sciendo
DOI: 10.2478/awutm-2024-0005

Annals of West University of Timisoara
Mathematics and Computer Science
60 (2024), 73–84

Copson-type Inequalities via the k-Hadamard Operator

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Abstract In this paper, we present a novel class of Copson-type inequalities involving the right-sided k-Hadamard fractional integral operator with all parameters of integrability $p \neq 0$ and obtain some classical special cases of Copson inequalities. The main results will be proved by employing the Hölder inequality and the Fubini theorem.

AMS Subject Classification (2020) 26D10; 26A33; 26D15
Keywords Copson-type inequality; k-Hadamard operator; Hölder inequality

1 Introduction

In literature, the following inequality known by the Copson integral inequality has been proven in [2, Theorem C], for $p > 1$ and a non-negative continuous function f on $(0, \infty)$

$$\int_0^\infty \left(\int_x^\infty f(t) \frac{dt}{t} \right)^p dx \leq p^p \int_0^\infty f^p(x) dx, \tag{1.1}$$

where the operator

$$Cf(x) = \int_x^\infty f(t) \frac{dt}{t},$$

represents the Copson operator, also known as the Hardy adjoint operator.
In 1976, E.T.Copson proved the following inequalities [3] for $c \neq 1$, $0 < b \leq \infty$ and a non-negative continuous function ϕ on $(0, \infty)$

$$\int_0^b F^p(x) \Phi^{-c}(x) \phi(x) dx \leq \left(\frac{p}{|c-1|} \right)^p \int_0^b f^p(x) \Phi^{p-c}(x) \phi(x) dx, \quad \text{for } p \geq 1, \tag{1.2}$$

and

$$\int_0^b F^p(x)\Phi^{-c}(x)\phi(x)dx \geq \left(\frac{p}{|c-1|}\right)^p \int_0^b f^p(x)\Phi^{p-c}(x)\phi(x)dx, \quad \text{for } 0 < p \leq 1. \quad (1.3)$$

where

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \begin{cases} \displaystyle\int_0^x f(t)\phi(t)dt & \text{for } c > 1, \\ \displaystyle\int_x^\infty f(t)\phi(t)dt & \text{for } c < 1. \end{cases}$$

In [1] the following Copson inequality for $p < 0$ and $\alpha \neq p-1$ has been proved:

$$\int_0^b F^p(x)\Phi^{\alpha-p}(x)\phi(x)dx \leq \left(\frac{p}{|p-1-\alpha|}\right)^p \int_0^b f^p(x)\Phi^{\alpha}(x)\phi(x)dx, \quad (1.4)$$

where

$$\Phi(x) = \int_0^x \phi(t)dt, \quad F(x) = \begin{cases} \displaystyle\int_0^x f(t)\phi(t)dt & \text{for } \alpha > p-1, \\ \displaystyle\int_x^\infty f(t)\phi(t)dt & \text{for } \alpha < p-1. \end{cases}$$

During the 20th century, Copson-type inequality (1.1) has been the subject of particular attention by various authors, which has led to important lines of study. In recent decades, the study of dynamic inequalities over time scales has become a prominent area in applied and pure mathematics. While the authors have presented some generalizations of the Hardy-Copson integral inequalities that involve dynamic inequalities on time scales, the majority of research findings focus primarily on Hardy-type and Copson-type dynamic inequalities on time scales [4,5,8–10,13,14,16].

On the other hand, fractional inequalities have become an active field of research, with several authors interested in proving Copson inequalities of fractional type using the conformable calculus on time scales [15]. In particular, Saker et al. [12] proved some Copson and converse Copson-type inequalities via conformable calculus.

Motivated by the literature above, in this study we provide a novel version of the Copson-type inequalities in fractional calculus employing the right-sided k-Hadamard fractional integral operator on the interval $[a,b] \subset (0,\infty)$ defined as follows:

$$H_{b^-}^{\alpha,k}(f)(x) = \frac{1}{k\,\Gamma_k(\alpha)} \int_x^b \left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f(t)\,\frac{dt}{t}, \quad a \leq x < b, \quad (1.5)$$

where f is an integrable function on (a,b) and the k-gamma function is defined for all $\alpha,\ k > 0$ by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt\ ,$$

and verifies

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha).$$

For $k = 1$, we get the right sided Hadamard fractional integral operator

$$H_{b^-}^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{t}{x}\right)^{\alpha-1} f(t) \, \frac{dt}{t}, \quad a \leq x < b. \tag{1.6}$$

Putting $\alpha = 1$ in (1.6), we get the Copson operator Cf as follows:

$$Cf\,(x) = \int_x^b f(t)\frac{dt}{t}, \quad a \leq x < b. \tag{1.7}$$

The Hölder inequality is one of the most well-known integral inequalities. The following statement was proved.

