DOI: 10.2478/auom-2022-0028
An. Şt. Univ. Ovidius Constanţa

sciendo
Vol. 30(2),2022, 231–257

Sombor index of zero-divisor graphs of commutative rings

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Abstract

In this paper, we investigate the Sombor index of the zero-divisor graph of $\mathbb{Z}_n$ which is denoted by $\Gamma(\mathbb{Z}_n)$ for $n \in \{p^\alpha, pq, p^2q, pqr\}$ where p, q and r are distinct prime numbers. Moreover, we introduce an algorithm which calculates the Sombor index of $\Gamma(\mathbb{Z}_n)$. Finally, we give Sombor index of product of rings of integers modulo n.

1 Introduction

Zero-divisor graphs of commutative rings entered the area of algebraic combinatorics by the work of I. Beck [11]. His definition of zero-divisor graph has vertex set on R and any two elements $x, y \in R$ are adjacent whenever $xy = 0$. Later, this definition of a zero-divisor graph of a commutative ring was modified on non-zero zero-divisors by Anderson and Livingston in [9]. After the introduction of zero-divisor graphs, different types of graphs related to commutative rings emerged such as annihilating-ideal graphs, comaximal graphs, total graphs [1, 2, 8, 37, 40, 42, 43, 45, 49].

The technique of encoding information using topological molecular descriptors on the molecular structure has a low computational cost and a good predictive potential. Moreover, these molecular descriptors give ideas about structural characteristics with easy identification. Hence, the number of topological

Key Words: Topological indices, Sombor index, zero-divisor graph, vertex degree, algorithm.
2010 Mathematics Subject Classification: Primary 13A70, 05C09, 05C07; Secondary 68R10, 05C25.
Received: 11.08.2021
Accepted: 30.11.2021
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molecular descriptors which are called graph invariants is huge, and they are mathematical values calculated from a graph representation of a molecule. A graph invariant is a number that is invariant under graph isomorphisms in graph theory. The graphical invariant is considered as a structural invariant related to a graph. In molecular graph theory, the topological index is constructed as a graphical invariant. For this reason, the computing of topological indices of many graph structures has been an attractive research area for scientists especially chemists and mathematicians for a long time. Topological indices play an important role in mathematical chemistry such as the QSPR/QSAR modeling [26, 44].

The Wiener index which is the oldest topological index and a distance-based index was studied for zero-divisor graphs in [10, 41, 47]. In 1972, the first Zagreb index and the second Zagreb index of graph G were suggested by Gutman and Trinajstić [27]. We attain more recent results on Zagreb index in [5, 7, 12, 13, 15, 28, 29, 38, 39, 41, 48]. In 1975, Randić introduced the Randić index of a graph G [34]. Fajtlowicz proposed two topological indices which are called the harmonic index and the inverse degree index [18]. Furtula and Gutman introduced the forgotten topological index [21].

In 2021, the Sombor index of a graph G is defined by the mathematical chemist Ivan Gutman [24]. Then, Cruz et al. examine graphs extremal over the set of all chemical graphs, connected chemical graphs, chemical trees, and hexagonal systems using the Sombor index [14]. The Sombor index can be used successfully on modeling thermodynamic properties of compounds demonstrated by Redžepović [35]. Alikhani et al. consider Sombor index of polymer graphs and show that the Sombor index of some graphs is computed from their monomer units [6]. The Sombor index has attracted important consideration from researchers within a very short time and many results about it can be found in [16, 17, 19, 20, 22, 23, 25, 30, 31, 32, 33, 36, 46, 51].

In this paper, we study Sombor index of zero-divisor graphs of some commutative rings. In Section 2, we give fundamental definitions and notions which will be used rest of the paper. Also, we calculate Sombor index of zero-divisor graphs of $\mathbb{Z}_n$ in Section 3. Finally, in Section 4, we calculate Sombor indices of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ for distinct prime numbers p, q and r.

2 Preliminaries

In this section, we recall some basic definitions and notions which will be used rest of the paper.

Let $G = (V(G), E(G))$ be an undirected graph. The number of vertices of G is the *order* and number of edges of G is the *size* of G. Let $x, y \in V(G)$.

The degree of a vertex x is the number of vertices adjacent to x and denoted by d_x.

Let $Z(R)$ denote the set of all zero divisors of a commutative ring R. The zero-divisor graph of R is an undirected graph which has a vertex set on $R\backslash\{0\}$ and for any $u, v \in Z(R)\backslash\{0\}$, and the vertices u and v are adjacent whenever $uv = 0$ in $R\backslash\{0\}$.

Next, we give definition of Sombor index of a graph which is a novel topological index.

