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Decorated Nonlinear Flags, Pointed Vortex Loops and the Dihedral Group

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Abstract We identify pointed vortex loops in the plane with low dimensional nonlinear flags decorated with volume forms. We show how submanifolds of such decorated nonlinear flags can be identified with coadjoint orbits of the area preserving diffeomorphism group and relate them to coadjoint orbits of pointed vortex loops. The subgroup of the dihedral group preserving the vorticity data plays a role in the description of these coadjoint orbits.

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1 Introduction

The nonlinear Grassmannian $\mathrm{Gr}_S(M)$ is an infinite dimensional smooth manifold consisting of all submanifolds of M diffeomorphic to a given closed manifold S; it represents the nonlinear version of the Grassmannian of linear subspaces of fixed dimension in a given vector space. It can also be seen as the base space of a principal bundle with total space $\mathrm{Emb}(S, M)$, the space of embeddings of S into M, and structure group $\mathrm{Diff}(S)$, the Lie group of diffeomorphisms of S ([12]).

The nonlinear Grassmannian generalizes to the manifold of nonlinear flags as follows. Let $\mathscr{S} = (S_1, \ldots, S_r)$. A nonlinear flag of type $\mathscr{S}$ in M is an r-tuple $(N_1, \ldots, N_r)$ of submanifolds of M, each N_i diffeomorphic to the corresponding S_i, that are embedded in each other: $N_1 \subseteq N_2 \subseteq \cdots \subseteq N_r$. The manifold of nonlinear frames, consisting of a suitable collection of embeddings from S_i into M, represents the total space of a principal bundle over the nonlinear flag manifold $\mathrm{Flag}_{\mathscr{S}}(M)$, with structure group $\prod_{i=1}^{r} \mathrm{Diff}(S_i)$ ([12]). In this paper, we will consider a submanifold of $\mathrm{Flag}_{\mathscr{S}}(M)$ given by a fixed sequence of embeddings $S_1 \overset{\iota_i}{\hookrightarrow} S_2 \overset{\iota_2}{\hookrightarrow} \ldots S_{r-1} \overset{\iota_{r-1}}{\hookrightarrow} S_r$.

Both the nonlinear Grassmannian and nonlinear flag manifold can be decorated with various structures in order to enrich their properties. This is done in an elegant way via associated bundle constructions. In this paper, we consider nonlinear flags decorated with densities, referred to as weighted nonlinear flags ([14]), and nonlinear flags decorated with volume forms.

Applications of nonlinear Grassmannians and their generalizations can be found in a wide array of fields. They play an important role in computer vision, where they appear as shape spaces (see [3, 20] for nonlinear Grassmannians as shape spaces and [5, 15, 22] for examples of manifolds of low dimensional flags as shape spaces), and in continuum mechanics [21]. However, our main motivation for studying them comes from their usefulness in describing coadjoint orbits of diffeomorphism groups. Coadjoint orbits of the volume preserving group of diffeomorphisms described by nonlinear Grassmannians and decorated nonlinear Grassmannians appear in [11], [16] and [9]. Weighted nonlinear Grassmannians of isotropic submanifolds in a symplectic manifold have been used to describe coadjoint orbits of the Hamiltonian group in [8, 17, 24], and weighted nonlinear Grassmannians of isotropic submanifolds in a contact manifold have been identified with coadjoint orbits of the contact group [13]. Symplectic nonlinear flags have been used to describe coadjoint orbits of the Hamiltonian group in [12].

The purpose of this paper is to form a connection between two results regarding manifolds of nonlinear flags. In [14], the authors establish a one-to-one correspondence between connected components of $\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M)$, the manifold of isotropic weighted nonlinear flags of type $(\mathscr{S},\iota,\mu)$ in a symplectic manifold M, and coadjoint orbits of $\mathrm{Ham}_c(M)$, the group of compactly supported Hamiltonian diffeomorphisms of M, by identifying these connected components with leaves of Weinstein's isodrastic foliation [24]. Example 3.19 showcases the lowest dimensional example of coadjoint orbits of $\mathrm{Ham}_c(M)$ consisting of weighted nonlinear flags: manifolds of weighted curves endowed with weighted points can be realized as coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$. In [6], the authors study a type of singular vorticity named pointed vortex loop, consisting of a curve endowed with a volume form and point vortices. They regard them as elements in the dual of the Lie algebra of $\mathrm{Ham}_c(\mathbb{R}^2)$ and show which spaces of pointed vortex loops can be put in one-to-one correspondence with coadjoint orbits of this group, by identifying them with spaces of embeddings from S^1 to $\mathbb{R}^2$.

Inspired by Example 3.19 in [14], we consider a model $\mathscr{S}$ given by $\iota : \{1,\dots,k\} \hookrightarrow S^1$, with $\iota(i) = t_i$ denoting k consecutive points on the unit circle S^1. We study manifolds of nonlinear flags of type $(\mathscr{S},\iota,\mu)$ in $\mathbb{R}^2$ in two cases: in the first one, μ denotes a pair of nowhere zero densities $\mu = (\mu_0,\mu_1)$ on $\mathscr{S} = (\{1,\dots,k\}, S^1)$ and in the second, μ denotes a pair of volume forms. We show how in the first case, $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$ can be identified with a quotient space $\mathrm{Emb}(S^1,\mathbb{R}^2)/K$, with K denoting a subgroup of the dihedral group of order $2k$ and how, in the second case, $\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$ can be identified with a quotient space $\mathrm{Emb}(S^1,\mathbb{R}^2)/\mathbb{Z}_m$, where $\mathbb{Z}_m$ represents a subgroup of the cyclic group $\mathbb{Z}_k$. In both cases, connected submanifolds can be identified with coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$, and these coadjoint orbits are the same as the ones obtained in [6] for pointed vortex loops.

2 Weighted nonlinear flag manifolds

In this section we describe the notion of weighted nonlinear flag manifolds, as it appears in [14].

The term nonlinear flag describes a sequence of nested submanifolds $N_1 \subseteq \cdots \subseteq N_r \subseteq M$ of a smooth manifold M, where each N_i is diffeomorphic to a prescribed closed (i.e. compact and without boundary) model manifold S_i, for $i = 1, \dots, r$.

2.1 Manifolds of nonlinear flags

We denote the model for the nonlinear flag by $\mathscr{S} = (S_1, \dots, S_r)$ and fix a sequence of embeddings $\iota := (\iota_1, \dots, \iota_{r-1})$ such that

$$S_1 \overset{\iota_1}{\hookrightarrow} S_2 \overset{\iota_2}{\hookrightarrow} S_3 \dots \overset{\iota_{r-1}}{\hookrightarrow} S_r.$$

Moreover, we introduce the notation $j_i := \iota_{r-1} \circ \iota_{r-2} \circ \cdots \circ \iota_i$, where $i = \overline{1, r-1}$. Thus we can define $Q_i := j_i(S_i)$ and $\mathcal{Q} := (Q_1, \dots, Q_{r-1})$, to denote a nonlinear flag in S_r. The manifold of nonlinear flags of type $(\mathscr{S}, \iota)$ in M is the space

$$\begin{aligned}\operatorname{Flag}_{\mathscr{S},\iota}(M) = \Big\{ (N_1, \dots, N_r) \in \prod_{i=1}^{r} \operatorname{Gr}_{S_i}(M) : \exists f \in \operatorname{Emb}(S_r, M) \text{ such that} \\ N_i = f(Q_i), \forall i = \overline{1, r-1} \text{ and } f(S_r) = N_r \Big\},\end{aligned}$$

where $\operatorname{Gr}_{S_i}(M)$ is the Grassmannian of type S_i in M, i.e. the space of all submanifolds of M diffeomorphic to S_i.

