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III-harmonic Curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ Space

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Abstract Some work has been done in the study of non-geodesic III-harmonic curves in some model spaces. In this paper, we study III-harmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space. We give necessary and sufficient conditions for helices to be III-harmonic. Also, we characterize III-harmonic curves in terms of their curvature and torsion.

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1 Introduction

First we should recall some notions and results related to the harmonic and the poly-harmonic (r-harmonic $r \geq 1$) maps between Riemannian manifolds. Harmonic maps $\psi : (\mathcal{M}, g) \to (\mathcal{N}, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$\mathcal{E}_1 : C^\infty(\mathcal{M}, \mathcal{N}) \to \mathbb{R}, \quad \mathcal{E}_1(\psi) = \frac{1}{2} \int_{\mathcal{M}} |\mathrm{d}\psi|^2 \, v_g,$$

and are characterized by the vanishing of the first tension field

$$\tau_1(\psi) = -\mathrm{d}^*\mathrm{d}\psi = trace\nabla\mathrm{d}\psi,$$

where d is the exterior differentiation and d^* is the codifferentiation.

We remind that the bienergy of ψ is given by

$$\mathcal{E}_2 : C^\infty(\mathcal{M}, \mathcal{N}) \to \mathbb{R}, \quad \mathcal{E}_2(\psi) = \frac{1}{2} \int_{\mathcal{M}} |\tau(\psi)|^2 \, v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^\psi \tau(\psi) - trace_g \mathrm{R}^{\mathcal{N}}(\mathrm{d}\psi, \tau(\psi))\mathrm{d}\psi,$$

where $\Delta^{\psi} = -trace(\nabla^{\psi})^2 = -trace(\nabla^{\psi}\nabla^{\psi} - \nabla^{\psi}_{\nabla})$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional $\mathcal{E}_2$. We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\psi'}\psi' - \mathrm{R}(\psi', \nabla_{\psi'}\psi')\psi' = 0. \tag{1.1}$$

Eells and Lemaire [8] proposed the problem to consider the polyharmonic (r-harmonic $r \geq 1$) maps of order r, these are critical points of the r-energy functional defined by

$$\mathcal{E}_r(\psi) = \int_{\mathcal{M}} e_r(\psi) v_g, \quad r \geq 1, \tag{1.2}$$

where $e_r(\psi) = \frac{1}{2}\|(\mathrm{d} + \mathrm{d}^*)^r \psi\|^2$ for smooth maps ψ. A map ψ is r-harmonic if it is a critical point of the functional $\mathcal{E}_r(\psi)$ defined in (1.2).

Every harmonic map is a solution of the polyharmonic map, see [4] for a recent classification result. In [27], S. B. Wang studied the first variation formula of the k-energy $\mathcal{E}_k$ whose critical maps are called k-harmonic maps. In [13], S. Maeta showed the second variation formula of the k-energy. Triharmonic curves with constant curvature in space forms were studied by Maeta in [13]. Recently, for the study of triharmonic curves in 3-dimensional homogeneous spaces see [21–23] respectively. The main goal of this article is to define and characterize III-harmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space.

2 Preliminaries

In [14] Emil Molnár proposed a projective spherical model as unified geometrical model of homogeneous geometries. $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry is one of the eight homogeneous Thurston 3-geometries. The hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbb{R}}$ geometry is described in details in [7, 10, 14–16]. In $\widetilde{\mathbf{SL}_2\mathbb{R}}$, the metric is given by

$$ds^2 = dr^2 + (\cosh r \sinh r)^2 d\vartheta^2 + (d\phi + \sinh^2 r d\vartheta)^2,$$

where ϕ is a fiber coordinate and $(r,\ \vartheta)$ are polar coordinates of the intersection point of a fiber and the hyperbolic base plane.

