Accepted: 10.03.2025