Definition 2.1. *[24] Let G be a graph and $u, v \in V(G)$, then the Sombor index of G is defined by*

$$\mathrm{SO}(G) = \sum_{uv \in E(G)} \sqrt{(d_u)^2 + (d_v)^2}.$$

Lemma 2.2. *[50] Let $\Gamma(\mathbb{Z}_n)$ be a zero-divisor graph of a commutative ring $\mathbb{Z}_n$. Then, the vertex set of $\Gamma(\mathbb{Z}_n)$ is the disjoint union of vertex subsets of A_i such that i is a proper divisor of n. Moreover, $|A_i| = \phi(\frac{n}{i})$.*

Proposition 2.3. *[10] Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ where p_is are distinct prime numbers, and $t, \alpha_i \in \mathbb{N}$ for all i. Let $d = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t}$ be a divisor of n with $d \neq n$. If $u \in A_d$, then*

$$d_u = \begin{cases} d-2, & \text{if } \beta_i \geq \lceil \frac{\alpha_i}{2} \rceil \text{ for all } i \\ d-1, & \text{otherwise.} \end{cases}$$

3 Sombor index of zero-divisor graph of $\mathbb{Z}_n$

Recently, the zero-divisor graph of the ring $\mathbb{Z}_n$ is a popular research in spectral graph and chemical graph theory. Many researchers have studied in this area. Singh and Bhat have examined adjacency matrix and Wiener index of zero-divisor graph $\Gamma(\mathbb{Z}_n)$ [41]. Later, Asir and Rabikka have studied Wiener index of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ [10]. Now, we analyze Sombor index of zero-divisor graph $\Gamma(\mathbb{Z}_n)$ in this section.

Theorem 3.1. *Let p be a prime number, then followings hold:*

(i) If $p = 2$, then $\mathrm{SO}(\Gamma(\mathbb{Z}_{p^2})) = 0$.

(ii) If $p > 2$, then $\mathrm{SO}(\Gamma(\mathbb{Z}_{p^2})) = \sqrt{2}\binom{p-1}{2}(p-2)$.

Proof. (i) It is clear that $\mathbb{Z}_4$ has only one non-zero zero-divisor which is 2. Then, $\Gamma(\mathbb{Z}_4)$ is a one-vertex graph, and the graph has no edge.

(ii) $\Gamma(\mathbb{Z}_{p^2}) \cong K_{p-1}$, and K_{p-1} has $\binom{p-1}{2}$ edges with each vertex having degree $p-2$. Therefore, we have $\mathrm{SO}(\Gamma(\mathbb{Z}_{p^2})) = \sqrt{2}\binom{p-1}{2}(p-2)$.

□

Now we are about to calculate Sombor index of zero-divisor graph for powers of p greater than or equal to 3.

Theorem 3.2. *Let $p > 2$ be a prime number and $\alpha \in \mathbb{N}$ with $\alpha \geq 3$, then Sombor index of $\Gamma(\mathbb{Z}_{p^\alpha})$ is*

$$\begin{aligned}\mathrm{SO}(\Gamma(\mathbb{Z}_{p^\alpha})) = p^{\alpha-1}(p-1)\Bigg[&\Big(1-\frac{1}{p}\Big)\sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor}\sum_{j=1}^{i}\frac{1}{p^{i-j}}\sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2}\\ &+p^{\alpha-1}(p-1)\sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-2}\sum_{j=i+1}^{\alpha-1}\frac{1}{p^{i+j}}\sqrt{(p^i-2)^2+(p^j-2)^2}\\ &+\frac{1}{\sqrt{2}}\sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1}\Big(1-\frac{2}{p^i}\Big)\Big(p^{\alpha-i-1}(p-1)-1\Big)\Bigg].\end{aligned}$$

Proof. We demonstrate the zero-divisor sets of $\mathbb{Z}_{p^\alpha}$ as follows:

$$\begin{aligned}A_1 &= \{px \mid x = 1,\ldots,p^{\alpha-1}-1, p\nmid x\}\\ &.\\ &.\\ &.\\ A_i &= \{p^i x \mid x = 1,\ldots,p^{\alpha-i}-1, p\nmid x\}\\ &.\\ &.\\ &.\\ A_{\alpha-1} &= \{p^{\alpha-1}x \mid x = 1,\ldots,p-1, p\nmid x\}\end{aligned}$$

The vertex set of the graph $\Gamma(\mathbb{Z}_{p^\alpha}) = \bigcup_{i=1}^{\alpha-1} A_i$ where $\bigcap_{i=1}^{\alpha-1} A_i = \emptyset$. Besides, $|A_i|$ means the number of vertices of A_i. We calculate the number of vertices of all zero-divisor sets as $|A_1| = p^{\alpha-1} - p^{\alpha-2}$, $|A_2| = p^{\alpha-2} - p^{\alpha-3}$, $\ldots$, $|A_i| = p^{\alpha-i} - p^{\alpha-i-1}$, $\ldots$, $|A_{\alpha-1}| = p-1$. Moreover, the degree of each vertex in these zero-divisor sets can be defined such that

$$d_u = \begin{cases} p^i - 1, & i < \alpha/2 \\ p^i - 2, & i \geq \alpha/2 \end{cases}$$

for all $u \in A_i$ and $i = 1, 2, ..., \alpha - 1$.

We indicate the proof of this theorem by examining the sub-states of α such that α is odd and even.

Suppose that $p > 2$ is a prime number, $\alpha \in \mathbb{N}$ with $\alpha \geq 3$ and α is even. In this situation, we have three sub-cases as follows:

Case 1:
Each vertex from A_i and each vertex from $A_{\alpha-j}$ are adjacent where $i = 1, 2, \ldots, \frac{\alpha}{2} - 1$ and $j = 1, 2, \ldots, i$. For any edge $e = uv$, we have $d_u = p^i - 1$ and $d_v = p^{\alpha-j} - 2$ where $u \in A_i$ and $v \in A_{\alpha-j}$. So, we get

$$\sum_{i=1}^{\frac{\alpha}{2}-1} \sum_{j=1}^{i} |A_i||A_{\alpha-j}| \sqrt{(p^i - 1)^2 + (p^{\alpha-j} - 2)^2}. \tag{1}$$

