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Solving a system of Caputo-Hadamard fractional differential equations via Perov's fixed point theorem

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Abstract In this study, we discuss the existence and the uniqueness of the solution to Caputo-Hadamard Cauchy problems for a system of fractional differential equations, by using Perov's fixed point theorem. Finally, two examples are provided to illustrate our results.

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1 Introduction and preliminaries

Because of their significance in describing a variety of events, differential equations with fractional order have recently attracted the attention of numerous authors. For further information, we direct the reader to [10, 12] and the references therein. The concept of the Riemann-Liouville derivative served as the foundation for its development into the Caputo or Hadamard derivative type, from which various findings were obtained (see, for instance, [2, 8, 9, 21]). Later, Jarad et al. [11] introduced a new derivative type by combining Caputo derivative with Hadamard one called Caputo-Hadamard, at least one of its advantages being that the derivative of the constant is equal to zero, while in the case of Hadamard or Riemann-Liouville derivative is not. Several results found using the Caputo derivative are still valid when employing the Caputo-Hadamard derivative.

The fixed point technique is a useful tool for studying the existence of solutions, while the uniqueness can be based on either Banach's theorem or on one of its generalizations, and one of these is Perov's fixed point point theorem. Some fixed point results in Perov's sense are provided in [3, 7, 15, 17]. Using the vector version fixed point theorem, numerous authors investigated the existence of solutions for systems of differential equations, see [4, 13, 14, 16].

In [6], the authors studied the existence, boundedness, and compactness of the set of solutions to the following system

$$\begin{cases} \mathrm{D}^q\zeta(t) = f(t,\zeta(t),\xi(t)), \ t \geq 0, \\ \mathrm{D}^p\xi(t) = g(t,\zeta(t),\xi(t)), \\ \zeta(0) = \zeta_0, \xi(0) = \xi_0, \end{cases}$$

where $\mathrm{D}^q, \mathrm{D}^p$ denote the Caputo derivative, $0 < p, q \leq 1$ and $f, g : [1,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

The problem of integro-differential equation with delay:

$$\begin{cases} \zeta'(t) = f(t,\zeta(t),\zeta'(t-\tau)) + \int_0^t g(t,s,\zeta(s),\zeta'(s))ds, \ 0 \leq t \leq T, \\ \zeta(t) = \varphi(t), t \in [-\tau, 0], \end{cases}$$

was studied in [2] via Perov's fixed point theorem.

Alsulami et al. [1] examined the existence and uniqueness of the solution for a nonlinear coupled system of Caputo type fractional differential equations:

$$\begin{cases} \mathrm{D}\zeta^q(t) = f(t,\zeta(t),\xi), \ 0 \leq t \leq T, 1 < q \leq 2 \\ \mathrm{D}\xi^p(t) = g(t,\zeta(t),\xi), \ 0 \leq t \leq T, 1 < p \leq 2 \end{cases}$$

with the following coupled boundary conditions

$$\begin{cases} \zeta(0) = \lambda_1\xi(T), \ \ \zeta'(0) = \lambda_2\xi'(T), \\ \xi(0) = \kappa_1\zeta(T), \ \ \xi'(0) = \kappa_2\zeta'(T), \end{cases}$$

where $\mathrm{D}^q, \mathrm{D}^p$ denote the Caputo derivative, f, g are given functions and $\lambda_i, \kappa_i, \ \ i = 1, 2$, are real numbers with $\lambda_i\kappa_i \neq 1$ for $i = 1, 2$.

Inspired by the previous results, we discuss the existence and uniqueness of the solution for the following system of fractional integro-differential equations :

$$\begin{cases} \mathcal{D}^q\zeta(t) = f_1(t,\zeta(t),\xi(t)) + \int_1^t k_1(t,s,\zeta(s),\xi(s))ds, \ 0 < t \leq T, \\ \mathcal{D}^p\xi(t) = f_2(t,\zeta(t),\xi(t)) + \int_1^t k_2(t,s,\zeta(s),\xi(s))ds, \\ \zeta(1) = \zeta_0, \xi(1) = \xi_0, \end{cases} \tag{1.1}$$

where $\mathcal{D}^q, \mathcal{D}^p$ denote the Caputo-Hadamard derivative, $0 < p, q \leq 1$, $f_i : [1,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $k_i : [1,\infty) \times [1,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$, are given continuous functions.

First, we give some definitions and basic elements of vector-valued metric spaces.

For two elements $\xi, \zeta \in \mathbb{R}^n$, where $\xi = (\xi_1, \xi_2, ..., \xi_n)$ and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_n)$ the inequality $\xi < \zeta$ means $\xi_i < \zeta_i$ for all $i \in \{1, 2, 3, ..., n\}$. We have also $|\xi| = (|\xi_1|, ..., |\xi_n|)$. Two vectors in $\mathbb{R}^n$ are considered equal if their corresponding coordinates are equal.