We consider the group of diffeomorphisms in $\operatorname{Diff}(\mathscr{S}) = \prod_{i=1}^{r} \operatorname{Diff}(S_i)$ that are compatible with ι:

$$\operatorname{Diff}(\mathscr{S}; \iota) := \left\{ (g_1, \dots, g_r) \in \prod_{i=1}^{r} \operatorname{Diff}(S_i) : g_{i+1} \circ \iota_i = \iota_i \circ g_i, \forall i \right\}.$$

There exists a principal $\operatorname{Diff}(\mathscr{S}; \iota)$-bundle with total space $\operatorname{Emb}(S_r, M)$ over $\operatorname{Flag}_{\mathscr{S},\iota}(M)$,

$$\begin{aligned}\operatorname{Emb}(S_r, M) &\to \operatorname{Flag}_{\mathscr{S},\iota}(M) \\ f &\mapsto (f(Q_1), f(Q_2), \dots, f(Q_{r-1}), f(S_r))\end{aligned} \tag{2.1}$$

which allows us to describe the manifold of nonlinear flags of type $(\mathscr{S}, \iota)$ as a homogeneous space:

$$\operatorname{Flag}_{\mathscr{S},\iota}(M) = \operatorname{Emb}(S_r, M) / \operatorname{Diff}(\mathscr{S}; \iota).$$

2.2 Manifolds of weighted nonlinear flags

Furthermore, we are interested in endowing the submanifolds N_i with a weight, i.e. a nowhere zero density. A density on a vector space V is a function $\mu : V \times \cdots \times V \to \mathbb{R}$ satisying the condition $\mu(T(v_1), \dots, T(v_n)) = |\det T| \mu(v_1, \dots, v_n)$, for $T : V \to V$ a linear map. For S a smooth manifold without boundary, we denote by $\mathscr{D}(T_pS)$ the set of all densities on the tangent space at $p \in S$ and by $\mathscr{D}S = \amalg_{p\in S}\mathscr{D}(T_pS)$ the density bundle of S. Note that $\mathscr{D}S = \Lambda^k T^*S \otimes \mathcal{O}_S$, where $\mathcal{O}_S$ denotes the orientation bundle of S. A density on S is a section of $\mathscr{D}S$ (see [18]).

To this purpose, we introduce the notations $\mathrm{Den}_\times(S_i)$, to represent the space of nowhere zero densities on S_i, and $\mathrm{Den}_\times(\mathscr{S}) = \prod_{i=1}^r \mathrm{Den}_\times(S_i)$. The manifold of weighted flags of type $(\mathscr{S},\iota)$ in M can be defined as the associated bundle to (2.1), via the following $\mathrm{Diff}(M)$ equivariant bijection:

$$\begin{aligned} \mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota}(M) &= \mathrm{Emb}(S_r,M)\times_{\mathrm{Diff}(\mathscr{S};\iota)}\mathrm{Den}_\times(\mathscr{S})\\ ((N_1,\nu_1),\dots,(N_r,\nu_r)) &\leftrightarrow [f,(\mu_1,\dots,\mu_r)] \end{aligned} \tag{2.2}$$

where $N_i = f(Q_i)$, $\nu_i = (f\circ j_i)_*\mu_i$ for all $i = \overline{1,r-1}$ and $N_r = f(S_r)$, $\nu_r = f_*\mu_r$.

For an element $\mu = (\mu_1,\dots,\mu_r) \in \mathrm{Den}_\times(\mathscr{S})$ and a sequence of embeddings ι, we define the submanifold of weighted nonlinear flags of type $(\mathscr{S},\iota,\mu)$ in M by

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M) := \Big\{((N_1,\nu_1),\dots,(N_r,\nu_r)) \in \mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota}(M) : (\mu_1,\dots,\mu_r) \cong (\nu_1,\dots,\nu_r)\Big\}$$

that is, the space of all weighted nonlinear flags $((N_1,\nu_1),\dots,(N_r,\nu_r))$ in M such that the diffeomorphism mapping S_i to N_i is also taking μ_i to ν_i. The bijection (2.2) restricts to

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}(S_r,M)\times_{\mathrm{Diff}(\mathscr{S};\iota)}\mathrm{Den}(\mathscr{S})_{\iota,\mu},$$

where $\mathrm{Den}(\mathscr{S})_{\iota,\mu}$ denotes the $\mathrm{Diff}(\mathscr{S};\iota)$ orbit of μ.

Remark 2.1. Let

$$\mathrm{Diff}(\mathscr{S};\iota,\mu) := \big\{(g_1,\dots,g_r)\in\mathrm{Diff}(\mathscr{S},\iota) : g_i^*\mu_i = \mu_i, \forall i\big\}$$

denote the isotropy subgroup of μ in $\mathrm{Diff}(\mathscr{S};\iota)$. As remarked in [14], there is no proof yet that this is a splitting Lie subgroup of $\mathrm{Diff}(\mathscr{S};\iota)$. Assuming that it is (see Section 3.2 for a case when this is happens), then the surjective and $\mathrm{Diff}(M)$ equivariant map

$$\begin{aligned} &\mathrm{Emb}(S_r,M)\to\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M),\\ &\qquad f\mapsto\Big(\big(f(Q_1),(f\circ j_1)_*\mu_1\big),\dots,\big(f(Q_{r-1}),(f\circ j_{r-1})_*\mu_{r-1}\big)\big(f(S_r),f_*\mu_r\big)\Big), \end{aligned}$$

is a locally trivial smooth principal fiber bundle with structure group $\mathrm{Diff}(\mathscr{S};\iota,\mu)$. Moreover,

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}(S_r,M)/\mathrm{Diff}(\mathscr{S};\iota,\mu).$$

2.3 Manifolds of nonlinear flags decorated with volume forms

Similarly to Section 2.2, we can endow the submanifolds N_i with volume forms instead of densities. To this aim we need the notations $\mathrm{Vol}(S_i)$ to represent the space of volume forms on a manifold S_i and $\mathrm{Vol}(\mathscr{S}) = \prod_{i=1}^r \mathrm{Vol}(S_i)$. The manifold of nonlinear flags of type $(\mathscr{S},\iota)$ decorated with volume forms can be defined as an associated bundle to the principal bundle (2.1)

$$\begin{aligned} \mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota}(M) &= \mathrm{Emb}(S_r,M)\times_{\mathrm{Diff}(\mathscr{S};\iota)}\mathrm{Vol}(\mathscr{S})\\ ((N_1,\nu_1),\dots,(N_r,\nu_r)) &\leftrightarrow [f,(\mu_1,\dots,\mu_r)] \end{aligned} \tag{2.3}$$

where $N_i = f(Q_i)$, $\nu_i = (f \circ j_i)_* \mu_i$ for all $i = \overline{1, r-1}$ and $N_r = f(S_r)$, $\nu_r = f_* \mu_r$.

Let us fix an element $\mu = (\mu_1, \ldots, \mu_r)$ in $\mathrm{Vol}(\mathscr{S})$. We define the manifold of nonlinear flags of type $(\mathscr{S}, \iota, \mu)$ decorated with volume forms as

$$\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}(S_r, M) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Vol}(\mathscr{S})_{\iota,\mu} \tag{2.4}$$

by restricting (2.3) to the $\mathrm{Diff}(\mathscr{S};\iota)$ orbit of μ in $\mathrm{Vol}(\mathscr{S})$, which we denote by $\mathrm{Vol}(\mathscr{S})_{\iota,\mu}$. Formally, we can also write this space as the quotient

$$\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}(S_r, M) / \mathrm{Diff}(\mathscr{S};\iota,\mu), \tag{2.5}$$

where $\mathrm{Diff}(\mathscr{S};\iota,\mu)$ denotes the isotropy subgroup of μ in $\mathrm{Diff}(\mathscr{S};\iota)$ (see Remark 2.1).