Case 2:
Each vertex from A_i and each vertex from A_j are adjacent where $i = \frac{\alpha}{2}, \ldots, \alpha - 2$ and $j = i + 1, \ldots, \alpha - 1$. For any edge $e = uv$, we have $d_u = p^i - 2$ and $d_v = p^j - 2$ where $u \in A_i$ and $v \in A_j$. Hence, we attain

$$\sum_{i=\frac{\alpha}{2}}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} |A_i||A_j| \sqrt{(p^i - 2)^2 + (p^j - 2)^2}. \tag{2}$$

Case 3:
Each vertex from A_i is adjacent to each other vertices from A_i where $i = \frac{\alpha}{2}, \ldots, \alpha - 1$. For any edge $e = uv$, we have $d_u = d_v = p^i - 2$ where $u, v \in A_i$. So, we have

$$\sum_{i=\frac{\alpha}{2}}^{\alpha-1} \frac{|A_i|(|A_i| - 1)}{2} \sqrt{(p^i - 2)^2 + (p^i - 2)^2}. \tag{3}$$

The Sombor index of $\Gamma(\mathbb{Z}_{p^\alpha})$ is calculated by summing Equations (1), (2), and (3) where α is even as follows:

$$\begin{aligned}\mathrm{SO}(\Gamma(\mathbb{Z}_{p^\alpha})) &= \sum_{i=1}^{\frac{\alpha}{2}-1}\sum_{j=1}^{i}|A_i||A_{\alpha-j}|\sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\ &+ \sum_{i=\frac{\alpha}{2}}^{\alpha-2}\sum_{j=i+1}^{\alpha-1}|A_i||A_j|\sqrt{(p^i-2)^2+(p^j-2)^2} \\ &+ \sum_{i=\frac{\alpha}{2}}^{\alpha-1}\frac{|A_i|(|A_i|-1)}{2}\sqrt{(p^i-2)^2+(p^i-2)^2}.\end{aligned} \tag{4}$$

Now, we suppose that $p > 2$ is a prime number, $\alpha \in \mathbb{N}$ with $\alpha \geq 3$ and α is odd. In this circumstance, we have also three sub-cases including different boundaries as follows:

Case 1:
Each vertex from A_i and each vertex from $A_{\alpha-j}$ are adjacent where $i = 1, \ldots, \frac{\alpha-1}{2}$ and $j = 1, \ldots, i$. For any edge $e = uv$, we have $d_u = p^i - 1$ and $d_v = p^{\alpha-j} - 2$ where $u \in A_i$ and $v \in A_{\alpha-j}$. From this, we have

$$\sum_{i=1}^{\frac{\alpha-1}{2}}\sum_{j=1}^{i}|A_i||A_{\alpha-j}|\sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2}. \tag{5}$$

Case 2:
Each vertex from A_i and each vertex from A_j are adjacent where $i = \frac{\alpha+1}{2}, \ldots,$ $\alpha - 2$ and $j = i+1, \ldots, \alpha-1$. For any edge $e = uv$, we have $d_u = p^i - 2$ and $d_v = p^j - 2$ where $u \in A_i$ and $v \in A_j$. Hence, we attain

$$\sum_{i=\frac{\alpha+1}{2}}^{\alpha-2}\sum_{j=i+1}^{\alpha-1}|A_i||A_j|\sqrt{(p^i-2)^2+(p^j-2)^2}. \tag{6}$$

Case 3:
Each vertex from A_i is adjacent to each other vertices from A_i where $i = \frac{\alpha+1}{2}, \ldots, \alpha-1$. For any edge $e = uv$, we have $d_u = d_v = p^i - 2$ where $u, v \in A_i$. So, we have

$$\sum_{i=\frac{\alpha+1}{2}}^{\alpha-1}\frac{|A_i|(|A_i|-1)}{2}\sqrt{(p^i-2)^2+(p^i-2)^2}. \tag{7}$$

The Sombor index of $\Gamma(\mathbb{Z}_{p^\alpha})$ is calculated by using Equations (5), (6), and (7) where α is odd as follows:

$$\begin{aligned}\mathrm{SO}(\Gamma(\mathbb{Z}_{p^\alpha})) &= \sum_{i=1}^{\frac{\alpha-1}{2}} \sum_{j=1}^{i} |A_i||A_{\alpha-j}|\sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\ &+ \sum_{i=\frac{\alpha+1}{2}}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} |A_i||A_j|\sqrt{(p^i-2)^2+(p^j-2)^2} \\ &+ \sum_{i=\frac{\alpha+1}{2}}^{\alpha-1} \frac{|A_i|(|A_i|-1)}{2}\sqrt{(p^i-2)^2+(p^i-2)^2}.\end{aligned} \tag{8}$$

According the Sombor indices in Equations (4) and (8), we represent Sombor index of the graph $\Gamma(\mathbb{Z}_{p^\alpha})$ in a single form as follows:

$$\begin{aligned}\mathrm{SO}(\Gamma(\mathbb{Z}_{p^\alpha})) &= \sum_{i=1}^{\lfloor\frac{\alpha-1}{2}\rfloor} \sum_{j=1}^{i} |A_i||A_{\alpha-j}|\sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\ &+ \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} |A_i||A_j|\sqrt{(p^i-2)^2+(p^j-2)^2} \\ &+ \sum_{i=\lceil\frac{\alpha}{2}\rceil}^{\alpha-1} \frac{|A_i|(|A_i|-1)}{2}\sqrt{(p^i-2)^2+(p^i-2)^2}.\end{aligned} \tag{9}$$