Definition 1.1. *[18] Let X be a nonempty set. A generalized (vector-valued) metric on X is a function $\mathfrak{d} : X \times X \to \mathbb{R}^n_+$ satisfying the following assertions:*

1. *$\mathfrak{d}(\zeta, \xi) \geq 0$, for every $\zeta, \xi \in X$ and $\mathfrak{d}(\zeta, \xi) = 0$ if and only if $\zeta = \xi$.*
2. *For all $\zeta, \xi \in X$, $\mathfrak{d}(\zeta, \xi) = \mathfrak{d}(\xi, \zeta)$.*

3. *For all* $\zeta, \xi, \varsigma \in X$, $\mathfrak{d}(\zeta, \xi) \leq \mathfrak{d}(\zeta, \varsigma) + \mathfrak{d}(\xi, \varsigma)$.

$(X, \mathfrak{d})$ is called a generalized (vector-valued) metric space.

Example 1.1. For $X = \mathbb{R}^2$, the function

$$\mathfrak{d}(\xi, \zeta) = (|\xi_1 - \zeta_1|, |\xi_2 - \zeta_2|),$$

where $\xi = (\xi_1, \xi_2)$ and $\zeta = (\zeta_1, \zeta_2)$, is a generalized (vector-valued) metric on X.

Definition 1.2. *[18] In a generalized (vector-valued) metric space, an open ball* $B_\rho(\xi_0)$ *centered at the point* ξ_0 *with radius* ρ *(where* ρ *is a vector in this case) is the set of all points* ξ *in the space such that the vector-valued distance* $\mathfrak{d}(\xi_0, \xi)$ *is less than* ρ.

Remark 1.1. In a generalized (vector-valued) metric space, concepts such as Cauchy sequences, convergent sequences, closed and open sets are defined similarly as in a metric space.

For the standard notations and terminology pertaining to matrix analysis, we refer the reader to [19, 20].

Theorem 1.1. *[18] Let* $(X, \mathfrak{d})$ *be a complete generalized (vector-valued) metric space and let* $\mathcal{T} : X \to X$ *be a self mapping. If there exists a matrix* $M \in \mathcal{M}(\mathbb{R})$ *convergent to zero such that for all* $\zeta, \xi \in X$ *we have*

$$\mathfrak{d}(\mathcal{T}\zeta, \mathcal{T}\xi) \leq M\mathfrak{d}(\zeta, \xi),$$

then $\mathcal{T}$ *has a unique fixed point.*

Now, we give some background of fractional calculus. For an interval I, $AC(I, \mathbb{R})$ is the space of functions $\zeta : I \to \mathbb{R}$ that are absolutely continuous. Let $\delta = t\frac{d}{dt}$. We set

$$AC^n_\delta(I, \mathbb{R}) = \{\zeta : I \to \mathbb{R} : \delta^{n-1}\zeta(t) \in AC(I, \mathbb{R})\}.$$

$AC^1(I, \mathbb{R})$ is the space of functions $\zeta : I \to \mathbb{R}$ that are absolutely continuous, and have an absolutely continuous first derivative.

Definition 1.3. *[12] The Hadamard fractional integral of order* $q > 0$ *for a function* $\zeta \in L^1[1, +\infty)$ *is defined as*

$${}^H I^q \zeta(t) = \frac{1}{\Gamma_1(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{\zeta(s)}{s} ds,$$

provided the integral exists.

Definition 1.4. *[11] For a given function* $\zeta \in AC^n_\delta([a, b], \mathbb{R})$ *such that* $0 < a < b$, *the Caputo-Hadamard fractional derivative of order* $q > 0$ *is defined as follows:*

$$\mathcal{D}^q \zeta(t) = {}^H D^q \left[\zeta(s) - \sum_{m=0}^{n-1} \frac{\delta^m \zeta(a)}{m!} \left(\log \frac{s}{a}\right)^q\right](t),$$

where $\delta^n = (t\frac{d}{dt})^n, n - 1 < q < n$ *and* $n \in \mathbb{N}$.

Lemma 1.2. *[11] Let $n-1 < q < n$, $n \in \mathbb{N}$, and $\zeta \in AC^n_\delta([a,b],\mathbb{R})$ or $C^n_\delta([a,b],\mathbb{R})$. Then*

$$I^q\left(\mathcal{D}^q\zeta\right)(t) = \zeta(t) - \sum_{m=0}^{n-1} \frac{\delta^m \zeta(a)}{m!}\left(\log\frac{t}{a}\right)^q.$$

Theorem 1.3. *[5] Let X be a generalized (vector-valued) Banach space and let $\mathcal{T} : X \to X$ be a completely continuous operator. Then, either*

1. *$\mathcal{T}$ has at least a fixed point, or*
2. *the set $\mathcal{M} = \{\zeta \in X, \lambda\mathcal{T}\zeta = \zeta, \lambda \in [0,1]\}$ is unbounded.*