2.4 Homological description of $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M)$

Consider the finite dimensional vector space

$$H(\mathscr{S},\iota) := \prod_{i=1}^{r} H^{k_i}(S_i, \iota_{i-1}(S_{i-1}); \mathcal{O}_{S_i}) = \prod_{i=1}^{r} H_0(S_i \setminus \iota_{i-1}(S_{i-1}); \mathbb{R}),$$

where $S_0 = \emptyset$ by convention and k_i is the dimension of S_i. The left-hand side denotes the relative de Rham cohomology with coefficients in the orientation bundle. The $\mathrm{Diff}(\mathscr{S};\iota)$ equivariant identification on the right-hand side indicates the Poincaré-Lefschetz duality. The dimension of $H_0(S_i \setminus \iota_{i-1}(S_{i-1}); \mathbb{R})$ is given by the number of connected components of $S_i \setminus \iota_{i-1}(S_{i-1})$.

Let $\mathrm{Den}(\mathscr{S}) = \prod_{i=1}^{r} \mathrm{Den}(S_i)$, with $\mathrm{Den}(S_i)$ the space of densities on S_i and define a $\mathrm{Diff}(\mathscr{S};\iota)$ equivariant linear map

$$h_{\mathscr{S},\iota} : \mathrm{Den}(\mathscr{S}) \to H(\mathscr{S},\iota), \quad h_{\mathscr{S},\iota}(\mu_1, \ldots, \mu_r) := ([\mu_1], \ldots, [\mu_r]).$$

For $\mu \in \mathrm{Den}(\mathscr{S})$, describing the class $[\mu] := h_{\mathscr{S},\iota}(\mu)$ is equivalent to specifying the integrals of μ_i over each connected component of $S_i \setminus \iota_{i-1}(S_{i-1})$ for $i = 1, \ldots, r$.

Proposition 2.1. *([14]) The manifold of weighted flags of type $(\mathscr{S}, \iota, \mu)$ is identified with the quotient space*

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}(S_r, M) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Den}(\mathscr{S})_{\iota,\mu},$$

where

$$\mathrm{Den}(\mathscr{S})_{\iota,\mu} = \mathrm{Den}_\times(\mathscr{S}) \cap h^{-1}_{\mathscr{S},\iota}(H(\mathscr{S},\iota)_{[\mu]})$$

with $H(\mathscr{S},\iota)_{[\mu]}$ denoting the finite $\mathrm{Diff}(\mathscr{S};\iota)$ orbit of $[\mu]$.

3 Coadjoint orbits of $\mathrm{Ham}_c(M)$

In this section, we consider (M, ω) a symplectic manifold and denote by $\mathrm{Ham}_c(M)$ the group of compactly supported Hamiltonian diffeomorphisms of M. We are interested in finding manifolds of weighted nonlinear flags that can be realized as coadjoint orbits of this group.

3.1 Coadjoint orbits and isodrastic distributions

Let $\mathrm{Emb}^{\mathrm{iso}}(S, M)$ denote the space of isotropic embeddings from S to M. An element $f \in \mathrm{Emb}^{\mathrm{iso}}(S, M)$ is characterized by $f^*\omega = 0$. Weinstein's isodrastic distribution ([24]) on $\mathrm{Emb}^{\mathrm{iso}}(S, M)$ is given by

$$\mathcal{D}_f := \{u_f \in \Gamma(f^*TM) : f^*i_{u_f}\omega \in dC^\infty(S)\}.$$

This is an integrable distribution with codimension $\dim H^1(S;\mathbb{R})$, whose leaves are orbits of $\mathrm{Ham}(M)$. It gives rise to a smooth foliation of $\mathrm{Emb}^{\mathrm{iso}}(S, M)$ called the isodrastic foliation. The action of $\mathrm{Ham}_c(M)$ on each leaf $\mathcal{E} \subseteq \mathrm{Emb}^{\mathrm{iso}}(S, M)$ admits local smooth sections ([14]).

Let $\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M)$ denote the manifold of weighted isotropic nonlinear flags of type $(\mathscr{S}, \iota, \mu)$ in M. It is in bijection with an associated bundle over $\mathrm{Flag}^{\mathrm{iso}}_{\mathscr{S},\iota}(M)$:

$$\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M) = \mathrm{Emb}^{\mathrm{iso}}(S_r, M) \times_{\mathrm{Diff}(S;\iota)} \mathrm{Den}(\mathscr{S})_{\iota,\mu}.$$

The product $\mathcal{D} \times \ker Th_{\mathscr{S},\iota}$ defines an integrable distribution on $\mathrm{Emb}^{\mathrm{iso}}(S_r, M) \times \mathrm{Den}_\times(\mathscr{S})$ which descends to an integrable distribution

$$\overline{\mathcal{D}} = \mathcal{D} \times_{\mathrm{Diff}(\mathscr{S};\iota)} \ker Th_{\mathscr{S},\iota}$$

on $\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota}(M)$. It gives rise to a smooth foliation whose leaves are smooth submanifolds of codimension $\dim H^1(S_r;\mathbb{R})$ in $\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M)$. The action of $\mathrm{Ham}_c(M)$ on such a leaf admits local smooth sections [14].

Moreover, each leaf is a symplectic manifold. On each manifold $(\mathrm{Emb}^{\mathrm{iso}}(S_i, M) \times \mathrm{Den}_\times(S_i), \mathcal{D}_i \times \ker Th_{S_i})$ there exists a closed leafwise differential 2-form

$$\Omega_{(f,\alpha),i}((u_f, d\lambda), (v_f, d\gamma)) := \int_{S_i} (\omega(u_f, v_f)\alpha) - f^*i_{u_f}\omega \wedge \gamma + f^*i_{v_f}\omega \wedge \lambda),$$

where $f \in \mathrm{Emb}^{\mathrm{iso}}(S_i, M)$, $\alpha \in \mathrm{Den}_\times(S_i)$, $u_f, v_f \in \mathcal{D}_f$ and $d\lambda, d\gamma \in \ker T_\alpha h_{S_i}$.

We note that this 2-form can only be defined on the subbundle $\mathcal{D}_i \times \ker Th_{S_i}$ of $T\left(\mathrm{Emb}^{\mathrm{iso}}(S_i, M) \times \mathrm{Den}_\times(S_i)\right)$. Here is a brief explanation for the term $\int_{S_i} f^*i_{u_f}\omega \wedge \gamma$ (that also holds for the term $\int_{S_i} f^*i_{v_f}\omega \wedge \lambda$). We remark that $u_f \in T_f \mathrm{Emb}^{\mathrm{iso}}(S, M)$ means $f^*i_{u_f}\omega$ is a closed 1-form on S_i, while $\gamma \in T_\alpha \mathrm{Den}_\times(S_i)$ is a closed form on S_i of degree equal to the dimension of S_i. To pair these two forms in a meaningful way, at least one of them has to be exact. Thus,

(i) if $f^*i_{u_f}\omega$ is an exact form, i.e. $f^*i_{u_f}\omega = dh$ with $h \in C^\infty(S_i)$, then u_f is in $\mathcal{D}_f$;

(ii) if γ is exact, then $\gamma \in \ker Th_{S_i}$ and we have $\gamma = d\tilde{\gamma}$, with $\tilde{\gamma} \in \Omega^{\dim S_{i-1}}(S_i)$.

However, we want the integral to be independent of the choice of h or γ. In order to achieve this, we obtain, via the Stokes theorem, that both $f^*i_{u_f}\omega$ and γ have to be exact.