Theorem 1.1. *Let $[a, b] \subset (0, \infty)$, $p > 1$ and f, g be measurable functions on $[a, b]$. If $f \in L_p([a, b])$ and $g \in L_q([a, b])$ where $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(t)g(t)|dt \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}. \tag{1.8}$$

The inequality (1.8) is reversed for $0 < p < 1$ and $p < 0$.
For $p = 1$, we have

$$\int_a^b |f(t)g(t)|dt \leq \|g\|_\infty \cdot \left(\int_a^b |f(t)|dt\right), \tag{1.9}$$

where

$$\|g\|_\infty = ess \sup_{t\in[a,b]} |g(t)|\,.$$

Fore more details see [6, Theorem 189], [7] and [11].

2 Main results

We will use Hölder inequality, Fubini theorem, and the monotonicity of the function ϕ to demonstrate the main results. The functions f and ϕ used in this study are assumed to be positive measurable on the interval $[a, b] \subset (0, \infty)$. We denote by Ψ the function that is defined as $\Psi(x) = \log \frac{b}{x}$, $a \leq x < b$.

Theorem 2.1. *Let $p > 1$ and k, $\alpha > 0$ such that $\frac{\alpha}{k} \geq 1$.*

1. *If ϕ is a non-increasing function, then*

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\, dx \leq \frac{k\, b\, \phi(a)}{(\alpha(p-1)+k)\, \Gamma_k^{p-1}(\alpha+k)} \times \left[(\Psi(a))^{\frac{\alpha}{k}(p-1)+1} H_{b^-}^{\alpha,k}(f^p)(a) - H_{b^-}^{\alpha,k}\left(f^p \cdot \Psi^{\frac{\alpha}{k}(p-1)+1}\right)(a)\right]. \tag{2.1}$$

2. If ϕ is a non-decreasing function, then

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{k\,b}{(\alpha(p-1)+k)\,\Gamma_k^{p-1}(\alpha+k)} \times \left[(\Psi(a))^{\frac{\alpha}{k}(p-1)+1} H_{b^-}^{\alpha,k}\left(f^p\,\phi\right)(a) - H_{b^-}^{\alpha,k}\left(f^p\cdot\phi\cdot\Psi^{\frac{\alpha}{k}(p-1)+1}\right)(a)\right]. \tag{2.2}$$

Proof. Using the Hölder inequality for $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}
&\int_a^b (H_{b^-}^{\alpha,k}(f)(x))^p\,\phi(x)\,dx\\
&= \int_a^b \phi(x)\left[\int_x^b \left[\frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{1}{t}\right]^{\frac{1}{p}} f(t)\left[\frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{1}{t}\right]^{\frac{1}{q}} dt\right]^p dx\\
&\le \int_a^b \phi(x)\left[\frac{1}{k\,\Gamma_k(\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f^p(t)\frac{dt}{t}\right]\left[\frac{1}{k\,\Gamma_k(\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{dt}{t}\right]^{p-1} dx\\
&= \frac{1}{k\,\Gamma_k(\alpha)}\int_a^b \phi(x)\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f^p(t)\frac{dt}{t}\left[\frac{1}{\alpha\,\Gamma_k(\alpha)}\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}}\right]^{p-1} dx\\
&= \frac{1}{k\,\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)}\int_a^b \phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{f^p(t)}{t}\,dt\,dx.
\end{aligned}$$

Using Fubini's theorem, we get

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{1}{k\,\Gamma_k(\alpha)\Gamma_k^{p-1}(\alpha+k)}\int_a^b\frac{f^p(t)}{t}\int_a^t\phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}dx\,dt,$$

and since ϕ and $\log(\frac{t}{\bullet})$ are non-increasing functions, we obtain

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{\phi(a)}{k\,\Gamma_k(\alpha)\,\Gamma_k^{p-1}(\alpha+k)} \times \int_a^b\left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\left(\int_a^t\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}dx\right)\frac{dt}{t}. \tag{2.3}$$