2 Main results

Consider the space $C(J,\mathbb{R})$ of all continuous functions from J into $\mathbb{R}$, where $J = [1,T]$. Note that $C(J,\mathbb{R})$ is a Banach space with the norm $\|\zeta\|_\infty = \sup\limits_{1\le t\le T} |\zeta(t)|$

Lemma 2.1. *The function $\zeta \in C(J,\mathbb{R})$ is a solution of the problem*

$$\begin{cases} \mathcal{D}^q\zeta(t) = f_1(t,\zeta(t),\xi(t)) + \int_1^t k_1(t,s,\zeta(s),\xi(s))ds, \ 1 < t \le T, \\ \zeta(1) = \zeta_0, \end{cases} \tag{2.1}$$

if and only if it is a solution of the following integral equation:

$$\begin{aligned} \zeta(t) =& \zeta_0 + \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{f_1(s,\zeta(s),\xi(s))}{s}ds \\ &+ \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{ds}{s}\int_1^s k_1(s,\tau,\zeta(\tau),\xi(\tau))d\tau. \end{aligned} \tag{2.2}$$

Proof. From Lemma 1.2 we get

$$\zeta(t) = I^q f_1(t) + I^q\left(\int_1^t k_1(t,\tau,\zeta(\tau),\xi(\tau))d\tau\right) + c,$$

which implies that

$$\begin{aligned} \zeta(t) =& \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{f_1(s,\zeta(s),\xi(s))}{s}ds \\ &+ \frac{1}{\Gamma(q)}\int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{ds}{s}\int_1^s k_1(s,\tau,\zeta(\tau),\xi(\tau))d\tau + c. \end{aligned}$$

Using the initial condition, we get

$$\zeta(1) = c = \zeta_0,$$

whence

$$\zeta(t)=\zeta_0+\frac{1}{\Gamma(q)}\left[\int_1^t\left(\frac{1}{s}\log\frac{t}{s}\right)^{q-1}\left(f_1(s,\zeta(s),\xi(s))ds+\int_1^s k_1(s,\tau,\zeta(\tau),\xi(\tau))d\tau\right)ds\right].$$

Conversely, if ζ satisfies (2.2), then we have $\zeta(1)=\zeta_0$ and

$$\mathcal{D}^q\zeta(t)=\mathcal{D}^q\left(I^q f_1(t)+I^q\left(\int_1^t k_1(t)dt\right)\right),$$

thus

$$\mathcal{D}^q\zeta(t)=f_1(t)+\int_1^t k_1(t)dt.$$

□

The pair $(\zeta,\xi)\in(C(J,\mathbb{R}))^2$ is a solution of the problem (1.1) if it is a solution of the following system

$$\begin{cases}\zeta(t)=\zeta_0+\dfrac{1}{\Gamma(q)}\displaystyle\int_1^t\frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1}\left(f_1(s,\zeta(s),\xi(s))+\int_1^s k_1(s,\tau,\zeta(\tau),\xi(\tau))d\tau\right)ds,\\ \xi(t)=\xi_0+\dfrac{1}{\Gamma(p)}\displaystyle\int_1^t\frac{1}{s}\left(\log\frac{t}{s}\right)^{p-1}\left(f_2(s,\zeta(s),\xi(s))+\int_1^s k_2(s,\tau,\zeta(\tau),\xi(\tau))d\tau\right)ds.\end{cases}$$

Now, define an operator $\mathcal{T}:\mathbb{R}^2\to\mathbb{R}^2$ by

$$\mathcal{T}(\zeta,\xi)(t)=(\mathcal{T}_1,\mathcal{T}_2)^t$$

such that

$$\begin{cases}\mathcal{T}_1(\zeta,\xi)(t)= & \zeta_0+\dfrac{1}{\Gamma(q)}\left[\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{f_1(s,\zeta(s),\xi(s))}{s}ds\right.\\ & \left.+\displaystyle\int_1^t\frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1}ds\int_1^s k_1(s,\tau,\zeta(\tau),\xi(\tau))d\tau\right],\\ \mathcal{T}_2(\zeta,\xi)(t)= & \xi_0+\dfrac{1}{\Gamma(p)}\left[\displaystyle\int_1^t\left(\log\frac{t}{s}\right)^{p-1}\frac{f_2(s,\zeta(s),\xi(s))}{s}ds\right.\\ & \left.+\displaystyle\int_1^t\frac{1}{s}\left(\log\frac{t}{s}\right)^{p-1}ds\int_1^s k_2(s,\tau,\zeta(\tau),\xi(\tau))d\tau\right].\end{cases}$$

We introduce the following hypotheses:

(H_1) : There exist continuous functions $\psi_i,\phi_i:[1,T]\to\mathbb{R}_+$, $i=1,2$, such that

$$|f_i(t,\zeta(t),\xi(t))-f_i(t,\overline{\zeta}(t),\overline{\xi}(t))|\le\psi_i(t)|\zeta-\overline{\zeta}|+\phi_i(t)|\xi-\overline{\xi}|,$$

for all $\zeta,\xi,\overline{\zeta}$ and $\overline{\xi}$ in $C(J,\mathbb{R})$. Denote

$$\psi_i^*=\|\psi_i\|_\infty$$

and

$$\phi_i^*=\|\phi_i\|_\infty.$$

(H_2) : There exist continuous functions $\varphi_i, \eta : [1,T] \to \mathbb{R}_+, i = 1,2$, such that

$$|k_i(t,s,\zeta(t),\xi(t)) - k_i(t,\overline{\zeta}(t),\overline{\xi}(t))| \leq \varphi_i(t)|\zeta - \overline{\zeta}| + \eta_i(t)|\xi - \overline{\xi}|,$$

for all $\zeta, \xi, \overline{\zeta}$ and $\overline{\xi}$ in $C(J,\mathbb{R})$. Denote

$$\varphi_i^* = \|\varphi\|_\infty$$

and

$$\eta_i^* = \|\eta\|_\infty$$

Theorem 2.2. *Under the hypotheses* (H_1) *and* (H_2)*, the problem* (1.1) *has a unique solution provided that the matrix* $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ *converges to zero, where*

$$a_1 = \frac{\psi_1^*(\log T)^{q+1}}{\Gamma(q+1)} + \frac{\varphi_1^*(\log T)^{q+1}}{\Gamma(q+2)}, \quad a_2 = \frac{\psi_1^*(\log T)^{p+1}}{\Gamma(p+1)} + \frac{\varphi_1^*(\log T)^{p+1}}{\Gamma(p+2)},$$

$$b_1 = \frac{\phi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\eta_1^*(\log T)^{q+1}}{\Gamma(q+2)}, \quad b_2 = \frac{\phi_2^*(\log T)^{p}}{\Gamma(p+1)} + \frac{\eta_2^*(\log T)^{p+1}}{\Gamma(p+2)}.$$

Proof. First, we prove that $\mathcal{T}$ is well defined. In fact for $(\zeta,\xi) \in (C(J,\mathbb{R}))^2$ we have

$$|\mathcal{T}_1(\zeta,\xi)(t)| \leq \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} |f_1(s,\zeta(s),\xi(s))|ds$$
$$+ \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} \int_1^s |k_1(s,\zeta(\tau),\xi(\tau))|d\tau ds,$$

which implies:

$$\|\mathcal{T}_1\| \leq \left(\frac{\psi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\varphi_1^*(\log T)^{q+1}}{\Gamma(q+2)}\right)\|\zeta\|_\infty + \left(\frac{\phi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\eta_1^*(\log T)^{q+1}}{\Gamma(q+2)}\right)\|\xi\|_\infty < \infty.$$

By the same reasoning, we can find $\|\mathcal{T}_2\|_\infty < \infty$. Then $\mathcal{T}$ is well defined.

Now, we prove $\mathcal{T}$ satisfies the conditions of Theorem 2.2. We have:

$$|\mathcal{T}_1(\zeta,\xi) - \mathcal{T}_1(\overline{\zeta},\overline{\xi})| \leq \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} |f_1(s,\zeta,(s),\xi,(s)) - f_1(s,\overline{\zeta}(s),\overline{\xi}(s))|ds$$
$$+ \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} \int_1^s |k_1(s,\zeta(\tau),\xi(\tau)) - k_1(s,\overline{\zeta}(\tau),\overline{\xi}(\tau))|d\tau ds$$
$$\leq \left(\frac{\psi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\varphi_1^*(\log T)^{q}}{\Gamma(q+2)}\right)|\zeta - \overline{\zeta}|$$
$$+ \left(\frac{\phi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\eta_1^*(\log T)^{q}}{\Gamma(q+2)}\right)|\xi - \overline{\xi}|,$$

which implies that

$$\|\mathcal{T}(\zeta,\xi)(t) - \mathcal{T}(\overline{\zeta},\overline{\xi})(t)\| \leq \begin{pmatrix} \frac{\psi_1^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\varphi_1^*(\log T)^{q+1}}{\Gamma(q+2)} & \frac{\phi_1^*(\log T)^{p}}{\Gamma(p+1)} + \frac{\eta_1^*(\log T)^{p+1}}{\Gamma(p+2)} \\ \frac{\psi_2^*(\log T)^{q}}{\Gamma(q+1)} + \frac{\phi_2^*(\log T)^{q+1}}{\Gamma(q+2)} & \frac{\varphi_2^*(\log T)^{p}}{\Gamma(p+1)} + \frac{\eta_2^*(\log T)^{p+1}}{\Gamma(p+2)} \end{pmatrix} \begin{pmatrix} \|\zeta - \overline{\zeta}\| \\ \|\xi - \overline{\xi}\| \end{pmatrix}.$$

Thus

$$\mathfrak{d}(\mathcal{T}\zeta, \mathcal{T}\xi) \leq M\mathfrak{d}(\zeta, \xi),$$

where M converges to zero. Then, from Perov's theorem the mapping $\mathcal{T}$ has a unique fixed point, which is a solution of the problem (1.1). □

Now, we establish an existence result by using Schaefer's fixed point theorem. For this, consider the space $(C(J, \mathbb{R}))^2$ endowed with the norm

$$\|(\zeta, \xi)\| = \|\zeta\|_\infty + \|\xi\|_\infty,$$

which is a Banach space.