Let

$$q_i := j_i^* \times \mathrm{pr}_i : \mathrm{Emb}^{\mathrm{iso}}(S_r, M) \times \mathrm{Den}_\times(\mathscr{S}) \to \mathrm{Emb}^{\mathrm{iso}}(S_i, M) \times \mathrm{Den}_\times(S_i).$$

Then

$$\Omega := \sum_{i=1}^{r} q_i^* \Omega_i \tag{3.1}$$

is a closed leafwise differential 2-form on $(\mathrm{Emb}^{\mathrm{iso}}(S_r, M) \times \mathrm{Den}_\times(\mathscr{S}), \mathcal{D} \times \ker Th_{\mathscr{S},\iota})$ which descends to a leafwise symplectic form $\bar{\Omega}$ on the associated bundle

$$(\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota}(M), \bar{\mathcal{D}}) = (\mathrm{Emb}^{\mathrm{iso}}(S_r, M) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Den}_\times(\mathscr{S}), \mathcal{D} \times_{\mathrm{Diff}(\mathscr{S};\iota)} \ker Th_{\mathscr{S},\iota}).$$

The restriction of $\bar{\Omega}$ to $(\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M), \bar{\mathcal{D}})$ is a leafwise symplectic form as well, and we will use the same notation to refer to it in the following sections. See [14] for proofs.

Theorem 3.1. *([14]) Each leaf $\mathcal{F} \subseteq \mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota,\mu}(M)$ of the isodrastic distribution $\overline{\mathcal{D}}$ is in one-to-one correspondence with a coadjoint orbit of the group of compactly supported Hamiltonian diffeomorphisms of M, which we denote by $\mathrm{Ham}_c(M)$.*

Remark 3.1. The results in this section also hold for the manifolds of nonlinear flags decorated with volume forms described in Section 2.3.

3.2 Coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$

This subsection showcases a low dimensional example for Theorem 3.1. This result first appeared as Example 3.19 in [14].

Let (M, ω) be the symplectic manifold $(\mathbb{R}^2, dx \wedge dy)$ and $k \geq 3$ a natural number. We consider the following objects: $\mathscr{S} = (\{1, \ldots, k\}, S^1)$ and $\iota : \{1, \ldots, k\} \to S^1$ an embedding with $\iota(i) := t_i \in S^1$. We choose nowhere zero densities $\mu_0 = \sum_{i=1}^k \Gamma_i \delta_i$ on $\{1, \ldots, k\}$, with $(\Gamma_i)_{i \in \{1,\ldots,k\}}$ real numbers and δ_i the Dirac delta distribution, and $\mu_1 = \frac{\mathsf{w}}{2\pi} |\mathrm{d}t|$ on S^1, with $\mathsf{w} \in \mathbb{R}$. We write $\mu := (\mu_0, \mu_1)$.

Moreover, let D_{2k} denote the dihedral group of order $2k$:

$$D_{2k} = \{e, r, \ldots, r^{k-1}, s, rs, \ldots, r^{k-1}s\}.$$

We regard the elements of this group as the symmetries of a regular polygon with k vertices labeled $1, 2, \ldots, k$, with r denoting the rotation of angle $\frac{2\pi}{k}$ and s denoting a symmetry (reflection) with respect to a given axis. We note that the dihedral group is generated by the elements r, s with the relations $r^k = s^2 = 1$ and $rs = sr^{-1}$, giving rise to the following presentation of the group:

$$D_{2k} = \langle r, s | r^k = s^2 = 1, rs = sr^{-1} \rangle$$

The group of diffeomorphisms compatible with ι reads

$$\mathrm{Diff}(\mathscr{S};\iota) = \left\{ g \in \mathrm{Diff}(S^1) : g(\{t_1, \ldots, t_k\}) = \{t_1, \ldots, t_k\} \right\}.$$

Proposition 3.2. *There exists a surjective homomorphism*

$$\Phi : \mathrm{Diff}(\mathscr{S};\iota) \to D_{2k} \tag{3.2}$$

where $\Phi(g) = \gamma \in D_{2k}$ is given by the relations $g(t_i) = t_{\gamma(i)}$.

Proof. To prove Φ is a group homomorphism, we must prove it is compatible with the group structures. For $g_1, g_2 \in \mathrm{Diff}(\mathscr{S}, \iota)$ and $\gamma_1, \gamma_2 \in D_{2k}$ we have $g_1(g_2(t_i)) = g_1(t_{\gamma_2(i)}) = t_{\gamma_1(\gamma_2(i))}$, so the map Φ is a group homomorphism.

To prove the surjectivity, we take an arbitrary $\gamma \in D_{2k}$ and look for a diffeomorphism in $\mathrm{Diff}(\mathscr{S}; \iota)$ such that $g(t_i) = t_{\gamma(i)}$ for all i. Since D_{2k} is generated by the elements r and s, we have only these two cases to consider.

Let $\gamma \in D_{2k}$ be the rotation r, thus for each vertex i of the polygon, $\gamma(i) = i + 1$. There exist orientation preserving diffeomorphisms $g_i : [t_i, t_{i+1}] \to [t_{i+1}, t_{i+2}]$. In order to glue them to an orientation preserving diffeomorphism $g \in \mathrm{Diff}(\mathscr{S}; \iota)$, we choose g_i such that, for a small $\varepsilon > 0$, their restrictions to $[t_i, t_i + \varepsilon]$, respectively to $[t_{i+1} - \varepsilon, t_{i+1}]$ are the unique rigid rotations that relate t_i and t_{i+1}, respectively t_{i+1} and t_{i+2}. Clearly, $g(t_i) = t_{i+1} = t_{\gamma(i)}$ for all i.

Let $\gamma \in D_{2k}$ be an axial symmetry s. Without loss of generality we can assume that this is the symmetry satisfying $s(1) = 1$ and $s(i) = k + 2 - i$. There exist orientation reversing diffeomorphisms $g_i : [t_i, t_{i+1}] \to [t_{s(i+1)}, t_{s(i)}]$. In order to glue them to an orientation reversing diffeomorphism $g \in \mathrm{Diff}(\mathscr{S}; \iota)$, we choose g_i such that, for a small $\varepsilon > 0$, their restrictions to $[t_i, t_i + \varepsilon]$, respectively to $[t_{i+1} - \varepsilon, t_{i+1}]$ are the unique rigid axial symmetries that relate t_i and $t_{s(i)}$, respectively t_{i+1} and $t_{s(i+1)}$. □

In this low dimensional setting, the finite dimensional vector space $H(\mathscr{S}, \iota)$ from Section 2.4 takes the following form:

$$\begin{aligned} H(\mathscr{S}, \iota) &= H_0(\{1, \ldots, k\}) \times H_0(S^1 \setminus \iota(\{1, \ldots, k\})) \\ &\cong \mathbb{R}^k \times \mathbb{R}^k. \end{aligned} \tag{3.3}$$

The isomorphism (3.3) maps a class $[\alpha] \in H(\mathscr{S}, \iota)$ to $\big((\Gamma_i)_{i\in\{1,\ldots,k\}}, (\mathsf{w}_i)_{i\in\{1,\ldots,k\}}\big)$, with Γ_i and w_i representing the integrals of α_0, respectively α_1 over connected components of $\{1, \ldots, k\}$, respectively $S^1 \setminus \{t_1, \ldots, t_k\}$. In particular, for $\mu_0 = \Sigma_{i=1}^k \Gamma_i \delta_i$ and $\mu_1 = \frac{\mathsf{w}}{2\pi}|\mathrm{dt}|$, the class $[\mu]$ corresponds to $\Big(\Gamma_i := \int_{\{i\}} \mu_0\Big)$ and $\Big(\mathsf{w}_i := \int_{t_i}^{t_{i+1}} \mu_1 = \frac{\mathsf{w}}{2\pi}(t_{i+1} - t_i)\Big)$.

Section 2 showcases two descriptions of $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(M)$, which we adapt for this low dimensional case.

