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From Pikovsky-Rabinovich Dynamical System to Lagrange-Hamilton Geometrical Objects

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Abstract The aim of this paper is to develop, via the least squares variational method, the Lagrange-Hamilton geometries (in the sense of nonlinear connections, d-torsions and Lagrangian Yang-Mills electromagnetic-like energy) produced by Pikovsky-Rabinovich dynamical system from cryptography. From a geometrical point of view, the Jacobi stability of the Pikovsky-Rabinovich system is discussed.

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1 Pikovsky-Rabinovich dynamical system in cryptography

In cryptography, a useful differential system for transmitting information using chaotic signals was introduced in [10] by A. S. Pikovsky and M. I. Rabinovich:

$$\begin{cases} \dfrac{dx^1}{dt} = -\nu_1 x^1 + hx^2 + x^2x^3 \\ \dfrac{dx^2}{dt} = hx^1 - \nu_2 x^2 - x^1x^3 \\ \dfrac{dx^3}{dt} = -\nu_3 x^3 + x^1x^2, \end{cases} \tag{1.1}$$

where ν_1, ν_2, ν_3 and h are some parameters. This Pikovsky-Rabinovich system is closely associated with the Lorenz chaotic system from meteorology (see Tong et al. [11]). From a geometrical point of view, a particular form of the Pikovsky-Rabinovich DEs system (1.1) was studied, via Hamilton-Poisson structures, by Chiş and Puta in [4].

2 From Pikovsky-Rabinovich system to Lagrange geometry

Obviously, in the context of Pikovsky-Rabinovich DEs system (1.1), we could consider that we work on the particular 3-dimensional manifold $M = \mathbb{R}^3$, whose coordinates are (x^1, x^2, x^3), and the corresponding tangent TM and cotangent T^*M bundles have the coordinates $(x^i, y^i)_{i=\overline{1,3}}$, respectively $(x^i, p_i)_{i=\overline{1,3}}$.

Moreover, if we take the vector field $X = (X^i(x^1, x^2, x^3))_{i=\overline{1,3}}$ on $M = \mathbb{R}^3$, which is given by

$$\begin{aligned} X^1(x^1, x^2, x^3) &= -\nu_1 x^1 + hx^2 + x^2x^3, \\ X^2(x^1, x^2, x^3) &= hx^1 - \nu_2 x^2 - x^1x^3, \\ X^3(x^1, x^2, x^3) &= -\nu_3 x^3 + x^1x^2, \end{aligned} \tag{2.1}$$

we can regard the Pikovsky-Rabinovich system (1.1) as the dynamical system

$$\frac{dx^i}{dt} = X^i(x(t)), \quad i = \overline{1,3}. \tag{2.2}$$

It is clear that the solutions of class C^2 of the dynamical system (2.2) are the global minimum points for the *least squares Lagrangian* (Udrişte's terminology) $L : TM \to \mathbb{R}$, which is expressed by[1]

$$\begin{gathered} L(x, y) = \delta_{ij}\left(y^i - X^i(x)\right)\left(y^j - X^j(x)\right) \Leftrightarrow \\ L(x, y) = \left(y^1 - X^1(x)\right)^2 + \left(y^2 - X^2(x)\right)^2 + \left(y^3 - X^3(x)\right)^2. \end{gathered} \tag{2.3}$$

The Euler-Lagrange equations of the least squares Lagrangian (2.3) are described by

$$\frac{\partial L}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^k}\right) = 0, \quad y^k = \frac{dx^k}{dt}, \quad k = \overline{1,3}.$$

These Euler-Lagrange equations can be rewritten in the following geometrical form:

$$\frac{d^2x^k}{dt^2} + 2G^k(x, y) = 0, \tag{2.4}$$

where

$$G^k = \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^j \partial y^k} y^j - \frac{\partial L}{\partial x^k}\right) = -\frac{1}{2}\left[\left(\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i}\right) y^j + \frac{\partial X^j}{\partial x^i} X^j\right]$$

is endowed with the geometrical meaning of *semispray* of L.

In the sequel, following the geometrical ideas exposed in the works Miron and Anastasiei [5], Udrişte and Neagu [12], [9], and Balan and Neagu [1], we can construct an entire natural collection of nonzero Lagrangian geometrical objects (such as nonlinear connection, d-torsions and Yang-Mills electromagnetic-like energy) that characterize the initial dynamical system (2.2) and implicitly (1.1).