Theorem 2.3. *Assume that* (H_1) *and the following hypothesis hold:*

(H_3) : *There exist nonnegative real numbers* a_i, b_i, c_i, d_i, $i = 1, 2, 3$, *such that for all* $\zeta, \xi \in \mathbb{R}$ *we have*

$$|f_1(t, \zeta, \xi)| \leq a_1|\zeta| + a_2|\xi| + a_3, \quad k_1(t, s, \zeta, \xi) \leq b_1|\zeta| + b_2|\xi| + b_3,$$

$$|f_2(t, \zeta, \xi)| \leq c_1|\zeta| + c_2|\xi| + c_3, \quad k_2(t, s, \zeta, \xi) \leq d_1|\zeta| + d_2|\xi| + d_3.$$

Then the problem (1.1) *has at least a solution.*

Proof. The proof will be given into four steps.

Step 1. $\mathcal{T}$ is continuous.

Let (ζ_n, ξ_n) be a sequence in $(C([1,T], \mathbb{R}))^2$ converging to (ζ, ξ). We have:

$$\begin{aligned}|\mathcal{T}_1(\zeta_n) - \mathcal{T}_1(\zeta)| \leq & \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} |f_1(s, \zeta, (s), \xi, (s)) - f_1(s, \overline{\zeta}(s), \overline{\zeta}(s))|ds \\ & + \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} \int_1^s |k_1(s, \zeta(\tau), \xi(\tau)) - k_1(s, \overline{\zeta}(\tau), \overline{\xi}(\tau))|d\tau ds.\end{aligned}$$

Then

$$\begin{aligned}\|\mathcal{T}_1(\zeta_n) - \mathcal{T}_1(\zeta)\| \leq & \left(\frac{\psi_1^*(\log T)^q}{\Gamma(q+1)} + \frac{\varphi_1^*(\log T)^{q+1}}{\Gamma(q+2)}\right)\|\zeta - \overline{\zeta}\|_\infty \\ & + \left(\frac{\phi_1^*(\log T)^q}{\Gamma(q+1)} + \frac{\eta_1^*(\log T)^{q+1}}{\Gamma(q+2)}\right)\|\xi_n - \xi\|_\infty.\end{aligned}$$

Since $\|\xi_n - \xi\|_\infty \to 0$ and $\|\zeta_n - \zeta\|_\infty \to 0$, we get $\|\mathcal{T}_1(t, \zeta_n, \xi_n)) - \mathcal{T}_1(t, \zeta, \xi_n)\|_\infty \to 0$.

Analogously, we find

$$\begin{aligned}\|\mathcal{T}_2(\zeta_n) - \mathcal{T}_2(\zeta)\|_\infty \leq & \left(\frac{\psi_2^*(\log T)^p}{\Gamma(p+1)} + \frac{\varphi_2^*(\log T)^{p+1}}{\Gamma(p+2)}\right)\|\zeta - \overline{\zeta}\|_\infty \\ & + \left(\frac{\phi_2^*(\log T)^p}{\Gamma(p+1)} + \frac{\eta_2^*(\log T)^{p+1}}{\Gamma(p+2)}\right)\|\xi_n - \xi\|_\infty,\end{aligned}$$

which implies $\|\mathcal{T}_2(t, \zeta_n, \xi_n)) - \mathcal{T}_2(t, \zeta, \xi_n)\|_\infty \to 0$. Then $\mathcal{T}$ is continuous.

Step 2. $\mathcal{T}$ is bounded.

Let $B_R = \{(\zeta,\xi) \in (C([1,T],\mathbb{R}))^2, \|(\zeta,\xi)\|_\infty \leq R\}$ be the closed ball of center 0 and radius $R = (R_1, R_2)$, and let $(\zeta,\xi) \in B_R$. We have

$$\begin{aligned}|\mathcal{T}_1| \leq & \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1} ||f_1(s,\zeta,(s),\xi,(s))|ds \\ &+ \int_1^t \frac{1}{s}\left(\log\frac{t}{s}\right)^{q-1}\int_1^s |k_1(s,\zeta_1(\tau),\xi(\tau)) - k_1(s,\overline{\zeta}(\tau),\overline{\xi}(\tau))|)|d\tau ds \\ \leq & (a_1|\zeta| + a_2|\xi| + a_3)\frac{(\log T)^q}{\Gamma(q+1)} + (b_1|\zeta| + b_2|\xi| + b_3)\frac{(\log T)^{q+1}}{\Gamma(q+2)},\end{aligned}$$

whence

$$\|\mathcal{T}_1\|_\infty \leq \left(\frac{a_1(\log T)^q}{\Gamma(q+1)} + b_1\right)(R_1 + R_2) + a_3 + b_3 < \infty.$$

Step 3. $\mathcal{T}$ maps a bounded set into a relatively compact set.