Note that $|A_i| = \phi(\frac{\alpha}{i}) = p^{\alpha-i} - p^{\alpha-i-1} = p^{\alpha-i-1}(p-1)$ by Lemma 2.2. Hence, we get

$$
\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{p^\alpha})) &= \sum_{i=1}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=1}^{i} |A_i||A_{\alpha-j}| \sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\
&+ \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} |A_i||A_j| \sqrt{(p^i-2)^2+(p^j-2)^2} \\
&+ \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-1} \frac{|A_i|(|A_i|-1)}{2} \sqrt{(p^i-2)^2+(p^i-2)^2} \\
&= \sum_{i=1}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=1}^{i} p^{\alpha-i-1}(p-1)p^{j-1}(p-1) \sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\
&+ \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} p^{\alpha-i-1}(p-1)p^{\alpha-j-1}(p-1) \sqrt{(p^i-2)^2+(p^j-2)^2} \\
&+ \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-1} \frac{p^{\alpha-i-1}(p-1)\Big(p^{\alpha-i-1}(p-1)-1\Big)}{2} \sqrt{(p^i-2)^2+(p^i-2)^2} \\
&= \sum_{i=1}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=1}^{i} p^{\alpha-i+j-2}(p-1)^2 \sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\
&+ \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} p^{2\alpha-i-j-2}(p-1)^2 \sqrt{(p^i-2)^2+(p^j-2)^2} \\
&+ \frac{1}{\sqrt{2}} \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-1} p^{\alpha-i-1}(p-1)(p^i-2)\Big(p^{\alpha-i-1}(p-1)-1\Big) \\
&= p^{\alpha-1}(p-1)\Bigg[\Big(1-\frac{1}{p}\Big) \sum_{i=1}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{j=1}^{i} \frac{1}{p^{i-j}} \sqrt{(p^i-1)^2+(p^{\alpha-j}-2)^2} \\
&+ p^{\alpha-1}(p-1) \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-2} \sum_{j=i+1}^{\alpha-1} \frac{1}{p^{i+j}} \sqrt{(p^i-2)^2+(p^j-2)^2} \\
&+ \frac{1}{\sqrt{2}} \sum_{i=\lceil \frac{\alpha}{2} \rceil}^{\alpha-1} \Big(1-\frac{2}{p^i}\Big)\Big(p^{\alpha-i-1}(p-1)-1\Big)\Bigg].
\end{aligned}
$$

□

In the next theorem, we give Sombor index of a zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ for distinct primes p and q.

Theorem 3.3. *Let p and q be prime numbers with $p \neq q$. Then, Sombor index of the graph $\Gamma(\mathbb{Z}_{pq})$ is*

$$\mathrm{SO}(\Gamma(\mathbb{Z}_{pq})) = \sqrt{(p-1)^4(q-1)^2 + (p-1)^2(q-1)^4}.$$

Proof. The graph $\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph. The bipartitions of $\Gamma(\mathbb{Z}_{pq})$ are $A_1 = \{px \mid x = 1, 2, \ldots, q-1\}$ and $A_2 = \{qx \mid x = 1, 2, \ldots, p-1\}$. Since $|A_1| = \phi(\frac{pq}{p}) = q-1$ and $|A_2| = \phi(\frac{pq}{q}) = p-1$, then the size of this graph is $(p-1)(q-1)$. It follows that

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{pq})) &= \sum_{uv \in E(\Gamma(\mathbb{Z}_{pq}))} \sqrt{{d_u}^2 + {d_v}^2} \\
&= \sum_{uv \in E(\Gamma(\mathbb{Z}_{pq}))} \sqrt{(q-1)^2 + (p-1)^2} \\
&= |A_1||A_2|\sqrt{(p-1)^2 + (q-1)^2} \\
&= (q-1)(p-1)\sqrt{(p-1)^2 + (q-1)^2} \\
&= \sqrt{(p-1)^4(q-1)^2 + (p-1)^2(q-1)^4}
\end{aligned}$$

where $u \in A_1$ and $v \in A_2$. $\square$

Theorem 3.4. *Let $\Gamma(\mathbb{Z}_{p^2q})$ be a zero-divisor graph and p and q be distinct prime numbers. Then, Sombor index of $\Gamma(\mathbb{Z}_{p^2q})$ is*

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{p^2q})) = (p-1)(q-1)\Bigg[&(p-1)\sqrt{(p-1)^2 + (pq-2)^2} \\
&+ p\sqrt{(p^2-1)^2 + (q-1)^2} \\
&+ \sqrt{(p^2-1)^2 + (pq-2)^2} \\
&+ \frac{(p-2)(pq-2)}{\sqrt{2}(q-1)}\Bigg].
\end{aligned}$$

Proof. Since proper divisors of $n = p^2q$ are p, p^2, q and pq, then the vertex set can be partitioned as $V(\Gamma(\mathbb{Z}_n)) = A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_i \cap A_j = \emptyset$ where

$i, j = 1, \ldots, 4$, $i \neq j$ and

$$\begin{aligned}
A_1 &= \{px \mid x = 1, 2, \ldots, pq-1, p \nmid x, q \nmid x, \}, \\
A_2 &= \{qx \mid x = 1, 2, \ldots, p^2-1, p \nmid x\}, \\
A_3 &= \{p^2 x \mid x = 1, 2, \ldots, q-1\}, \\
A_4 &= \{pqx \mid x = 1, 2, \ldots, p-1\}.
\end{aligned}$$