If

$$\{\theta^1 = dr, \theta^2 = \cosh r \sinh r d\vartheta, \theta^3 = d\phi + \sinh^2 r d\vartheta\}$$

is an orthonormal coframe then the dual orthonormal frame fields are given by

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sinh 2r} & -\tanh r \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\vartheta \\ \partial_\phi \end{pmatrix},$$

where $\partial_r = \frac{\partial}{\partial r}$, $\partial_\vartheta = \frac{\partial}{\partial \vartheta}$, $\partial_\phi = \frac{\partial}{\partial \phi}$.

One can easily check that $g(e_i, e_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta.

The symmetric metric tensor field g is given by

$$G = (g_{ij})_{1\leq i,j\leq 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\cosh^2 r + \sinh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}.$$

The inverse matrix $(g^{ij})_{1\leq i,j\leq 3}$ of $(g_{ij})_{1\leq i,j\leq 3}$ is given by

$$G^{-1} = \frac{4}{\sinh^2 2r} \begin{pmatrix} \sinh^2 r \cosh^2 r & 0 & 0 \\ 0 & 1 & -\sinh^2 r \\ 0 & -\sinh^2 r & \sinh^2 r(\cosh^2 r + \sinh^2 r) \end{pmatrix}.$$

The Levi-Civita connection ∇ is defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$, where the Cristoffel symbols Γ^k_{ij} are given by

$$\begin{aligned}
\Gamma^1_{ij} &= -\frac{1}{2}\sinh 2r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1+4\sinh^2 r & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\Gamma^2_{ij} &= \frac{2}{\sinh 2r} \begin{pmatrix} 0 & 1+3\sinh^2 r & 1 \\ 1+3\sinh^2 r & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\Gamma^3_{ij} &= -\tanh r \begin{pmatrix} 0 & 2\sinh^2 r & 1 \\ 2\sinh^2 r & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

The Levi-Civita connection ∇ is given by

$$\begin{aligned}
\nabla_{e_1} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \\
\nabla_{e_2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 2\coth 2r & 1 \\ -2\coth 2r & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \\
\nabla_{e_3} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.
\end{aligned} \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[e_1, e_2] = -2\coth(2r)e_2 - 2e_3, \quad [e_1, e_3] = [e_2, e_3] = 0.$$

The Riemannian curvature tensor is given by

$$\mathrm{R}(X_1,\ X_2,\ X_3,\ X_4) = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\mathrm{R}(X_2,\ X_1)X_3,\ X_4) = -g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\mathrm{R}(X_1,\ X_2)X_3,\ X_4), \tag{2.2}$$

where X_1, X_2, X_3, X_4 are smooth vector fields on $\widetilde{\mathbf{SL}_2\mathbb{R}}$.

The components of the Riemannian curvature tensor defined by

$$\mathrm{R}(X_1, X_2)X_3 = (\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})X_3$$

are [9]

$$\begin{aligned} \mathrm{R}(e_1, e_2)e_1 &= 7e_2, \quad \mathrm{R}(e_1, e_2)e_2 = -7e_1, \quad \mathrm{R}(e_1, e_3)e_1 = -e_3, \\ \mathrm{R}(e_1, e_3)e_3 &= e_1, \quad \mathrm{R}(e_3, e_2)e_2 = e_3, \quad \mathrm{R}(e_3, e_2)e_3 = -e_2. \end{aligned} \tag{2.3}$$

The Riemannian curvature tensor field R is described by

$$\mathrm{R}_{1212} = -7, \quad \mathrm{R}_{1313} = \mathrm{R}_{2323} = 1. \tag{2.4}$$

The mixed product of three vectors X, Y, Z of a $\widetilde{\mathbf{SL}_2\mathbb{R}}$ is

$$[X, Y, Z] = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(X \wedge Y, Z). \tag{2.5}$$

3 Polyharmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space

3.1 Biharmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space

In [6] and [11], the authors studied biharmonic curves in BCV-spaces and they obtained interesting classification results. In [5] the authors proved that any proper biharmonic curve in the Heisenberg group $\mathcal{H}_3$ is a helix and gave its explicit parametrization.

Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parametrized by arc length and let $\{T = \psi'(s),\ N,\ B\}$ be the orthonormal moving Frenet frame along the curve ψ in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ such that $T = \psi'$ is the unit vector field tangent to ψ, N is the unit vector field in the direction $\nabla_T T$ normal to ψ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations:

$$\nabla_T \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.1}$$

where

$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(T,\ T) = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(N,\ N) = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(B,\ B) = 1,$$
$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(T,\ N) = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(T,\ B) = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(N,\ B) = 0,$$

and k ($k^2 = g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T T, \nabla_T T)$) is the curvature of ψ and τ is its torsion.

From (3.1) we have

$$\nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B, \tag{3.2}$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \\ B_1 & B_2 & B_3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{3.3}$$

A direct computation using (2.4) and (3.3) gives the following:

$$\begin{cases} \mathrm{R}(T,N,T,N) = \sum_{1\leq i,j,k,l\leq 3} T_i N_j T_k N_l \mathrm{R}_{ijkl} = 1 - 8B_3^2 \\ \mathrm{R}(T,N,T,B) = \sum_{1\leq i,j,k,l\leq 3} T_i N_j T_k B_l \mathrm{R}_{ijkl} = 8N_3B_3 \\ \mathrm{R}(B,N,T,N) = \sum_{1\leq i,j,k,l\leq 3} B_i N_j T_k N_l \mathrm{R}_{ijkl} = 8T_3B_3 \\ \mathrm{R}(B,N,T,B) = \sum_{1\leq i,j,k,l\leq 3} B_i N_j T_k B_l \mathrm{R}_{ijkl} = -8T_3N_3 \\ \mathrm{R}(B,T,T,B) = \sum_{1\leq i,j,k,l\leq 3} B_i T_j T_k B_l \mathrm{R}_{ijkl} = 8N_3^2 - 1. \end{cases} \tag{3.4}$$

We can use (2.1) and compute the covariant derivatives of the vector fields T, N and B as:

$$\begin{aligned} \nabla_T T =& (T_1' - 2T_2^2 \coth 2r - 2T_2T_3)e_1 + (T_2' + 2T_1T_2 \coth 2r + 2T_1T_3)e_2 + (T_3')e_3, \\ \nabla_T N =& (N_1' - 2T_2N_2 \coth 2r - T_2N_3 - T_3N_2)e_1 + (N_2' + 2T_2N_1 \coth 2r + T_1N_3 + T_3N_1)e_2 \\ &+ (N_3' + T_2N_1 - T_1N_2)e_3, \\ \nabla_T B =& (B_1' - 2T_2B_2 \coth 2r - T_2B_3 - T_3B_2)e_1 + (B_2' + 2T_2B_1 \coth 2r + T_1B_3 + T_3B_1)e_2 \\ &+ (B_3' + T_2B_1 - T_1B_2)e_3. \end{aligned} \tag{3.5}$$

Theorem 3.1 ([9]). *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic biharmonic curve if and only if*

$$\begin{cases} k = constant \neq 0 \\ k^2 + \tau^2 = 1 - 8B_3^2 \\ \tau' = -8N_3B_3 \end{cases}. \tag{3.6}$$

3.2 III-harmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ space

To study III-harmonic curves in $\widetilde{\mathbf{SL}_2\mathbb{R}}$, we shall use their Frenet vector fields and equations.

Let us denote by $\psi : I \to \mathcal{M}^n$ an arc length parametrized curve in $\mathcal{M}^n$. Assume that ψ is non-geodesic.