Noting that

$$\int_a^t \left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx \le \int_a^t \frac{b}{x}\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx$$

$$= \frac{k\,b}{\alpha(p-1)+k}\left(\left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}(p-1)+1} - \left(\log\frac{b}{t}\right)^{\frac{\alpha}{k}(p-1)+1}\right),$$

replacing in the inequality (2.3), we get

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{k\,b\,\phi(a)}{(\alpha(p-1)+k)\,\Gamma_k^{p-1}(\alpha+k)\,k\,\Gamma_k(\alpha)}$$

$$\times\left[\left(\log\tfrac{b}{a}\right)^{\frac{\alpha}{k}(p-1)+1}\int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\,\frac{dt}{t} - \int_a^b\left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\left(\log\frac{b}{t}\right)^{\frac{\alpha}{k}(p-1)+1}\frac{dt}{t}\right],$$

this yields the inequality (2.1).

The proof of the inequality (2.2) is the same as the proof of the inequality (2.1). □

Theorem 2.2. *Let f, $\phi > 0$ be positive measurable functions on $[a,b] \subset (0,\infty)$, $0 < p < 1$ and k, $\alpha > 0$ and such that $0 < \frac{\alpha}{k} \le 1$.*

1. *If ϕ is a non-increasing function, then*

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \ge \frac{k\,a}{(k-\alpha(1-p))\,\Gamma_k^{p-1}(\alpha+k)}$$
$$\times\left[(\Psi(a))^{1-\frac{\alpha}{k}(1-p)}\,H_{b^-}^{\alpha,k}\left(f^p\cdot\phi\right)(a) - H_{b^-}^{\alpha,k}\left(f^p\cdot\phi\cdot\Psi^{1-\frac{\alpha}{k}(1-p)}\right)(a)\right]. \tag{2.4}$$

2. *If ϕ is a non-decreasing function, then*

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \ge \frac{k\,a\,\phi(a)}{(k-\alpha(1-p))\,\Gamma_k^{p-1}(\alpha+k)}$$
$$\times\left[(\Psi(a))^{1-\frac{\alpha}{k}(1-p)}\,H_{b^-}^{\alpha,k}\left(f^p\right)(a) - H_{b^-}^{\alpha,k}\left(f^p\cdot\Psi^{1-\frac{\alpha}{k}(1-p)}\right)(a)\right]. \tag{2.5}$$

Proof. The Hölder inequality for $0 < p < 1$ yields

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \ge$$

$$\frac{1}{k\,\Gamma_k(\alpha)\,\Gamma_k^{p-1}(\alpha+k)}\int_a^b \phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{f^p(t)}{t}\,dt\,dx,$$

since

$$\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} \ge \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1},$$

then

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \geq$$
$$\frac{1}{k\,\Gamma_k(\alpha)\,\Gamma_k^{p-1}(\alpha+k)} \int_a^b \phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} \int_x^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} \frac{f^p(t)}{t}\,dt\,dx,$$

using the Fubini theorem, we get

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \geq$$
$$\frac{1}{k\,\Gamma_k(\alpha)\,\Gamma_k^{p-1}(\alpha+k)} \int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} \frac{f^p(t)}{t} \int_a^t \phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx\,dt,$$

given that the function ϕ is non-increasing, we deduce

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \geq$$
$$\frac{1}{k\,\Gamma_k(\alpha)\,\Gamma_k^{p-1}(\alpha+k)} \int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} \frac{f^p(t)}{t}\phi(t) \int_a^t \left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx\,dt. \tag{2.6}$$

We notice that

$$\begin{aligned}\int_a^t \left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx &\geq \int_a^t \left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} \frac{a}{x}\,dx \\ &= \frac{k\ a}{k-\alpha(1-p)}\left[\left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}(p-1)+1} - \left(\log\frac{b}{t}\right)^{\frac{\alpha}{k}(p-1)+1}\right],\end{aligned}$$

replacing in the inequality (2.6), we obtain

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \geq \frac{k\,a}{(k-\alpha(1-p))\,k\,\Gamma_k(\alpha)\ \Gamma_k^{p-1}(\alpha+k)}$$
$$\times\left[\left(\log\tfrac{b}{a}\right)^{1-\frac{\alpha}{k}(1-p)} \int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\phi(t)\frac{dt}{t} - \int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\phi(t)\left(\log\frac{b}{t}\right)^{1-\frac{\alpha}{k}(1-p)}\frac{dt}{t}\right],$$

which is the desired inequality (2.4). The proof of the inequality (2.5) is similar to the proof of the inequality (2.4).