One can calculate the number of vertices of all zero-divisor sets as $|A_1| = (p-1)(q-1)$, $|A_2| = p(p-1)$, $|A_3| = (q-1)$, and $|A_4| = (p-1)$. Also, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_4|, & u \in A_1 \\ |A_3|, & u \in A_2 \\ |A_2| + |A_4|, & u \in A_3 \\ |A_1| + |A_3| + |A_4| - 1, & u \in A_4 \end{cases}.$$

Note that, any two vertices $u \in A_i$ and $v \in A_j$ are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $u \cdot v$. This implies that we have the following cases for any edge e in $\Gamma(\mathbb{Z}_n)$:

Case 1:
If $e = uv$, then $u \in A_1$ and $v \in A_4$. In this case, $d_u = p - 1$ and $d_v = pq - 2$. The number of edges which has one endpoint in A_1 and the other in A_4 is $|A_1||A_4|$. So, we have

$$|A_1||A_4|\sqrt{(p-1)^2 + (pq-2)^2}.$$

Case 2:
If $e = uv$, then $u \in A_2$ and $v \in A_3$. In this case, $d_u = p^2 - p$ and $d_v = q - 1$. The number of edges which has one endpoint in A_2 and other in A_3 is $|A_2||A_3|$. Hence, we attain

$$|A_2||A_3|\sqrt{(p^2-1)^2 + (q-1)^2}.$$

Case 3:
If $e = uv$, then $u \in A_3$ and $v \in A_4$. In this case, $d_u = q - 1$ and $d_v = p - 1$. The number of edges which has one endpoint in A_3 and other in A_4 is $|A_3||A_4|$. Therefore, we obtain

$$|A_3||A_4|\sqrt{(p^2-1)^2 + (pq-2)^2}.$$

Case 4:
If $e = uv$, then $u, v \in A_4$. In this case, $d_u = d_v = p - 2$ and the number of edges which has endpoints are in A_4 is $\frac{|A_4|(|A_4|-1)}{2}$. So, we have

$$\frac{|A_4|(|A_4|-1)}{2}\sqrt{(pq-2)^2+(pq-2)^2}.$$

Thus summing up all these cases respectively, one can conclude that

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{p^2q})) &= |A_1||A_4|\sqrt{(p-1)^2+(pq-2)^2} \\
&+ |A_2||A_3|\sqrt{(p^2-1)^2+(q-1)^2} \\
&+ |A_3||A_4|\sqrt{(p^2-1)^2+(pq-2)^2} \\
&+ \frac{|A_4|(|A_4|-1)}{2}\sqrt{(pq-2)^2+(pq-2)^2} \\
&= (p-1)^2(q-1)\sqrt{(p-1)^2+(pq-2)^2} \\
&+ (p^2-p)(q-1)\sqrt{(p^2-1)^2+(q-1)^2} \\
&+ (p-1)(q-1)\sqrt{(p^2-1)^2+(pq-2)^2} \\
&+ \frac{(p-1)(p-2)}{2}\sqrt{(pq-2)^2+(pq-2)^2}.
\end{aligned}$$

Using this identity, Sombor index of $\Gamma(\mathbb{Z}_{p^2q})$ is

$$\begin{aligned}
&(p-1)(q-1)\Bigg[(p-1)\sqrt{(p-1)^2+(pq-2)^2} \\
&+ p\sqrt{(p^2-1)^2+(q-1)^2} \\
&+ \sqrt{(p^2-1)^2+(pq-2)^2} \\
&+ \frac{(p-2)(pq-2)}{\sqrt{2}(q-1)}\Bigg].
\end{aligned}$$

□

Example 3.5. *For the graph* $\Gamma(\mathbb{Z}_{75})$*, we have* $p = 5$ *and* $q = 3$*. Then,* $\mathrm{SO}(\Gamma(\mathbb{Z}_{75})) \cong 1727.24$*, and the set of zero-divisors can be written as follows:*

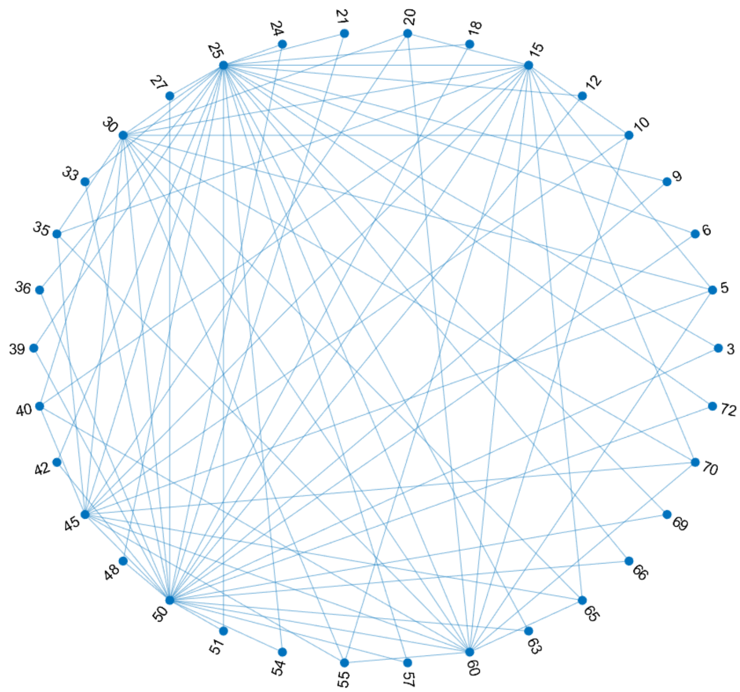

Figure 1: The graph $\Gamma(\mathbb{Z}_{75})$

$$A_1 = \{3, 6, 9, 12, 18, 21, 24, 27, 33, 36, 39, 42, 48, 51, 54, 57, 63, 66, 69, 72\},$$
$$A_2 = \{5, 10, 20, 35, 40, 55, 65, 70\},$$
$$A_3 = \{15, 30, 45, 60\},$$
$$A_4 = \{25, 50\}.$$