Associated bundle description. This description is based on Proposition 2.1. We are therefore interested in the action of $\mathrm{Diff}(\mathscr{S}; \iota)$ on $H(\mathscr{S}, \iota)$. This action factorizes through an action of D_{2k}, via the group homomorphism Φ (3.2). Let $\Phi(g) = \gamma \in D_{2k}$ and $[\mu] = ((\Gamma_i), (\mathsf{w}_i)) \in H(\mathscr{S}, \iota) = \mathbb{R}^{2k}$. The element $g \cdot [\mu]$ under the action of $\mathrm{Diff}(\mathscr{S}; \iota)$ is identified with

$$g \cdot [\mu] = \begin{cases} ((\Gamma_{\gamma(i)}), (\mathsf{w}_{\gamma(i)})), & \text{if } \gamma \text{ is a rotation,} \\ ((\Gamma_{(\gamma(i))}), (\mathsf{w}_{\gamma(i+1)})), & \text{if } \gamma \text{ is a symmetry} \end{cases}$$

and we denote by $H(\mathscr{S}, \iota)_{[\mu]}$ the finite $\mathrm{Diff}(\mathscr{S}; \iota)$ orbit of $[\mu]$. We have a $\mathrm{Diff}(\mathscr{S}; \iota)$ equivariant linear map

$$h_{\mathscr{S},\iota} : \mathrm{Den}(\mathscr{S}) \to H(\mathscr{S}, \iota), \quad h_{\mathscr{S},\iota}(\mu) = \big((\Gamma_i)_{i\in\{1,\ldots,k\}}), (\mathsf{w}_i)_{i\in\{1,\ldots,k\}}\big).$$

According to Proposition 2.1, the manifold of nonlinear flags of type $(\mathscr{S}, \iota, \mu)$ in $\mathbb{R}^2$ is defined as

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Den}(\mathscr{S})_{\iota,\mu},$$

where $\mathrm{Den}(\mathscr{S})_{\iota,\mu}$ denotes the $\mathrm{Diff}(\mathscr{S};\iota)$ orbit of μ in $\mathrm{Den}(\mathscr{S})$ which can be identified with

$$\mathrm{Den}(\mathscr{S})_{\iota,\mu} = \mathrm{Den}_\times(\mathscr{S}) \cap h^{-1}_{\mathscr{S},\iota}(H(\mathscr{S},\iota)_{[\mu]}).$$

Principal bundle description. This description is based on Remark 2.1. Therefore, we turn our attention to the subgroup of $\mathrm{Diff}(\mathscr{S};\iota)$ consisting of diffeomorphisms that preserve the densities μ_0 and μ_1:

$$\mathrm{Diff}(\mathscr{S};\iota,\mu) = \left\{ g \in \mathrm{Diff}(S^1) : g^*\mu_1 = \mu_1,\ g|_{\{t_1,\ldots,t_k\}} \text{ preserves } \mu_0 \right\}.$$

It is also a subgroup of the group of rigid transformations of S^1 (which consist of rotations and axial symmetries):

$$\mathrm{Diff}(S^1;\mu) = \mathrm{Rot}(S^1) \cup \sigma\, \mathrm{Rot}(S^1) \cong O(2)$$

with $\sigma \in \mathrm{Diff}(S^1;\mu)$ denoting any orientation reversing element.

Lemma 3.3. *The map*

$$\Psi : \mathrm{Diff}(\mathscr{S};\iota,\mu) \to D_{2k}$$

defined by $g(t_i) = t_{\gamma(i)}$, *for* $\gamma \in D_{2k}$ *and* $g \in \mathrm{Diff}(\mathscr{S};\iota,\mu)$, *is an injective group homomorphism.*

Proof. The map is a group homomorphism because it is a restriction of the group homomorphism Φ in Proposition 3.2. To prove the injectivity, we show that its kernel is trivial. A diffeomorphism $g \in \mathrm{Diff}(\mathscr{S};\iota,\mu)$ is in the kernel of Ψ if $g(t_i) = t_i$. Assume that $g \in s\,\mathrm{Rot}(S^1)$ (i.e. g is a symmetry of the circle). A symmetry that fixes all points t_i requires them to belong to the symmetry axis. On the circle, that is impossible when we have more than two points, and we have $k \geq 3$ points. Thus $g \in \mathrm{Rot}(S^1)$. The only rotation of the circle that preserves all the points t_i (here a single point is enough) is the identity, hence the kernel of Ψ is trivial. □

We will denote the image of Ψ in D_{2k} by K. By Lemma 3.3, K is a subgroup of the dihedral group, hence a finite group. It is isomorphic with $\mathrm{Diff}(\mathscr{S};\iota,\mu)$, thus we can use Remark 2.1 to identify the manifold $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$ with the quotient space $\mathrm{Emb}(S^1,\mathbb{R}^2)/K$. We can summarize these facts in the following theorem.

Theorem 3.4. *Let* $\mathscr{S} = (\{1,\ldots,k\}, S^1)$ *with* $\iota : \{1,\ldots,k\} \to S^1$ *given by* $\iota(i) = t_i$ *and* $\mu = (\mu_0,\mu_1)$ *a pair of nowhere zero densities on* $\{1,\ldots,k\}$ *respectively on* S^1. *Then, the manifold of nonlinear weighted flags* $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$ *is a quotient space of* $\mathrm{Emb}(S^1,\mathbb{R}^2)$ *by a finite group.*

Remark 3.2. The elements of $\mathrm{Flag}_{\mathscr{S},\iota}(\mathbb{R}^2)$ are pointed closed curves in the plane, which are isotropic for reasons of dimension. According to Section 3.1, there exists an isodrastic distribution on $\mathrm{Flag}^{\mathrm{wt\,iso}}_{\mathscr{S},\iota}(\mathbb{R}^2) = \mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota}(\mathbb{R}^2)$ and each leaf is a submanifold $\mathfrak{F} \subseteq \mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$ for a certain μ.

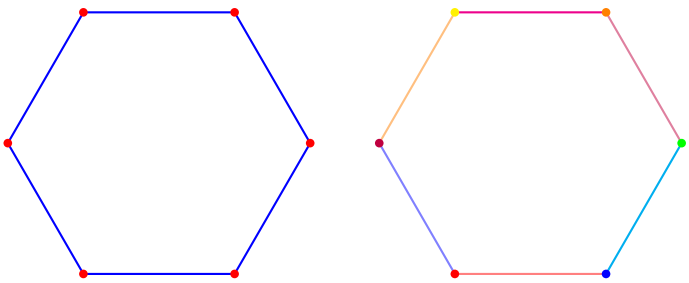
It is known [14] that the isodrastic leaves in $\mathrm{Emb}(S^1, \mathbb{R}^2)$ are $\mathrm{Emb}_a(S^1, \mathbb{R}^2)$, where a denotes a fixed area enclosed by the embeddings. Using Theorem 3.4, we get that the coadjoint orbits $(\mathcal{F}, \omega_{\mathrm{KKS}})$ of $\mathrm{Ham}_c(\mathbb{R}^2)$ in Theorem 3.1 are symplectically diffeomorphic to $(\mathrm{Emb}_a(S^1, \mathbb{R}^2)/K, \bar{\Omega})$, with K the image of Ψ from Lemma 3.3. Here the symplectic form $\bar{\Omega}$ descends from the closed 2-form $\Omega_f(u_f, v_f) = \int_{S^1} \omega(u_f, v_f)\mu_1 + \int_{\{t_1,\ldots,t_k\}} (\omega(u_f, v_f) \circ \iota)\mu_0$ on $\mathrm{Emb}(S^1, \mathbb{R}^2)$, with $u_f, v_f \in T_f \,\mathrm{Emb}(S^1, \mathbb{R}^2)$. This symplectic form $\bar{\Omega}$ can be also seen as descending from (3.1) in the associated bundle description.

3.3 Examples

The integrals $\Gamma_i = \int_{\{i\}} \mu_0$ and $\mathsf{w}_i = \int_{S^1 \setminus \iota(\{1,\ldots,k\})} \mu_1$ which characterize the class $[\mu] \in H(\mathscr{S}, \iota)$ are not necessarily distinct. We can represent this data on a regular polygon as follows: the vertices are labeled by $(\Gamma_i)_{i \in \{1,\ldots,k\}}$ and the edges are labeled by $(\mathsf{w}_i)_{i \in \{1,\ldots,k\}}$. We use colors to indicate identical labels: two vertices, respectively two edges, have the same color if and only if the Γ_is, respectively w_is, associated to them are equal. Figure 1 shows an example for $k = 6$ and showcases the most extreme cases for $((\Gamma_i), (\mathsf{w}_i))$:

(a) all Γ_is and all w_is are equal. Then, in the associated bundle description of $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$, the orbit $H(\mathscr{S}, \iota)_{[\mu]}$ is a one-point set and, in the principal bundle description, $\mathrm{Diff}(\mathscr{S}; \iota, \mu) \cong D_{2k}$.