If $r = 2t,\ t \geq 1$, then (1.2) takes the form [12, 27]

$$\mathcal{E}_{2t}(\psi) = \frac{1}{2}\int_I < \underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi > v_g. \tag{3.7}$$

If $r = 2t + 1$, then (1.2) takes the form

$$\mathcal{E}_{2t+1}(\psi) = \frac{1}{2}\int_I < \mathrm{d}\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \mathrm{d}\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi > v_g. \tag{3.8}$$

The Euler-Lagrange equations of (3.7) and (3.8), reduce to the equation

$$\tau_r(\psi) = \nabla_T^{2r-1}T + \sum_{j=0}^{r-1}(-1)^j \mathrm{R}(\nabla_T^{2r-3-j}T,\ \nabla_T^{j}T)T, \quad r \geq 1. \tag{3.9}$$

Solutions of equation (3.9) are called as r-harmonic curves.

The r-energy functionals defined in (3.7), (3.8) have been intensively studied (see [12, 17–19, 27]).

Remark 3.1. i) If $r = 1$, the functional (3.8) is just the energy.

ii) If $r = 2$, the functional (3.7) is called bienergy and its critical points are the so-called biharmonic maps.

iii) r-harmonic curves called as proper r-harmonic if they are not harmonic.

iv) Any harmonic curve is a r-harmonic curve, for any $r \geq 1$.

An arc-length parametrized curve $\psi : I \to \mathcal{M}^n$ from $I \subset \mathbb{R}$ to a Riemannian manifold $\mathcal{M}^n$ of dimension n is called *triharmonic* ($r = 3$) if [17]

$$\nabla_T^5 T + \mathrm{R}(\nabla_T^3 T,\ T)T - \mathrm{R}(\nabla_T^2 T,\ \nabla_T T)T = 0. \tag{3.10}$$

With respect to the Frenet frame, the III-harmonic curve equation takes the following form.

Proposition 3.2. *Let $\psi : I \subset \mathbb{R} \to \mathcal{M}^3$ be a differentiable curve parametrized by arc length. Then ψ is III-harmonic curve if and only if*

$$\begin{cases} \zeta_1(s) = 0 \\ \zeta_2(s) - \zeta_4(s)R(N,T,T,N) - \zeta_5(s)R(B,T,T,N) + \zeta_6(s)R(B,N,T,N) = 0 \\ \zeta_3(s) - \zeta_4(s)R(N,T,T,B) - \zeta_5(s)R(B,T,T,B) + \zeta_6(s)R(B,N,T,B) = 0, \end{cases} \tag{3.11}$$

where

$$\begin{aligned} \zeta_1(s) &= -10k'k'' - 5kk^{(3)} + 5kk'(2k^2+\tau^2) + 5k^2\tau\tau' \\ \zeta_2(s) &= k^5 + k^{(4)} - 15kk'^2 - 10k^2k'' + k\tau^2(2k^2+\tau^2) - 6\tau^2k'' \\ &\quad -12k'\tau\tau' - 3k\tau'^2 - 4k\tau\tau'' \\ \zeta_3(s) &= 4\tau k^{(3)} + k\tau^{(3)} - 9k^2k'\tau - 4k'(\tau^3 - \tau'') + 6\tau'(k'' - k\tau^2) - \tau'k^3 \\ \zeta_4(s) &= k'' - k(2k^2+\tau^2) \\ \zeta_5(s) &= 2k'\tau + k\tau', \\ \zeta_6(s) &= k^2\tau. \end{aligned}$$

Proof. From (3.1) we have

$$\nabla_T^2 T = (-k^2)T + (k')N + (k\tau)B,$$

$$\nabla_T^3 T = (-3kk')T + (k'' - k(k^2+\tau^2))N + (2k'\tau + k\tau')B,$$

$$\nabla_T^5 T = \zeta_1(s)T + \zeta_2(s)N + \zeta_3(s)B.$$

By (3.10) we see that ψ is a III-harmonic curve if and only if

$$\zeta_1(s)T+\zeta_2(s)N+\zeta_3(s)B+\zeta_4(s)R(N,T)T+\zeta_5(s)R(B,T)T-\zeta_6(s)R(B,N)T = 0. \quad (3.12)$$