□

Theorem 2.3. *Let f, $\phi > 0$ be positive measurable functions on $[a,b] \subset (0,\infty)$, $p < 0$ and k, $\alpha > 0$, such that $\frac{\alpha}{k} \geq 1$.*

1. If ϕ is a non-increasing function, then

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{k\,b\,\phi(a)\,\Gamma_k^{1-p}(\alpha+k)}{\alpha(1-p)-k}$$
$$\times\left[H_{b^-}^{\alpha,k}\left(f^p\cdot\Psi^{\frac{\alpha}{k}(p-1)+1}\right)(a) - (\Psi(a))^{\frac{\alpha}{k}(p-1)+1} H_{b^-}^{\alpha,k}(f^p)(a)\right]. \tag{2.7}$$

2. If ϕ is a non-decreasing function, then

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{k\,b\,\Gamma_k^{1-p}(\alpha+k)}{\alpha(1-p)-k}$$
$$\times\left[H_{b^-}^{\alpha,k}\left(f^p\cdot\phi\cdot\Psi^{\frac{\alpha}{k}(p-1)+1}\right)(a) - (\Psi(a))^{\frac{\alpha}{k}(p-1)+1} H_{b^-}^{\alpha,k}(f^p\cdot\phi)(a)\right]. \tag{2.8}$$

Proof. Utilizing the Hölder inequality for $p<0$ yields

$$\begin{aligned}H_{b^-}^{\alpha,k}(f)(x) &= \int_x^b \left(\frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{1}{t}\right)^{\frac{1}{p}} f(t)\left(\frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{1}{t}\right)^{\frac{1}{q}}dt\\ &\ge \left(\int_x^b \frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f^p(t)\frac{dt}{t}\right)^{\frac{1}{p}}\left(\int_x^b \frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{dt}{t}\right)^{\frac{1}{q}},\end{aligned}$$

therefore

$$\begin{aligned}&\int_a^b \left(H_{b^-}^{\alpha,k}f(x)\right)^p \phi(x)dx\\ &\le \int_a^b \phi(x)\left(\int_x^b \frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1} f^p(t)\frac{dt}{t}\right)\left(\int_x^b \frac{1}{k\,\Gamma_k(\alpha)}\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{dt}{t}\right)^{p-1}dx\\ &= \frac{\Gamma_k^{1-p}(\alpha+k)}{k\,\Gamma_k(\alpha)}\int_a^b \phi(x)\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}\frac{f^p(t)}{t}\,dt\,dx.\end{aligned}$$

Using the Fubini theorem and the fact that ϕ and $\log(\frac{t}{\bullet})$ are non-increasing functions, as in (2.3), we obtain

$$\int_a^b \left(H_{b^-}^{\alpha,k}f(x)\right)^p \phi(x)dx \le$$
$$\frac{\Gamma_k^{1-p}(\alpha+k)\,\phi(a)}{k\,\Gamma_k(\alpha)}\int_a^b\left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\left(\int_a^t\left(\log\frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)}dx\right)\frac{dt}{t}. \tag{2.9}$$

As $\frac{\alpha}{k}(p-1) < -1$ and

$$\int_a^t \left(\log \frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx \le \int_a^t \frac{b}{x}\left(\log \frac{b}{x}\right)^{\frac{\alpha}{k}(p-1)} dx$$
$$= \frac{k\,b}{\alpha(1-p)-k}\left(\left(\log \frac{b}{t}\right)^{\frac{\alpha}{k}(p-1)+1} - \left(\log \frac{b}{a}\right)^{\frac{\alpha}{k}(p-1)+1}\right),$$

by applying the inequality (2.9), we obtain

$$\int_a^b \left(H_{b^-}^{\alpha,k}(f)(x)\right)^p \phi(x)\,dx \le \frac{k\,b\,\phi(a)\,\Gamma_k^{1-p}(\alpha+k)}{(\alpha(1-p)-k)\;k\,\Gamma_k(\alpha)}$$
$$\times\left[\int_a^b \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\left(\log\frac{b}{t}\right)^{\frac{\alpha}{k}(p-1)+1}\frac{dt}{t} - \left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}(p-1)+1}\int_a^b\left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} f^p(t)\,\frac{dt}{t}\right],$$

this is the needed inequality (2.7).
The proof of the inequality (2.8) is the same as the proof of the inequality (2.7).