Moreover, these sets give rise to the graph which can be shown in Figure 1.

In the next theorem, the relation of Sombor index of $\Gamma(\mathbb{Z}_{pqr})$ is represented.

Theorem 3.6. *Let $\Gamma(\mathbb{Z}_{pqr})$ be a zero-divisor graph and p, q and r be distinct prime numbers. Then, Sombor index of $\Gamma(\mathbb{Z}_{pqr})$ is*

$$
\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{pqr})) = (p-1)(q-1)(r-1)\Bigg[& \sqrt{(p-1)^2+(qr-1)^2} \\
& + \sqrt{(q-1)^2+(pr-1)^2} \\
& + \sqrt{(r-1)^2+(pq-1)^2} \\
& + \frac{\sqrt{(pq-1)^2+(pr-1)^2}}{(p-1)} \\
& + \frac{\sqrt{(pq-1)^2+(qr-1)^2}}{(q-1)} \\
& + \frac{\sqrt{(pr-1)^2+(qr-1)^2}}{(r-1)}\Bigg].
\end{aligned}
$$

Proof. Since proper divisors of $n = pqr$ are p, q, r, pq, pr and qr, then the vertex set can be partitioned as $V(\Gamma(\mathbb{Z}_n)) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ and $A_i \cap A_j = \emptyset$ where $i, j = 1, \ldots, 6$, $i \neq j$ and

$$
\begin{aligned}
A_1 &= \{px \mid x = 1, 2, \ldots, qr-1, q \nmid x, r \nmid x\}, \\
A_2 &= \{qx \mid x = 1, 2, \ldots, pr-1, p \nmid x, r \nmid x\}, \\
A_3 &= \{rx \mid x = 1, 2, \ldots, pq-1, p \nmid x, q \nmid x\}, \\
A_4 &= \{pqx \mid x = 1, 2, \ldots, r-1\}, \\
A_5 &= \{prx \mid x = 1, 2, \ldots, q-1\}, \\
A_6 &= \{qrx \mid x = 1, 2, \ldots, p-1\}.
\end{aligned}
$$

The number of vertices of all zero-divisor sets can be calculated as $|A_1| = (q-1)(r-1)$, $|A_2| = (p-1)(r-1)$, $|A_3| = (p-1)(q-1)$, $|A_4| = (r-1)$, $|A_5| = (q-1)$, and $|A_6| = (p-1)$. Besides, the degree of each vertex in these zero-divisor sets can be determined as

$$
d_u = \begin{cases}
|A_6|, & u \in A_1 \\
|A_5|, & u \in A_2 \\
|A_4|, & u \in A_3 \\
|A_3| + |A_5| + |A_6|, & u \in A_4 \\
|A_2| + |A_4| + |A_6|, & u \in A_5 \\
|A_1| + |A_4| + |A_5|, & u \in A_6
\end{cases}.
$$

Remark that, any two vertices $u \in A_i$ and $v \in A_j$ are adjacent in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $u \cdot v$. This implies that we have six cases as follows for any edge e in $\Gamma(\mathbb{Z}_n)$:

Case 1:
If $e = uv$, then $u \in A_1$ and $v \in A_6$. In this case, $d_u = p-1$ and $d_v = (q-1)(r-1)+(q-1)+(r-1)$. The number of edges which has one endpoint in A_1 and the other in A_6 is $|A_1||A_6|$. So, we attain

$$|A_1||A_6|\sqrt{(p-1)^2 + ((q-1)(r-1)+(q-1)+(r-1))^2}.$$

Case 2:
If $e = uv$, then $u \in A_2$ and $v \in A_5$. In this case, $d_u = q-1$ and $d_v = (p-1)(r-1)+(p-1)+(r-1)$. The number of edges which has one endpoint in A_2 and the other in A_5 is $|A_2||A_5|$. So, we have

$$|A_2||A_5|\sqrt{(q-1)^2 + ((p-1)(r-1)+(p-1)+(r-1))^2}.$$

Case 3:
If $e = uv$, then $u \in A_3$ and $v \in A_4$. In this case, $d_u = r-1$ and $d_v = (p-1)(q-1)+(p-1)+(q-1)$. The number of edges which has one endpoint in A_3 and the other in A_4 is $|A_3||A_4|$. Hence, we get

$$|A_3||A_4|\sqrt{(r-1)^2 + ((p-1)(q-1)+(p-1)+(q-1))^2}.$$

Case 4:
If $e = uv$, then $u \in A_4$ and $v \in A_5$. In this case, $d_u = (p-1)(q-1)+(p-1)+(q-1)$ and $d_v = (p-1)(r-1)+(p-1)+(r-1)$. The number of edges which has one endpoint in A_4 and other in A_5 is $(q-1)(r-1)$. Accordingly, we attain