Using (2.2), we obtain (3.11). □

Theorem 3.3. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic III-harmonic curve if and only if the following equations hold,*

$$\begin{cases} \zeta_1(s) = 0 \\ \zeta_2(s) + \zeta_4(s)(1-8B_3^2) + 8\zeta_5(s)N_3B_3 + 8\zeta_6(s)T_3B_3 = 0 \\ \zeta_3(s) + 8\zeta_4(s)N_3B_3 + \zeta_5(s)(1-8N_3^2) - 8\zeta_6(s)T_3N_3 = 0. \end{cases} \quad (3.13)$$

Proof. Combining (3.4) and (3.11), we obtain (3.13). □

Corollary 3.4. *If $\tau = 0$ and $N_3B_3 \neq 0$, then $k = 0$.*

Proof. From (3.13), we get

$$\begin{cases} \zeta_1(s) = 0 \\ \zeta_2(s) + (1-8B_3^2)(k'' - 2k^3) = 0 \\ N_3B_3(k''-2k^3) = 0. \end{cases}$$

Then the following hold:

$$2k'k'' + kk^{(3)} - 2k^3k' = 0, \quad (3.14)$$

$$k'' = 2k^3, \quad (3.15)$$

$$k^5 + k^{(4)} - 15kk'^2 - 10k^2k'' = 0. \quad (3.16)$$

The equation (3.14) is rewritten as

$$k'k'' + (kk'')' - 2k^3k' = 0. \quad (3.17)$$

From (3.15) and (3.17) we obtain $k =$ constant. Hence (3.16) implies $k = 0$. □

4 III-harmonic helices in $\widetilde{\mathbf{SL}_2\mathbb{R}}$

A circular helix is a geometric curve with non-vanishing constant curvature and non-vanishing constant torsion. We give the following theorem

Theorem 4.1. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic proper III-harmonic curve with constant curvature k and nonzero torsion τ. Then $\psi(s)$ is a circular helix. Further, k and τ satisfy the differential equation system:*

$$\begin{cases} (k^2+\tau^2)^2 - (2k^2+\tau^2)(1-8B_3^2) + 8k\tau T_3B_3 = 0 \\ N_3((2k^2+\tau^2)B_3 + k\tau T_3) = 0. \end{cases} \quad (4.1)$$

Proof. From the first equation of (3.13), we obtain that τ = constant. Then $\psi(s)$ is a circular helix. If we substitute k = constant and τ = constant in (3.13), we obtain (4.1). □

Theorem 4.2. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic III-harmonic helix parametrized by arc length. If $N_3 = 0$, then $B_3 = constant$.*

Proof. Using (3.5), we have

$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T B, e_3) = B_3' - (T_1 B_2 - T_2 B_1).$$

Then from the Frenet equations (3.1) we obtain

$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T B, e_3) = -\tau N_3. \tag{4.2}$$

Since $N_3 = 0$ and $T_1 B_2 - T_2 B_1 = -N_3$, we have

$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T B, e_3) = B_3'. \tag{4.3}$$

These, together with (4.2) and (4.3) give $B_3' = 0$. □

Corollary 4.3. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic III-harmonic helix parametrized by arc length. If $N_3 = 0$, then $T_3 = constant$.*

Proof. By Theorem 4.2 and (4.1), we have $T_3 = constant$. □

Theorem 4.4. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic III-harmonic helix parametrized by arc length. If $N_3 \neq 0$, then*

$$\tau^2 = \frac{1 - 2k^2 - \sqrt{1 + 4k^2}}{2}, \quad or, \quad \tau^2 = \frac{1 - 2k^2 + \sqrt{1 + 4k^2}}{2}. \tag{4.4}$$

Proof. Since $N_3 \neq 0$, from (4.1), we obtain

$$(k^2 + \tau^2)^2 - (2k^2 + \tau^2) = 0. \tag{4.5}$$

Hence (4.5) implies (4.4). □

5 General helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$

In this section we give a characterization of general helices with respect to the Frenet frame $\{T, N, B\}$ in $\widetilde{\mathbf{SL}_2\mathbb{R}}$. A necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion be constant (see [1–3, 20, 24–26] for details).