(b) all Γ_is and all w_is are distinct. Then, in the associated bundle description of $\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$, the orbit $H(\mathscr{S}, \iota)_{[\mu]}$ has $2k$ elements and, in the principal bundle description, $\mathrm{Diff}(\mathscr{S}; \iota, \mu) \cong \{\mathrm{id}\}$.



(a) (b)

Figure 1: Two hexagons representing the two extreme cases of $(\Gamma_i)_{\{1,\ldots,6\}}$ and $(\mathsf{w}_i)_{\{1,\ldots,6\}}$: in a), all the Γ_is are equal and all the w_is are equal, in b) all the Γ_is are distinct and all w_is are distinct.

Remark 3.3. There exists other configurations of Γ_is and w_is for which we obtain orbits with $2k$ elements and trivial isotropy group as in the second case described above. For example, for $k = 6$ with all Γ_is distinct and all w_is equal (or vice versa), the orbit $H(\mathscr{S}, \iota)_{[\mu]}$ also has $2k$ elements and the isotropy subgroup $\mathrm{Diff}(\mathscr{S}; \iota, \mu)$ is the identity.

Example 3.1. Let $(\mathscr{S}, \iota, \mu)$ be a model for a weighted nonlinear flags with the following components: $\mathscr{S} = (\{1, \ldots, 6\}, S^1)$, $\iota : \{1, \ldots, 6\} \to S^1$ defined by $\iota(i) = t_i$ and $\mu = \Big(\mu_0 = \sum_{i=1}^k \Gamma_i \delta_i, \mu_1 = \frac{\mathsf{w}}{2\pi} |\mathrm{dt}|\Big)$.

(a) Suppose the data $((\Gamma_i),(\mathsf{w}_i))$ present rotational symmetry, that is there exists an ℓ such that $\Gamma_{i+\ell}=\Gamma_i$ and $\mathsf{w}_{i+\ell}=\mathsf{w}_i$ (we denote by ℓ the smallest natural number with this property). Let $\ell=2$ in this case, i.e. $\Gamma_1=\Gamma_3=\Gamma_5$, $\Gamma_2=\Gamma_4=\Gamma_6$, $\mathsf{w}_1=\mathsf{w}_3=\mathsf{w}_5$, $\mathsf{w}_2=\mathsf{w}_4=\mathsf{w}_6$. We can represent this data through the regular hexagon in Figure 2a. By (3.2), the orbit of an element $[\mu]\in H(\mathscr{S},\iota)$ under the action of $\mathrm{Diff}(\mathscr{S};\iota)$ is

$$\begin{aligned}H(\mathscr{S},\iota)_{[\mu]}=\{&(\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2),\\&(\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1),\\&(\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2),\\&(\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_1)\}.\end{aligned}$$

and, thus, by Proposition 2.1,

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)=\mathrm{Emb}(S^1,\mathbb{R}^2)\times_{\mathrm{Diff}(\mathscr{S};\iota)}\mathrm{Den}(\mathscr{S})_{\iota,\mu}$$

with $\mathrm{Den}(\mathscr{S})_{\iota,\mu}=\mathrm{Den}_\times(\mathscr{S})\cap h^{-1}_{\mathscr{S},\iota}(H(\mathscr{S},\iota)_{[\mu]})$.

The isotropy subgroup $\mathrm{Diff}(\mathscr{S};\iota,\mu)$ is isomorphic to a subgroup $K\subseteq D_{12}$. In this example, K is the cyclic group generated by the rotation of angle $2\pi/3$, which is isomorphic with $\mathbb{Z}_3$. Thus, by Theorem 3.4,

$$\mathrm{Flag}^{\mathrm{wt}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)=\mathrm{Emb}(S^1,\mathbb{R}^2)/\mathbb{Z}_3.$$

(b) Suppose that (Γ_i) present the rotatation symmetry described above, i.e. $\Gamma_{i+2}=\Gamma_i$ but all w_is are equal, see Figure 2b. In this case

$$\begin{aligned}H(\mathscr{S},\iota)_{[\mu]}=\{&(\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1),\\&(\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\Gamma_2,\Gamma_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1,\mathsf{w}_1)\}\end{aligned}$$

and $K\subseteq D_{12}$ is isomorphic to $\mathbb{Z}_3\cup s\mathbb{Z}_3$, where s denotes an axial symmetry of the hexagon.

Remark 3.4. ([7]) A subgroup of D_{2k} is either

(1) cyclic, of the form $\langle r^\ell\rangle$, where $\ell|k$, or

(2) dihedral, of the form $\langle r^\ell, r^i s\rangle$, where $\ell|k$ and $0\le i\le \ell-1$.

Let $m=k/\ell$. In the first case the subgroup is isomorphic to $\mathbb{Z}_m$, while in the second case it is isomorphic to D_{2m}.

4 Pointed vortex loops

In this section we work in $\mathbb{R}^2$, which we regard as a symplectic manifold. The canonical symplectic form on $\mathbb{R}^2$ is exact and represents the area form; we denote it by $\omega=d\nu$.

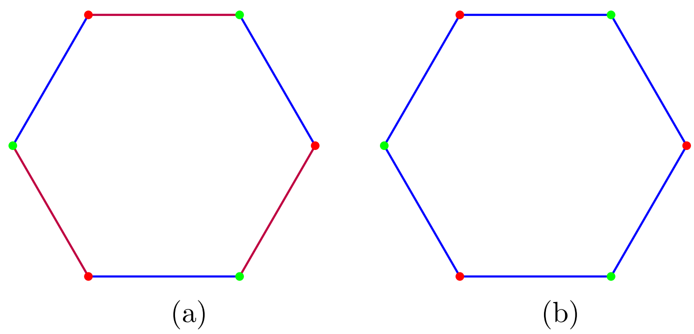


Figure 2: The data $(\Gamma_i)_{\{1,\ldots,6\}}$ and $(\mathsf{w}_i)_{\{1,\ldots,6\}}$ can be represented as a hexagon. The Γ_is denote the vertices, the w_i denote the edges, and we use the same colors to showcase which of these numbers are equal: $\Gamma_{i+2} = \Gamma_i$, $\mathsf{w}_{i+2} = \mathsf{w}_i$ in (a), and $\Gamma_{i+2} = \Gamma_i$, $\mathsf{w}_{i+1} = \mathsf{w}_i$ in (b).

Point vortices ([1] and [19]) are characterised by their position in the plane and their circulation. Vortex loops ([2] and [10]) are curves in the plane endowed with volume forms. The term pointed vortex loop describes an object of the form $\big(C, \beta, (x_i)_{i\in\{1,\ldots,k\}}, (\Gamma_i)_{i\in\{1,\ldots,k\}}\big)$, where (C, β) denotes a vortex loop and $\big((x_i)_{i\in\{1,\ldots,k\}}, (\Gamma_i)_{i\in\{1,\ldots,k\}}\big)$ are k point vortices attached to the curve C. In this section we recall how spaces of pointed vortex loops can be realized as coadjoint orbits, as proven in [6], then we establish a connection between these coadjoint orbits and the coadjoint orbits of nonlinear flags described in Section 3.

4.1 Pointed vortex loops as coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$.

We denote by $\mathsf{w} := \int_C \beta$ the total vorticity, and note that the point vortices x_i introduce k partial vorticities along the loop:

$$\mathsf{w}_i = \int_{x_i}^{x_{i+1}} \beta \quad \text{ with } \mathsf{w} = \mathsf{w}_1 + \cdots + \mathsf{w}_k.$$

We consider the orientation induced by β, hence all w_is are positive, and only consider consecutive points on the curve.