Firstly, let us compute the Jacobian matrix of X:

$$J = \left(\frac{\partial X^i}{\partial x^j}\right)_{i,j=\overline{1,3}} = \begin{pmatrix} -\nu_1 & h + x^3 & x^2 \\ h - x^3 & -\nu_2 & -x^1 \\ x^2 & x^1 & -\nu_3 \end{pmatrix}.$$

[1]The Latin indices $i, j, k, \ldots$ run from 1 to 3, and the Einstein convention of summation is adopted all over this paper.

Proposition 2.1. *The* ***Lagrangian canonical nonlinear connection*** *on the tangent bundle TM, produced by the Pikovsky-Rabinovich system (1.1), has the form*

$$N = \left(N^i_j\right)_{i,j=\overline{1,3}} = \begin{pmatrix} 0 & -x^3 & 0 \\ x^3 & 0 & x^1 \\ 0 & -x^1 & 0 \end{pmatrix}.$$

Proof. The entries of the canonical nonlinear connection are given by the formulas

$$N^i_j = \frac{\partial G^i}{\partial y^j},$$

see [5]. This fact implies the matrix relation (see [1])

$$N = -\frac{1}{2}\left[J - J^t\right].$$

□

The canonical Cartan linear connection on TM produced by the least squares Lagrangian (2.3) has all adapted components equal to zero. Further, we infer

Proposition 2.2. *The Lagrangian canonical Cartan linear connection, produced by the Pikovsky-Rabinovich system (1.1), is characterized by the* ***d-torsion skew-symmetric matrices****:*

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_2 = 0, \quad R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The general formulas which give the Lagrangian d-torsions are (see [5])

$$R_k = \left(R^i_{jk} := \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}\right)_{i,j=\overline{1,3}},$$

where

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^r_k \frac{\partial}{\partial y^r}.$$

By direct computations, it follows that we have (see [1])

$$R_k = \frac{\partial N}{\partial x^k}, \quad \forall\, k = \overline{1,3}.$$

□

Proposition 2.3. *The* ***Lagrangian Yang-Mills electromagnetic-like energy*** *produced by the Pikovsky-Rabinovich system (1.1) is given by*

$$\mathcal{EYM}(x) = \frac{1}{2} \cdot Trace\left[F \cdot F^t\right] = \left(x^1\right)^2 + \left(x^3\right)^2,$$

where the electromagnetic-like matrix is

$$F = -N = \begin{pmatrix} 0 & x^3 & 0 \\ -x^3 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix}.$$

For more details, see [5] and [1].

Open problem. In our context, from this new geometric-physical approach, the surfaces of constant level of the Lagrangian Yang-Mills electromagnetic-like energy produced by the Pikovsky-Rabinovich system (1.1) could have important connotations for the phenomena from cryptography taken in study. The shape of the surface of constant level

$$\Sigma_\rho : \left(x^1\right)^2 + \left(x^3\right)^2 = \rho > 0$$

is represented by a circular cylinder of radius $\sqrt{\rho}$, having as axis of symmetry the axis Ox^2. Is there a meaning of this fact, related to transmitting information using chaotic signals?

Theorem 2.4. *If the Lagrangian Yang-Mills electromagnetic-like energy is zero, it follows that we have:*

1. *If* $h \neq 0$, *then* $x^1 = x^2 = x^3 = 0$;
2. *If* $h = 0$, *then* $x^1 = x^3 = 0$ *and* $x^2 = Ce^{-\nu_2 t}$, *where* $C > 0$.

Proof. If we have $\mathcal{EYM}(x) = (x^1)^2 + (x^3)^2 = 0$, we infer that the Pikovsky-Rabinovich system (1.1) reduces to

$$\begin{cases} 0 = hx^2 \\ \dfrac{dx^2}{dt} = -\nu_2 x^2. \end{cases}$$

□

Remark 2.1. The *matrix of deviation curvature* from the Kosambi-Cartan-Chern (KCC) geometrical theory is given by the formula

$$P = \left(P^i_j\right)_{i,j=\overline{1,3}} = R_k y^k + \mathcal{E},$$

where, if

$$\mathcal{E}^i = 2G^i - N^i_j y^j = -\frac{1}{2}\left(\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i}\right) y^j - \frac{\partial X^j}{\partial x^i} X^j$$

is the *first invariant of the semispray* of the Lagrangian (2.3), then we have

$$\mathcal{E} = \left(\frac{\delta \mathcal{E}^i}{\delta x^j}\right)_{i,j=\overline{1,3}}.$$