$$|A_4||A_5|\sqrt{((p-1)(q-1)+(p-1)+(q-1))^2 + ((p-1)(r-1)+(p-1)+(r-1))^2}.$$

Case 5:
If $e = uv$, then $u \in A_4$ and $v \in A_6$. In this case, $d_u = (p-1)(q-1)+(p-1)+(q-1)$ and $d_v = (q-1)(r-1)+(q-1)+(r-1)$. The number of edges which has one endpoint in A_4 and other in A_6 is $(p-1)(r-1)$. Then, we have

$$|A_4||A_6|\sqrt{((p-1)(q-1)+(p-1)+(q-1))^2 + ((q-1)(r-1)+(q-1)+(r-1))^2}.$$

Case 6:
If $e = uv$, then $u \in A_5$ and $v \in A_6$. In this case, $d_u = (p-1)(r-1)+(p-1)+(r-1)$ and $d_v = (q-1)(r-1)+(q-1)+(r-1)$. The number of edges which has one endpoint in A_5 and other in A_6 is $(p-1)(q-1)$. Consequently, we get

$$|A_5||A_6|\sqrt{((p-1)(r-1)+(p-1)+(r-1))^2 + ((q-1)(r-1)+(q-1)+(r-1))^2}.$$

Thus summing up all these cases respectively, one can conclude that

$$\begin{aligned}
&\mathrm{SO}(\Gamma(\mathbb{Z}_{pqr})) = (p-1)(q-1)(r-1)\sqrt{(p-1)^2 + ((q-1)(r-1) + (q-1) + (r-1))^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(q-1)^2 + ((p-1)(r-1) + (p-1) + (r-1))^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(r-1)^2 + ((p-1)(q-1) + (p-1) + (q-1))^2} \\
&+ (q-1)(r-1)\sqrt{((p-1)(q-1) + (p-1) + (q-1))^2 + ((p-1)(r-1) + (p-1) + (r-1))^2} \\
&+ (p-1)(r-1)\sqrt{((p-1)(q-1) + (p-1) + (q-1))^2 + ((q-1)(r-1) + (q-1) + (r-1))^2} \\
&+ (p-1)(q-1)\sqrt{((p-1)(r-1) + (p-1) + (r-1))^2 + ((q-1)(r-1) + (q-1) + (r-1))^2} \\
&= (p-1)(q-1)(r-1)\sqrt{(p-1)^2 + (qr-1)^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(q-1)^2 + (pr-1)^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(r-1)^2 + (pq-1)^2} \\
&+ (q-1)(r-1)\sqrt{(pq-1)^2 + (pr-1)^2} \\
&+ (p-1)(r-1)\sqrt{(pq-1)^2 + (qr-1)^2} \\
&+ (p-1)(q-1)\sqrt{(pr-1)^2 + (qr-1)^2}.
\end{aligned}$$

From this identity, we get

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_{pqr})) = (p-1)(q-1)(r-1)\Bigg[&\sqrt{(p-1)^2 + (qr-1)^2} \\
&+ \sqrt{(q-1)^2 + (pr-1)^2} \\
&+ \sqrt{(r-1)^2 + (pq-1)^2} \\
&+ \frac{\sqrt{(pq-1)^2 + (pr-1)^2}}{(p-1)} \\
&+ \frac{\sqrt{(pq-1)^2 + (qr-1)^2}}{(q-1)} \\
&+ \frac{\sqrt{(pr-1)^2 + (qr-1)^2}}{(r-1)}\Bigg].
\end{aligned}$$

□

3.1 Matlab code for determining Sombor index of $\Gamma(\mathbb{Z}_n)$

In this subsection, we give an algorithm for calculating Sombor index of $\Gamma(\mathbb{Z}_n)$ when entering an integer n.

```
1  n=input("Enter n for Z_n:")
2  Vert=strings(1,n-2);
3  Adj=zeros(n-2);
4  Deg=zeros(1,n-2);
5  for i=2:n-1
6  Vert(i-1)=int2str(i);
7  for j=2:n-1
8  if (i==j), continue, end
9  if mod(i*j,n)==0
10 Adj(i-1,j-1)=1;
11 Deg(i-1)=Deg(i-1)+1;
12 end
13 end
14 end
15 for i=size(Deg,2):-1:1
16 if (Deg(i)==0)
17 Adj(i,:)=[];
18 Adj(:,i)=[];
19 Vert(i)=[];
20 Deg(i)=[];
21 end
22 end
23 si=0;
24 for i=1:size(Deg,2)-1
25 for j=i+1:size(Deg,2)
26 if (Adj(i,j)==1)
27 si= si + sqrt(Deg(i)^2+Deg(j)^2);
28 end
29 end
30 end
31 fprintf("Sombor Index of graph of Z_n: %f",si);
```

In the first four lines of the algorithm, n for $\mathbb{Z}_n$ is requested, and the vertex set ($Vert$), the adjacency matrix (Adj) and the degree array (Deg) are initialized. Next, in lines 5-14, all possible vertices in the graph are inserted to the set, and the adjacency matrix is filled while degree array is calculated under the condition $i \cdot j \equiv 0\ (mod\ n)$. After that, vertices having no neighbors are removed from vertex set, degree array and adjacency matrix in lines 15-22. Finally, in lines 23-31, Sombor index of graph $\Gamma(\mathbb{Z}_n)$ is computed and printed out.