Definition 5.1. *Let $\psi(s)$ be a curve in $\widetilde{\mathbf{SL}_2\mathbb{R}}$ and $\{T, N, B\}$ be the Frenet frame on $\widetilde{\mathbf{SL}_2\mathbb{R}}$ along ψ. A curve ψ such that*

$$\frac{\tau}{k} = \Lambda, \quad \Lambda \in \mathbb{R}, \tag{5.1}$$

is called a general helix with respect to Frenet frame.

Remark 5.1. i) If $k = 0$, then it indicates a straight line.

ii) If $k =$ constant $\neq 0$ and $\tau = 0$, then the curve ψ is a circle.

Theorem 5.1. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic III-harmonic general helix parametrized by arc length. If $N_3 = 0$, then ψ is a circular helix and $B_3 = constant$.*

Proof. From (5.1), we have

$$\begin{cases} \zeta_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(\Lambda^2+1) \\ \zeta_2(s) = (1+\Lambda^2)^2k^5 + k^{(4)} - 5(1+\Lambda^2)k(2kk'' + 3k'^2) \\ \zeta_3(s) = -\Lambda\xi_1(s) \\ \zeta_4(s) = k'' - k^3(\Lambda^2+2) \\ \zeta_5(s) = 3\Lambda k'k \\ \zeta_6(s) = \Lambda k^3. \end{cases} \tag{5.2}$$

By using equations (5.2) in (3.13), we can obtain a system of three differential equations characterizing III-harmonic general helix in $\widetilde{\mathbf{SL}_2\mathbb{R}}$

$$\begin{cases} \zeta_1(s) = 0 \\ \zeta_2(s) + \zeta_4(s) + 8B_3(-\zeta_4(s)B_3 + \zeta_5(s)N_3 + \zeta_6(s)T_3) = 0 \\ N_3(\zeta_4(s)B_3 - N_3\zeta_5(s) - \zeta_6(s)T_3) + \zeta_5(s) = 0. \end{cases} \tag{5.3}$$

Substituting $N_3 = 0$ into the third equation in (5.3) we have $\zeta_5(s) = 0$, which implies $k =$ constant and hence $\tau =$ constant. Then ψ is a circular helix. □

From (3.1) and (3.5), we have

$$g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T B,\ e_3) = -\tau N_3, \quad g_{\widetilde{\mathbf{SL}_2\mathbb{R}}}(\nabla_T B,\ e_3) = B_3' + T_2B_1 - T_1B_2.$$

Since $N_3 = T_2B_1 - T_1B_2 = 0$, it follows that $B_3 = constant$.
From the second equation of (5.3), we have

Corollary 5.2. *Let $\psi : I \to \widetilde{\mathbf{SL}_2\mathbb{R}}$ be a non-geodesic III-harmonic general helix parametrized by arc length. If $N_3 = 0$, then $T_3 = constant$.*

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References

[1] A. T. Ali, *Position vectors of general helices in Euclidean 3-space*, Bull. Math. Anal. Appl. **3** (2011), 1-8.

[2] M. Barros, *General helices and a theorem of Lancret*, Proc. Amer. Math. Soc. **125** (1997), 1503-1509.

[3] M. Barros, A. Ferrández, P. Lucas, M. A. Meroño, *General helices in the 3-dimensional Lorentzian space forms*, Rocky Mountain J. Math. **31** (2001), 373-388.

[4] V. Branding, *A structure theorem for polyharmonic maps between Riemannian manifolds*, arXiv preprint arXiv:1901.08445, 2019.

[5] R. Caddeo, C. Oniciuc, P. Piu, *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group*, Rend. Sem. Mat. Univ. Pol. Torino **62 (3)** (2004), 265-278.