□

3 Applications

Putting $k = 1$ in the above Theorems 2.1, 2.2 and 2.3 yields the following results related to the Hadamard operator.

Corollary 3.1. *Let $\alpha > 0$ and $f, \phi > 0$ be positive measurable functions on $[a, b] \subset (0, \infty)$.*

- *For all $p > 1$ and $\alpha \ge 1$, if ϕ is a non-increasing function, we get*

$$\int_a^b (H_{b^-}^{\alpha}(f)(x))^p\,\phi(x)\,dx \le \frac{b\,\phi(a)}{(\alpha(p-1)+1)\,\Gamma^{p-1}(\alpha+1)}$$
$$\times\left[(\Psi(a))^{\alpha(p-1)+1}\;H_{b^-}^{\alpha}(f^p)(a) - H_{b^-}^{\alpha}\left(f^p\cdot\Psi^{\alpha(p-1)+1}\right)(a)\right],$$

and if ϕ is a non-decreasing, we have

$$\int_a^b (H_{b^-}^{\alpha}(f)(x))^p\,\phi(x)\,dx \le \frac{b}{(\alpha(p-1)+1)\,\Gamma^{p-1}(\alpha+1)}$$
$$\times\left[(\Psi(a))^{\alpha(p-1)+1}\;H_{b^-}^{\alpha}(f^p\cdot\phi)(a) - H_{b^-}^{\alpha}\left(f^p\cdot\phi\cdot\Psi^{\alpha(p-1)+1}\right)(a)\right].$$

- *For all* $0 < p < 1$ *and* $0 < \alpha \leq 1$, *if* ϕ *is a non-increasing, we get*

$$\int_a^b \left(H_{b^-}^{\alpha}(f)(x)\right)^p \phi(x)\,dx \geq \frac{a}{(\alpha(p-1)+1)\,\Gamma^{p-1}(\alpha+1)}$$

$$\times\left[(\Psi(a))^{\alpha(p-1)+1} H_{b^-}^{\alpha}\left(f^p\cdot\phi\right)(a) - H_{b^-}^{\alpha}\left(f^p\cdot\phi\cdot\Psi^{\alpha(p-1)+1}\right)(a)\right],$$

and if ϕ *is a non-decreasing, we have*

$$\int_a^b \left(H_{b^-}^{\alpha}(f)(x)\right)^p \phi(x)\,dx \geq \frac{a\,\phi(a)}{(\alpha(p-1)+1)\,\Gamma^{p-1}(\alpha+1)}$$

$$\times\left[(\Psi(a))^{\alpha(p-1)+1} H_{b^-}^{\alpha}\left(f^p\right)(a) - H_{b^-}^{\alpha}\left(f^p\cdot\Psi^{\alpha(p-1)+1}\right)(a)\right].$$

- *For all* $p < 0$ *and* $\alpha \geq 1$, *if* ϕ *is a non-increasing, we get*

$$\int_a^b \left(H_{b^-}^{\alpha}(f)(x)\right)^p \phi(x)\,dx \leq \frac{b\,\phi(a)\,\Gamma^{1-p}(\alpha+1)}{\alpha(1-p)-1}$$

$$\times\left[H_{b^-}^{\alpha}\left(f^p\cdot\Psi^{\alpha(p-1)+1}\right)(a) - (\Psi(a))^{\alpha(p-1)+1} H_{b^-}^{\alpha}\left(f^p\right)(a)\right],$$

and if ϕ *is a non-decreasing, we have*

$$\int_a^b \left(H_{b^-}^{\alpha}(f)(x)\right)^p \phi(x)\,dx \leq \frac{b\,\Gamma^{1-p}(\alpha+1)}{\alpha(1-p)-1}$$

$$\times\left[H_{b^-}^{\alpha}\left(f^p\cdot\phi\cdot\Psi^{\alpha(p-1)+1}\right)(a) - (\Psi(a))^{\alpha(p-1)+1} H_{b^-}^{\alpha}\left(f^p\cdot\phi\right)(a)\right].$$

Putting $\alpha = 1$ in the previous Corollary 3.1, we get the following Copson-type inequalities.