From previous steps, $\mathcal{T}$ is uniformly bounded, so it suffices to prove $\mathcal{T}B_R$ is equicontinuous. In fact, for $(\zeta,\xi) \in B_R$ and $t_1, t_2 \in [1,T]$ with $t_1 < t_2$ we have

$$\begin{aligned}&|\mathcal{T}_1(\zeta(t_1),\xi(t_1)) - \mathcal{T}_1(\zeta(t_2),\xi(t_2))| \leq \\ &\leq \frac{1}{\Gamma(q)}\left[\int_1^{t_1}\frac{1}{s}\left(\left(\log\frac{t_2}{s}\right)^{q-1} - \left(\log\frac{t_1}{s}\right)^{q-1}\right)\left(|f_1(s,\zeta(s),\xi(s))| + \int_1^s |k_1(s,\zeta_1(\tau),\xi(\tau))|d\tau\right)ds\right. \\ &\left. + \int_{t_1}^{t_2}\frac{1}{s}\left(\log\frac{t_2}{s}\right)^{q-1}\left(|f_1(s,\zeta(s),\xi(s))| + \int_1^s |k_1(s,\zeta_1(\tau),\xi(\tau))|d\tau\right)ds\right].\end{aligned}$$

From the hypothesis (H_3) we get:

$$\begin{aligned}&\|\mathcal{T}_1(\zeta(t_1),\xi(t_1)) - \mathcal{T}_1(\zeta(t_2),\xi(t_2))\| \leq \\ &\leq \frac{(a_1+b_1)|\zeta| + (a_2+b_2)|\xi| + a_3 + b_3}{\Gamma(q+1)}\left(|(\log t_2)^q - (\log t_1)^q|\right) \\ &+ \frac{b_1|\zeta| + b_2|\xi| + b_3}{\Gamma(q+1)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{q-1} ds.\end{aligned}$$

Since the function $\zeta \mapsto (\log\zeta)^q$ is continuous on a compact set, it is uniformly continuous, which implies that $|(\log t_2)^q - (\log t_1)^q| \to 0$ and $|\log t_2 - \log t_1|^q \to 0$ as $t_1 \to t_2$. Also, the function $\zeta \mapsto (\log\zeta)^q$ is bounded (continuous on a compact set), so $\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{q-1} ds \to 0$ when $t_1 \to t_2$. Hence

$$\|\mathcal{T}_1(\zeta(t_1),\xi(t_1)) - \mathcal{T}_1(\zeta(t_2),\xi(t_2))\| \to 0 \quad \text{as } t_1 \to t_2.$$

Similarly we can obtain

$$\begin{aligned}&\|\mathcal{T}_2(\zeta(t_1),\xi(t_1)) - \mathcal{T}_2(\zeta(t_2),\xi(t_2))\| \leq \\ &\leq \frac{(c_1+d_1)|\zeta| + (c_2+d_2)|\xi| + c_3 + d_3}{\Gamma(p+1)}\left(|(\log t_2)^p - (\log t_1)^p\right) \\ &+ \frac{d_1|\zeta| + d_2|\xi| + d_3}{\Gamma(p+1)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{p-1} ds,\end{aligned}$$

which implies that $\|\mathcal{T}_2(\zeta(t_1), \xi(t_1)) - \mathcal{T}_2(\zeta(t_2), \xi(t_2))\|_\infty \to 0$ as $t_1 \to t_2$. Then $\mathcal{T}B_R$ is equicontiuous.

Step 4. The set $\mathcal{M} = \{(\xi, \zeta) \in (C([1,T], \mathbb{R}))^2, \lambda\mathcal{T}(\zeta, \xi) = (\zeta, \xi),\ \lambda \in [0,1]\}$ is bounded.

Let $(\zeta, \xi) \in \mathcal{M}$. We have

$$\begin{aligned}
|\zeta| \leq & \lambda \left(\zeta_0 + \int_1^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{q-1} |f_1(s, \zeta, (s), \xi, (s))| ds \right. \\
& \left. + \int_1^t \frac{1}{s} \left(\log \frac{t}{s} \right)^{q-1} \int_1^s |k_1(s, \zeta_1(\tau), \xi(\tau))| d\tau ds \right) \\
\leq & \lambda \left(a_1|\zeta| + a_2|\xi| + a_3 \right) \frac{(\log T)^q}{\Gamma(q+1)} + \lambda \left(b_1|\zeta| + b_2|\xi| + b_3 \right) \frac{(\log T)^{q+1}}{\Gamma(q+2)},
\end{aligned}$$

so

$$\|\zeta\|_\infty \leq \lambda\zeta_0 + \lambda \left(a_1R_1 + a_2R_2 + a_3 \right) \frac{(\log T)^q}{\Gamma(q+1)} + \lambda \left(b_1R_1 + b_2R_2 + b_3 \right) \frac{(\log T)^{q+1}}{\Gamma(q+2)}.$$