In the case of $\mathbb{R}^2$, the group of compactly supported Hamiltonian diffeomorphisms $\mathrm{Ham}_c(\mathbb{R}^2)$ coincides with the group of area-preserving diffeomorphisms. It is a Lie group, whose Lie algebra consists of Hamiltonian vector fields arising from compactly supported Hamiltonians, which we denote by $\mathfrak{X}_{\mathrm{ham},c}(\mathbb{R}^2)$. These vector fields can be identified with their corresponding compactly supported Hamiltonian functions, thus $\mathfrak{X}_{\mathrm{ham},c}(\mathbb{R}^2) \equiv C_c^\infty(\mathbb{R}^2)$ and the dual Lie algebra is $C_c^\infty(\mathbb{R}^2)^*$, the space of distributions on $\mathbb{R}^2$. A pointed vortex loop can be seen as an element in the dual Lie algebra $C_c^\infty(\mathbb{R}^2)^*$ by defining the pairing

$$\langle (C, \beta, (x_i)_{i\in\{1,\ldots,k\}}, (\Gamma_i)_{i\in\{1,\ldots,k\}}), h\rangle := \int_C h\beta + \sum_{i=1}^k \Gamma_i h(x_i), \tag{4.1}$$

where $h \in C_c^\infty(\mathbb{R}^2)$ represents an element of the Lie algebra.

The natural action of $\mathrm{Ham}_c(\mathbb{R}^2)$,

$$g \cdot (C, \beta, (x_i), (\Gamma_i)) = (g(C), g_*\beta, g(x_i), (\Gamma_i)), \tag{4.2}$$

leaves invariant the area a enclosed by the curve as well as the partial vorticities w_i. We denote by $\mathcal{O}_a^{\bar{\mathsf{w}}}$ the space of pointed vortex loops $(C, \beta, (x_i), (\Gamma_i))$ with these features:

$$\mathcal{O}_a^{\bar{\mathsf{w}}} = \left\{ (C, \beta, (x_i)_{i\in\{1,\ldots,k\}}, (\Gamma_i)_{i\in\{1,\ldots,k\}}) : a = \int_C \nu,\ \mathsf{w}_i = \int_{x_i}^{x_{i+1}} \beta \text{ for } i \in \{1, \ldots, k\} \right\}$$

Lemma 4.1. *([6]) Consider the manifold* $\mathrm{Emb}_a(S^1, \mathbb{R}^2)$ *of embeddings that enclose a fixed area* a *and fix a collection of positive numbers* $\bar{\mathsf{w}} = (\mathsf{w}_i)$ *with* $\mathsf{w} = \mathsf{w}_1 + \cdots + \mathsf{w}_k$. *Then the map*

$$\Phi : \mathrm{Emb}_a(S^1, \mathbb{R}^2) \to \mathcal{O}_a^{\bar{\mathsf{w}}}, \quad \Phi(f) = \big(f(S^1), f_*\mu, (f(t_i)), (\Gamma_i)\big) \tag{4.3}$$

is a bijection, where $\mu := \frac{\mathsf{w}}{2\pi} dt$,

$$t_1 = 0 \text{ and } t_i := \frac{\mathsf{w}_1 + \cdots + \mathsf{w}_{i-1}}{\mathsf{w}} 2\pi, \text{ for } i \in \{1, \ldots, k\}$$

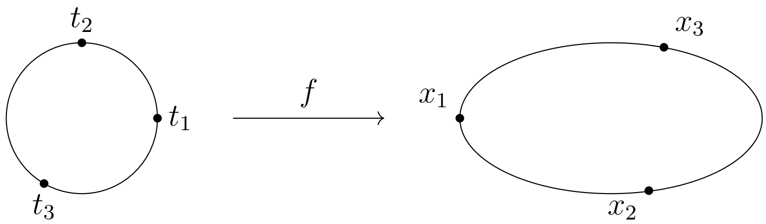


Figure 3: An embedding from (S^1, μ) to (C, β). We have that $C = f(S^1)$, $f_*\mu = \beta$ and $f(t_i) = x_i$, for all i.

Note that the inverse $\Phi^{-1}(C, \beta, (x_i), (\Gamma_i))$ is the embedding f characterised by $f(t) = x \in C$, for $t = \int_{x_1}^{x} \beta$. The bijection Φ intertwines the G action $g \cdot f = g \circ f$ with the natural action (4.2).

We are interested in finding coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$ that involve the space $\mathcal{O}_a^{\bar{\mathsf{w}}} \cong \mathrm{Emb}_a(S^1, \mathbb{R}^2)$, via (4.1). To this end, using the symplectic form ω on $\mathbb{R}^2$ and the volume form $\mu := \frac{\mathsf{w}}{2\pi}\mathrm{dt}$ on S^1, we can endow the manifold $\mathrm{Emb}_a(S^1, \mathbb{R}^2)$ with the following symplectic form

$$\Omega_f(u_f, v_f) = \frac{\mathsf{w}}{2\pi} \int_{S^1} \omega(u_f(t), v_f(t))\mathrm{dt} + \sum_{i=1}^{k} \Gamma_i \omega(u_f(t_i), v_f(t_i)), \tag{4.4}$$

where $u_f, v_f \in T_f \mathrm{Emb}_a(S^1, \mathbb{R}^2) = \left\{ u_f : S^1 \to \mathbb{R}^2 : \int_{S^1} f^* i_{u_f} \omega = 0 \right\}$ (see [6] for the complete proof and [23] for a study on induced differential forms on manifolds of mappings).

The action of the group $\mathrm{Ham}_c(\mathbb{R}^2)$ on $(\mathrm{Emb}_a(S^1, \mathbb{R}^2), \Omega)$ is transitive and Hamiltonian, with the equivariant following momentum map:

$$\mathbf{J} : \mathrm{Emb}_a(S^1, \mathbb{R}^2) \to C_c^\infty(\mathbb{R}^2)^*, \quad \langle \mathbf{J}(f), h \rangle = \frac{\mathsf{w}}{2\pi} \int_{S^1} h(f(t))dt + \sum_{i=1}^{k} \Gamma_i h(f(t_i)). \tag{4.5}$$

We remark that this is the relation (4.1), via the bijection Φ in (4.3).

This momentum map is injective when Γ_i and w_i are distinct for all i. Suppose this data configuration presents rotational symmetries, i.e. there exists an $\ell \in \{1, \dots, k\}$ such that

$$\Gamma_{i+\ell} = \Gamma_i, \quad \mathsf{w}_{i+\ell} = \mathsf{w}_i, \quad \text{for all } i \in \{1, \dots, k\}. \tag{4.6}$$

We choose the smallest ℓ satisfying the relations (4.6), we denote $m := k/\ell$ and we factor out the subgroup of $\mathrm{Rot}(S^1)$ generated by the rotation of angle $\frac{2\pi}{m}$, which is isomorphic to the cyclic group $\mathbb{Z}_m$. The action of $\mathrm{Ham}_c(\mathbb{R}^2)$ on the quotient manifold $(\mathrm{Emb}_a(S^1, \mathbb{R}^2)/\mathbb{Z}_m, \bar{\Omega})$, with $\bar{\Omega}$ the symplectic form descending from (4.4), is Hamiltonian, with the following injective and equivariant momentum map descending from (4.5):

$$\bar{\mathbf{J}} : \mathrm{Emb}_a(S^1, \mathbb{R}^2)/\mathbb{Z}_m \to C_c^\infty(\mathbb{R}^2)^*, \quad \langle \bar{\mathbf{J}}([f]), h \rangle = \frac{\mathsf{w}}{2\pi} \int_{S^1} h(f(t))dt + \sum_{i=1}^{k} \Gamma_i h(f(t_i)). \tag{4.7}$$

The following result emerges:

Theorem 4.2. *([6]) In the above setting,*

(i) if the data $(\Gamma_i), (\mathsf{w}_i)$ have no rotational symmetry, then $(\mathrm{Emb}_a(S^1, \mathbb{R}^2), \Omega)$ can be realized as a coadjoint orbit of $\mathrm{Ham}_c(\mathbb{R}^2)$, via the momentum map $\mathbf{J}$ in (4.5).