[6] R. Caddeo, S. Montaldo, C. Oniciuc, P. Piu, *The classification of biharmonic curves of Cartan-Vranceanu 3-dimensional spaces*, Modern trends in geometry and topology, Cluj Univ. Press, Cluj-Napoca (2006), 121-131.

[7] B. Divjak, Z. Erjavec, B. Szabolcs, B. Szilágyi, *Geodesics and geodesic spheres in $\widetilde{SL(2,\mathbb{R})}$ geometry*, Math. Commun. **14** (2009), 413-424.

[8] J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics **50**, 1983.

[9] Z. Erjavec, D. Horvat, *Biharmonic curves in $\widetilde{SL(2,\mathbb{R})}$ space*, Math. Commun. **19** (2014), 291-299.

[10] Z. Erjavec, *Minimal surfaces in $\widetilde{SL(2,\mathbb{R})}$ geometry*, Glas. Mat. **50** (2015), 207-221.

[11] D. Fetcu, *Biharmonic curves in Cartan-Vranceanu (2n+1)-dimensional spaces*, Beitr. Algebra Geom. **46 (2)** (2007), 513-521.

[12] S. Maeta, *k-Harmonic maps into a Riemannian manifold with constant sectional curvature*, Proc. Amer. Math. Soc. **140** (2012), 1835-1847.

[13] S. Maeta, *The second variational formula of the k-energy and k-harmonic curves*, Osaka J. Math. **49** (2012), 1035-1063.

[14] E. Molnár, *The projective interpretation of the eight 3-dimensional homogeneous geometries*, Beitr. Algebra Geom. **38** (1997), 261-288.

[15] E. Molnár, J. Szirmai, *Symmetries in the* 8 *homogeneous* $3-$ *geometries*, Symmetry Cult. Sci. **21** (2010), 87-117.

[16] E. Molnár, J. Szirmai, A. Vesnin, *The optimal packings by translation balls in* $SL(2,\mathbb{R})$, J. Geom. **105** (2014), 287-306.

[17] S. Montaldo, A. Pampano, *Triharmonic curves in 3-dimensional homogeneous spaces*, Mediterr. J. Math. **18** (2021), Article number 198.

[18] S. Montaldo, C. Oniciuc, A. Ratto, *Polyharmonic surfaces in 3-dimensional homogeneous spaces*, Manuscripta Math. **174** (2024), 31-58.

[19] S. Montaldo A. Ratto, *New examples of r- harmonic immersions into the sphere*, J. Math. Anal. Appl. **458 (1)** (2018), 849-859.

[20] J. Monterde, *Curves with constant curvature ratios*, Bol. Soc. Mat. Mexicana **13** (2007), 177-186.

[21] B. Senoussi, *Triharmonic curves in Heisenberg group* $\mathcal{H}_3$, J. Dyn. Syst. Geom. Theor. **20** (2022), 55-65.

[22] B. Senoussi, *Triharmonic curves in* SOL_3 *space*, Palestine J. Math. **11**, (2022), 264-273.

[23] B. Senoussi, *Triharmonic curves in Bianchi-Cartan-Vranceanu spaces*, Rendiconti Sem. Mat. Univ. Pol. Torino **80 (2)** (2022), 5-13.

[24] B. Senoussi, M. Bekkar, *Characterization of general helix in the 3- dimensional Lorentz-Heisenberg space*, Int. Electron. J. Geom. **6 (1)** (2013), 46-55.

[25] B. Senoussi, M. Bekkar, *Some characterization of curves in* $\widetilde{\mathbf{SL}_2\mathbb{R}}$ *space*, Sib. Elektron. Mat. Izv. **16** (2019), 902-915.

[26] D. J. Struik, *Lectures on classical differential geometry*, New York Dover Publications. Inc., 1988.

[27] S. B. Wang, *The first variation formula for k-harmonic mapping*, J. Nanchang Univ. **13** (1989).

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