Similarly, one can obtain

$$\|\xi\| \leq \lambda\xi_0 + \lambda \left(a_1R_1 + a_2R_2 + a_3 \right) \frac{(\log T)^p}{\Gamma(p+1)} + \lambda \left(b_1R_1 + b_2R_2 + b_3 \right) \frac{(\log T)^{p+1}}{\Gamma(p+2)}.$$

Then $\|(\zeta, \xi)\| = \|\zeta\|_\infty + \|\xi\|_\infty < \infty$, so the set $\mathcal{M}$ is bounded. According to Schaefer's fixed point theorem, $\mathcal{T}$ has a fixed point, which is a solution of the problem (1.1).

$\square$

Example 2.1. Consider the following system

$$\begin{cases}
\mathcal{D}^{\frac{1}{2}}\zeta(t) = \frac{1}{4t}\zeta + \frac{1}{2}\sin\xi + \frac{1}{4e^2}\int_1^t (\zeta(s) - \xi(s)) e^{-(t^2+s^2)} ds,\ 1 < t \leq e, \\
\mathcal{D}^{\frac{1}{2}}\xi(t) = \frac{1}{8}(\xi + \cos\zeta) + \frac{1}{4}\int_0^t (\xi - \zeta) e^{-(t+s)} ds, \\
\zeta(1) = 0, \xi(1) = 0.
\end{cases} \tag{2.3}$$

We check that all the hypotheses of Theorem 2.2 are satisfied. In fact, here we have

$$f_1(t, \zeta, \xi) = \frac{1}{4t}\zeta + \frac{1}{2}\sin\xi$$

and

$$k_1(t, s, \zeta, \xi) = \frac{e}{8}(\zeta(s) - \xi(s)) e^{-(t^2+s^2)}.$$

Then

$$\|f_1(t, \zeta, \xi) - f_1(t, \overline{\zeta}, \overline{\xi}))\| \leq \frac{1}{4t}|\zeta - \overline{\zeta}| + \frac{1}{2}|\xi - \overline{\xi}|,$$

so $\psi_1(t) = \frac{1}{4t}$ and $\phi_1(t) = \frac{1}{2}$.

On other hand we have

$$|k_1(t, s, \zeta, \xi) - k_1(t, \overline{\zeta}, \overline{\xi}))| \leq \frac{e}{8}e^{-t^2}|\zeta - \overline{\zeta}| + \frac{e}{8}e^{-t^2}|\xi - \overline{\xi}|,$$

so $\varphi_1(t) = \eta_1(t) = \frac{e^{-t^2}}{4e^2}$.

We have also

$$f_2(t,\zeta,\xi) = \frac{1}{8}(\xi + \cos\zeta)$$

and

$$k_2(t,s,\zeta,\xi) = \frac{1}{4}(\xi - \zeta)e^{-(t+s)},$$

thus

$$\|f_2(t,\zeta,\xi) - f_1(t,\overline{\zeta},\overline{\xi})\| \leq \frac{1}{8}|\zeta - \overline{\zeta}| + \frac{1}{8}|\xi - \overline{\xi}|$$

and

$$\|k_2(t,\zeta,\xi) - k_2(t,\overline{\zeta},\overline{\xi}))\| \leq \frac{1}{8e}e^{-t}(|\zeta - \overline{\zeta}| + |\xi - \overline{\xi}|),$$

so $\psi_2(t) = \phi_2(t) = \frac{1}{8}\log t$, $\varphi_2(t) = \eta_2(t) = \frac{e^{-t}}{4}$.

The matrix M is given by

$$\begin{pmatrix} \frac{\psi_1^*}{\Gamma(q+1)} + \frac{\varphi_1^*}{\Gamma(q+2)} & \frac{\phi_1^*}{\Gamma(q+1)} + \frac{\eta_1^*}{\Gamma(q+2)} \\ \frac{\psi_2^*}{\Gamma(p+1)} + \frac{\varphi_2^*}{\Gamma(p+2)} & \frac{\phi_2^*}{\Gamma(p+1)} + \frac{\eta_2^*}{\Gamma(p+2)} \end{pmatrix} = \begin{pmatrix} \frac{2}{3\sqrt{\pi}} & \frac{5}{6\sqrt{\pi}} \\ \frac{7}{12\sqrt{\pi}} & \frac{1}{2\sqrt{\pi}} \end{pmatrix}.$$

The characteristic polynomial of this matrix is given by

$$P(\lambda) = \lambda^2 - \frac{7}{6\sqrt{\pi}}\lambda + \frac{24\sqrt{\pi} - 35}{72\sqrt{\pi}}$$

with the roots

$$\lambda_1 = 0,6029436758 < 1, \quad \text{and} \quad \lambda_2 = 0,055277505 < 1.$$

Then the spectral radius is strictly less than one, which implies that M converges to zero. Consequently, all the hypotheses of Theorem 2.2 are satisfied. It follows that the problem (2.3) has a unique solution.