Corollary 3.2. *Let* f, $\phi > 0$ *be positive measurable functions on the* $[a,b] \subset (0,\infty)$ *and*

$$Cf(x) = \int_x^b f(t)\frac{dt}{t}.$$

- *For* $p > 1$, *if* ϕ *is a non-increasing, we have*

$$p\int_a^b (Cf(x))^p \phi(x)\,dx \leq b\,\phi(a)\left[\left(\log\frac{b}{a}\right)^p \int_a^b f^p(t)\,\frac{dt}{t} - \int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p \frac{dt}{t}\right],$$

if ϕ *is a non-decreasing, we get*

$$p\int_a^b (Cf(x))^p \phi(x)\,dx \leq b\left[\left(\log\frac{b}{a}\right)^p \int_a^b f^p(t)\,\phi(t)\,\frac{dt}{t} - \int_a^b f^p(t)\,\phi(t)\left(\log\frac{b}{t}\right)^p \frac{dt}{t}\right].$$

- *For $0 < p < 1$, if ϕ is a non-increasing, we have*
$$p\int_a^b (Cf(x))^p\,\phi(x)\,dx \geq a\left[\left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\phi(t)\,\frac{dt}{t} - \int_a^b f^p(t)\phi(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right],$$
if ϕ is a non-decreasing, we get
$$p\int_a^b (Cf(x))^p\,\phi(x)\,dx \geq a\,\phi(a)\left[\left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\frac{dt}{t} - \int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right].$$
- *For $p < 0$, if ϕ is a non-increasing, we have*
$$-p\int_a^b (Cf(x))^p\,\phi(x)\,dx \leq b\,\phi(a)\left[\int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t} - \left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\frac{dt}{t}\right],$$
if ϕ is a non-decreasing, we get
$$-p\int_a^b (Cf(x))^p\,\phi(x)\,dx \leq b\left[\int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\phi(t)\,\frac{dt}{t} - \left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\phi(t)\,\frac{dt}{t}\right].$$

The following results are particular cases of corollary 3.2, for a non-increasing function $\phi_1(x) = \dfrac{1}{x^q}$ and a non-decreasing function $\phi_2(x) = x^q$, where $q > 0$.

Corollary 3.3. *Let $f > 0$ be positive measurable function on the $[a, b] \subset (0, \infty)$ and*
$$Cf(x) = \int_x^b f(t)\frac{dt}{t}.$$

- *For $p > 1$, we have*
$$p\int_a^b \frac{(Cf(x))^p}{x^q}\,dx \leq \frac{b}{a^q}\left[\left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\frac{dt}{t} - \int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right],$$
and
$$p\int_a^b (Cf(x))^p\,x^q\,dx \leq b\left[\left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,t^q\,\frac{dt}{t} - \int_a^b f^p(t)\,t^q\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right].$$
- *For $0 < p < 1$, we have*
$$p\int_a^b \frac{(Cf(x))^p}{x^q}\,dx \geq a\left[\left(\log\frac{b}{a}\right)^p\int_a^b \frac{f^p(t)}{t^q}\,\frac{dt}{t} - \int_a^b \frac{f^p(t)}{t^q}\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right],$$
and
$$p\int_a^b (Cf(x))^p\,x^q\,dx \geq a^{q+1}\left[\left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\frac{dt}{t} - \int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t}\right].$$
- *For $p < 0$, we have*
$$-p\int_a^b \frac{(Cf(x))^p}{x^q}\,dx \leq \frac{b}{a^q}\left[\int_a^b f^p(t)\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t} - \left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,\frac{dt}{t}\right],$$
and
$$-p\int_a^b (Cf(x))^p\,x^q\,dx \leq b\left[\int_a^b f^p(t)\,t^q\left(\log\frac{b}{t}\right)^p\,\frac{dt}{t} - \left(\log\frac{b}{a}\right)^p\int_a^b f^p(t)\,t^q\,\frac{dt}{t}\right].$$

Acknowledgement

The authors express a great sense of gratitude and deep respect to the referees for their important remarks and comments, which allow us to correct and improve this paper.

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Received: 14.08.2023

Revised: 28.04.2024, 24.05.2024

Accepted: 4.06.2024