By direct computations, the entries of the matrix $\mathcal{E}$ are given by

$$\frac{\delta \mathcal{E}^i}{\delta x^j} = -\frac{1}{2}\left(\frac{\partial^2 X^i}{\partial x^j \partial x^k} - \frac{\partial^2 X^k}{\partial x^i \partial x^j}\right) y^k - \frac{\partial^2 X^k}{\partial x^i \partial x^j} X^k - \frac{\partial X^k}{\partial x^i}\frac{\partial X^k}{\partial x^j} -$$
$$-\frac{1}{4}\left(\frac{\partial X^k}{\partial x^j} - \frac{\partial X^j}{\partial x^k}\right)\left(\frac{\partial X^i}{\partial x^k} - \frac{\partial X^k}{\partial x^i}\right).$$

The behavior of the neighboring solutions of the Euler-Lagrange equations (2.4) is *Jacobi stable* if and only if the real parts of the eigenvalues of the deviation tensor P are strictly negative everywhere, and *Jacobi unstable*, otherwise. The Jacobi stability or instability has the geometrical meaning that the trajectories of the Euler-Lagrange equations (2.4) are bunching together or are dispersing (or even are chaotic). For more details about KCC theory and Jacobi stability, consult the works Böhmer et al. [2], Bucătaru-Miron [3] and Neagu-Ovsiyuk [8].

3 From Pikovsky-Rabinovich system to Hamilton geometry

Let us construct the *least squares Hamiltonian* $H : T^*M \to \mathbb{R}$, associated with the Lagrangian (2.3), whose expression is

$$\begin{gathered} H(x,p) = \frac{\delta^{ij}}{4} p_i p_j + X^k(x) p_k \Leftrightarrow \\ H(x,p) = \frac{1}{4}\left(p_1^2 + p_2^2 + p_3^2\right) + X^1(x)p_1 + X^2(x)p_2 + X^3(x)p_3, \end{gathered} \tag{3.1}$$

where $p_r = \partial L/\partial y^r$ and $H = p_r y^r - L$. Now, we can develop again a natural and distinct collection of nonzero Hamiltonian geometrical objects (such as nonlinear connection and d-torsions), which also characterize the Pikovsky-Rabinovich system (1.1). For all details about Hamilton geometry on cotangent bundles and the Hamiltonian least squares variational method for dynamical systems, see the monographs Miron et al. [6] and Neagu-Oană [7].

Proposition 3.1. *The* ***Hamiltonian canonical nonlinear connection*** *on the cotangent bundle T^*M produced by the Pikovsky-Rabinovich system (1.1) has the form*

$$\mathbf{N} = (N_{ij})_{i,j=\overline{1,3}} = 2\begin{pmatrix} -\nu_1 & h & x^2 \\ h & -\nu_2 & 0 \\ x^2 & 0 & -\nu_3 \end{pmatrix}.$$

Proof. The Hamiltonian nonlinear connection on the cotangent bundle T^*M, has the components (see [6])

$$N_{ij} = \frac{\partial^2 H}{\partial x^j \partial p_i} + \frac{\partial^2 H}{\partial x^i \partial p_j}.$$

By direct computations, we obtain

$$\mathbf{N} = J + J^t.$$

For more details, see [7]. □

The canonical Cartan linear connection on T^*M produced by the least squares Hamiltonian (3.1) has all adapted components equal to zero. Further, we obtain

Proposition 3.2. *The Hamiltonian canonical Cartan linear connection produced by the Pikovsky-Rabinovich system (1.1) is characterized by constant* ***d-torsion skew-symmetric matrices****:*

$$\mathbf{R}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{R}_2 = 0, \quad \mathbf{R}_3 = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The general formulas which give the Hamiltonian d-torsions are (see [6])

$$\mathbf{R}_k = \left(R_{kij} := \frac{\delta N_{ki}}{\delta x^j} - \frac{\delta N_{kj}}{\delta x^i} \right)_{i,j=\overline{1,3}},$$

where

$$\frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N_{rj}\frac{\partial}{\partial p_r}.$$

By direct computations, it follows that we have (see [7])

$$\mathbf{R}_k = \frac{\partial}{\partial x^k}\left[J - J^t\right] = -2R_k, \quad \forall\ k = \overline{1,3}.$$

□

Open problem. What is the real meaning in cryptography and transmitting information using chaotic signals for our Lagrange-Hamilton geometrical objects constructed in this paper?

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