Example 2.2. Consider the following system

$$\begin{cases} \mathcal{D}^{\frac{1}{4}}\zeta(t) = \frac{|\zeta|}{(t^2+3)(1+|\zeta|)} + \int_1^t \frac{1}{t+1}\sin\zeta(s)ds, \ 1 < t \leq e, \\ \mathcal{D}^{\frac{3}{4}}\xi(t) = \frac{|\xi|}{8t(1+|\xi|)} + \frac{\cos\zeta(s)}{8} + \int_1^t \frac{\sin\zeta(s)+\xi}{8t}ds, \\ \zeta(1) = 0, \xi(1) = 0. \end{cases} \tag{2.4}$$

In this example, we have

$$f_1(t,\zeta,\xi) = \frac{1}{t^2+3}\frac{|\zeta|}{1+|\zeta|}$$

and

$$k_1(t,s,\zeta,\xi) = \frac{1}{t+1}\sin\zeta(s).$$

Then

$$\|f_1(t,\zeta,\xi) - f_1(t,\overline{\zeta},\overline{\xi}))\| \leq \frac{1}{t^2+1}|\zeta - \overline{\zeta}|,$$

so $\psi_1(t) = \frac{1}{t^2+3}$, $\phi_1(t) = 0$, $\varphi_1(t) = \frac{1}{1+t}$ and $\eta_1(t) = 0$. Note that

$$k_1(t,\zeta,\xi) - k_1(t,\overline{\zeta},\overline{\xi}) \leq \frac{e}{8}e^{-t^2}|\zeta - \overline{\zeta}| + \frac{e}{8}e^{-t^2}|\xi - \overline{\xi}|,$$

so $\varphi_2(t) = \frac{1}{8t}$, $\eta_2(t) = \frac{1}{8}$ and $\varphi_2(t) = \eta_2(t) = \frac{1}{8t}$.
We have also

$$M = \frac{1}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{2}\right)} \begin{pmatrix} \psi_1^* + \frac{4\varphi_1^*}{5} & \phi_1^* + \frac{4}{5}\eta_1^* \\ \psi_2^* + \frac{4\varphi_2^*}{5} & \phi_2^* + \frac{4}{7}\eta_2^* \end{pmatrix} = \frac{1}{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{2}\right)} \begin{pmatrix} \frac{13}{20} & 0 \\ \frac{1}{4} & \frac{1}{14} \end{pmatrix},$$

where $\Gamma\left(\frac{5}{4}\right) = 0.90640...$, $\Gamma\left(\frac{5}{2}\right) = 1.32934...$ Then we have

$$P(\lambda) = \left(\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{2}\right)\lambda - \frac{13}{20}\right)\left(\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{5}{2}\right)\lambda - \frac{1}{14}\right),$$

which implies that there are two eigenvalues,

$$\lambda_1 = 0.5394576881 < 1, \ \lambda_2 = 0.05928106463 < 1.$$

Hence the matrix M converges to zero, so according to Perov's fixed point theorem the system has a unique solution.

Now, we show that this example satisfies both hypotheses of Theorem 2.3. In fact it suffices to check that the hypothesis (H_3) holds. We have

$$f_1(t, \zeta, \xi) = \frac{1}{t^2+3}\frac{|\zeta|}{1+|\zeta|} \leq \frac{1}{4}|\zeta|,$$

$$k_1(t, s, \zeta, \xi) = \frac{1}{t+1}\sin\zeta(s) \leq \frac{1}{2}|\zeta|,$$

$$f_2(t, \zeta, \xi) = \frac{|\xi|}{8t(1+|\xi|)} + \frac{\cos\zeta(s)}{8} \leq \frac{1}{8}|\xi| + \frac{1}{8}|\zeta|,$$

and

$$k_2(t, s, \zeta, \xi) = \frac{\sin\zeta(s) + \xi}{8t} \leq \frac{1}{8}|\zeta| + \frac{1}{8}|\xi|.$$

Then the hypothesis (H_3) holds with $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $c_1 = c_2 = d_1 = d_2 = \frac{1}{2}$ and $a_2 = a_3 = b_2 = b_3 = 0$, so from Theorem 2.3 the problem (2.4) has a solution.

Conclusion

In this work, we have used two fixed point theorems to prove the existence and the uniqueness of the solution for systems of fractional integro-differential equations. We hope this study can provide some contribution to the existence and uniqueness problem of initial boundary value problems. Based on these results one can find extensions to some other fractional systems of ordinary and partial differential equations.

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