(ii) If the data $(\Gamma_i), (\mathsf{w}_i)$ have rotational symmetry, then the space $(\mathrm{Emb}_a(S^1, \mathbb{R}^2)/\mathbb{Z}_m, \bar{\Omega})$, with $\mathbb{Z}_m$ the rotation subgroup generated by the rotation of angle $2\pi/m$, for $m = k/\ell$, can be realized as a coadjoint orbit of $\mathrm{Ham}_c(\mathbb{R}^2)$, via the momentum map $\bar{\mathbf{J}}$ in (4.7).

4.2 Pointed vortex loops in the setting of nonlinear flags

An early version of this result appears in [4].

Consider the nonlinear flag model $\mathscr{S} = (\{1, \dots, k\}, S^1)$ with $\iota : \{1, \dots, k\} \to S^1$ given by $\iota(i) = t_i$. The manifold of nonlinear flags of type $(\mathscr{S}, \iota)$ decorated with volume forms can be identified with an associated bundle as in (2.3):

$$\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota}(\mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Vol}(\mathscr{S}).$$

Furthermore, let $\mu = \left(\mu_0 = \sum_{i=1}^k \Gamma_i \delta_i,\ \mu_1 = \frac{\mathsf{w}}{2\pi}\mathrm{dt}\right)$ denote a pair of volume forms on $\{1, \dots, k\}$ respectively on S^1. The manifold of nonlinear flags of type $(\mathscr{S}, \iota, \mu)$ in $\mathbb{R}^2$ can be identified with the associated bundle as in (2.4):

$$\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2) = \mathrm{Emb}(S^1, \mathbb{R}^2) \times_{\mathrm{Diff}(\mathscr{S};\iota)} \mathrm{Vol}(\mathscr{S})_{\iota,\mu},$$

where $\mathrm{Vol}(\mathscr{S})_{\iota,\mu}$ denotes the $\mathrm{Diff}(\mathscr{S};\iota)$ orbit of μ in $\mathrm{Vol}(\mathscr{S})$.

The subgroup $\mathrm{Diff}(\mathscr{S}; \iota, \mu) \subset \mathrm{Diff}(S^1)$ consists only of diffeomorphisms preserving the volume form dt, hence preserving the orientation of S^1. We can formulate a lemma that follows from Proposition 3.2, analogous to Lemma 3.3.

Lemma 4.3. *Let $\mathbb{Z}_k$ denote the cyclic group of order k. The map*

$$\psi : \mathrm{Diff}(\mathscr{S}; \iota, \mu) \to \mathbb{Z}_k$$

defined by $g(t_i) = t_{\gamma(i)}$, for $\gamma \in \mathbb{Z}_k$ and $g \in \mathrm{Diff}(\mathscr{S}; \iota, \mu)$ is an injective group homomorphism.

Remark 4.1. Lemma 4.3 shows that $\mathrm{Diff}(\mathscr{S};\iota,\mu)$ is isomorphic to a subgroup of $K\subset\mathbb{Z}_k$ and we have $\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)=\mathrm{Emb}(S^1,\mathbb{R}^2)/K$, via (2.5). Analogous results to those in Remark 3.2 hold for nonlinear flags decorated with volume forms instead of densities. More precisely, there exists an isodrastic distribution on $\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota}(\mathbb{R}^2)$ whose leaves

$$\mathcal{F}=\mathrm{Emb}_a(S^1,\mathbb{R}^2)/K\subset\mathrm{Flag}^{\mathrm{vol}}_{\mathscr{S},\iota,\mu}(\mathbb{R}^2)$$

are coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$.

Let $\Gamma_i=\int_{\{i\}}\mu_0$ and $\mathsf{w}_i=\int_{S^1\setminus\iota(\{1,\dots,k\})}\mu_1$. We denote by ℓ the smallest natural number satisfying the relations $\Gamma_i=\Gamma_{i+\ell}$ and $\mathsf{w}_i=\mathsf{w}_{i+\ell}$, and $m:=k/\ell$. We note that K is isomorphic to the cyclic group $\mathbb{Z}_m$.

Corollary 4.4. *The coadjoint orbits $\mathcal{F}$ are the same as the coadjoint orbits consisting of pointed vortex loops which are diffeomorphic to $\mathrm{Emb}_a(S^1,\mathbb{R}^2)/\mathbb{Z}_m$, obtained in Theorem 4.2*

Remark 4.2. When the data (Γ_i) and (w_i) have no axial symmetry, the coadjoint orbits of weighted flags obtained in Remark 3.2 for the densities induced by the volume forms μ coincide with the coadjoint orbits of nonlinear flags of type $(\mathscr{S},\iota,\mu)$ decorated with volume forms.

Remark 4.3. Let $\mu=(\mu_1,\dots,\mu_r)\in\mathrm{Vol}(\mathscr{S})$. We denote by $|\mu|=(|\mu_1|,\dots,|\mu_r|)\in\mathrm{Den}_\times(\mathscr{S})$ the densities arising from the volume forms μ. A diffeomorphism preserving μ_i preserves both the orientation induced by it and $|\mu_i|$, while a diffeomorphism preserving only $|\mu_i|$ can reverse the orientation. We note that sometimes

$$\mathrm{Diff}(\mathscr{S};\iota,\mu)\subsetneq\mathrm{Diff}(\mathscr{S};\iota,|\mu|).$$

For example, consider $r=1$ and $(\mathscr{S},\mu)=(S^1,\mathrm{dt})$. In this case, $\mathrm{Diff}(S^1;|\mathrm{dt}|)$ and $\mathrm{Diff}(S^1;\mathrm{dt})$ do not coincide. The group of diffeomorphisms of S^1 preserving the volume form dt is isomorphic to the group of rigid rotations of S^1, while the group of diffeomorphisms of S^1 preserving the density $|\mathrm{dt}|$ is isomorphic to the group of rigid transformations of S^1:

$$\mathrm{Diff}(S^1;\mathrm{dt})\cong\mathrm{Rot}(S^1)\subsetneq\mathrm{Rot}(S^1)\cup s\,\mathrm{Rot}(S^1)\cong\mathrm{Diff}(S^1;|\mathrm{dt}|).$$

Here $s\in\mathrm{Diff}(S^1;|\mathrm{dt}|)$ denotes an orientation reversing element.

Remark 4.4. The motivation for studying nonlinear flags decorated with volume forms comes from the desire to include the orientation induced by a volume form in the input data. A low dimensional example is the best way to illustrate why this is interesting. Let (C,β) denote a vortex loop, where C is a curve in the plane and $\beta\in\Omega^1(C)$ is a volume form on the curve, and let $\mathsf{w}=\int_C\beta$ denote the total vorticity. We denote by $\mathcal{O}^{\mathsf{w}}_a$, respectively by $\bar{\mathcal{O}}^{\mathsf{w}}_a$, the space of vortex loops with the following properties: the orientation induced by β on the curve is counterclockwise, respectively clockwise, the curve encloses the area a and the total vorticity is w. In [6] the authors show how these spaces can be realized as coadjoint orbits of $\mathrm{Ham}_c(\mathbb{R}^2)$. Each of these two disjoint spaces is diffeomorphic to the coadjoint orbit of $\mathrm{Ham}_c(\mathbb{R}^2)$ consisting of weighted loops (C,ν), where $\nu=|\beta|$ is the density corresponding to the volume form β (see [14] for coadjoint orbits consisting of curves endowed with densities).

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