4 Sombor index of zero-divisor graph of products of rings of integers modulo n

In this section, we calculate Sombor index of the graphs $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ for distinct prime numbers p, q and r.

The zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q$ and some graph theoretical properties of it have been studied in [4]. In the following theorem, we give Sombor index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$.

Theorem 4.1. *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ be a zero-divisor graph and p, q be distinct prime numbers. Then, Sombor index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is*

$$\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = (p-1)(q-1)\sqrt{(p-1)^2 + (q-1)^2}.$$

Proof. Let $x \in {\mathbb{Z}_p}^*$ and $y \in {\mathbb{Z}_q}^*$ where $x = 1, 2, ..., p-1$ and $y = 1, 2, ..., q-1$. Since $(x, 0)(0, y) = (0, 0)$, the edge set of $x \in {\mathbb{Z}_p}^*$ contains only the edges between the vertices $(x, 0)$ and $(0, y)$.
The graph $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ is a complete bipartite graph which is isomorphic to $K_{p-1,q-1}$. Partitions of vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ are

$$A_1 = \{(x, 0) \mid 1 \leq x \leq p, x \in \mathbb{Z}_p\},$$

$$A_2 = \{(0, y) \mid 1 \leq y \leq q, y \in \mathbb{Z}_q\}$$

such that $A_1 \cup A_2 = V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$ and $A_1 \cap A_2 = \emptyset$. Since $|A_1| = p-1$ and $|A_2| = q-1$, the size of this graph is $(p-1)(q-1)$. Also, $d_u = |A_2|$ for all $u \in A_1$ and $d_v = |A_1|$ for all $v \in A_2$. Hence, we obtain

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) &= \sum_{uv \in E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q))} \sqrt{{d_u}^2 + {d_v}^2} \\
&= \mid A_1 \mid\mid A_2 \mid \sqrt{\mid A_2 \mid^2 + \mid A_1 \mid^2} \\
&= (p-1)(q-1)\sqrt{(p-1)^2 + (q-1)^2}.
\end{aligned}$$

□

Example 4.2. *For zero-divisor graph of $Z_7 \times Z_{11}$, we attain $p = 7$ and $q = 11$. Then, $\mathrm{SO}(\Gamma(\mathbb{Z}_7 \times \mathbb{Z}_{11})) \cong 699.71$, and the set of zero-divisors as follows:*

$$A_1 = \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0)\},$$
$$A_2 = \{(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (0, 8), (0, 9), (0, 10)\}.$$

These sets give rise to the graph depicted in Figure 2.

Akgunes and Nacaroglu have studied some properties of zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ [3]. Moreover, they have calculated irregularity index and Zagreb indices of this graph. We obtain Sombor index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ in the following theorem.

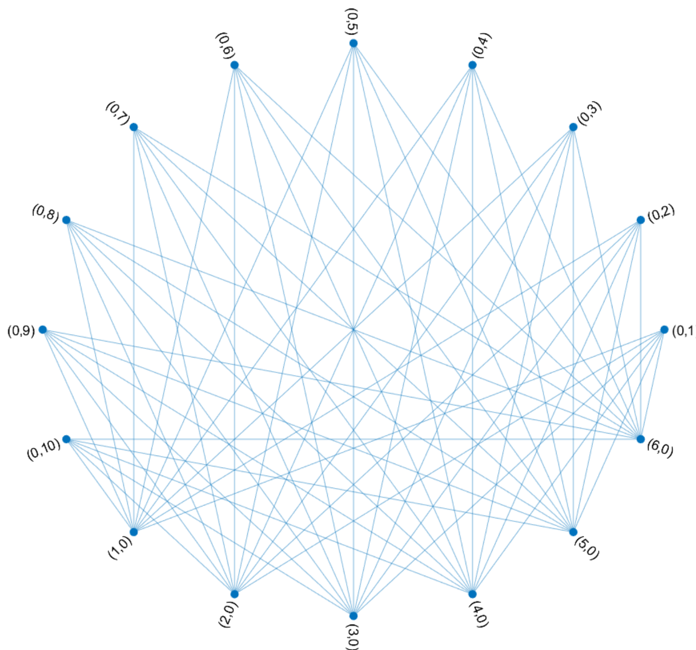

Figure 2: The graph $\Gamma(\mathbb{Z}_7 \times \mathbb{Z}_{11})$

Theorem 4.3. *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ be a zero-divisor graph and p, q, r be distinct prime numbers. Then, Sombor index of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ is*

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = (p-1)(q-1)(r-1)\Bigg[& \sqrt{(pq-1)^2 + (r-1)^2} \\
& + \sqrt{(pr-1)^2 + (q-1)^2} \\
& + \sqrt{(qr-1)^2 + (p-1)^2} \\
& + \frac{\sqrt{(pq-1)^2 + (pr-1)^2}}{(p-1)} \\
& + \frac{\sqrt{(pq-1)^2 + (qr-1)^2}}{(q-1)} \\
& + \frac{\sqrt{(pr-1)^2 + (qr-1)^2}}{(r-1)}\Bigg].
\end{aligned}$$

Proof. We divide the vertex set of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ into six subsets such that $V(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = \bigcup_{i=1}^{6} A_i$ and $A_i \cap A_j = \emptyset$ where $i = 1, 2, ..., 5$ and $j = i+1, ..., 6$. We show that these vertex subsets are as follows:

$$\begin{aligned}
A_1 &= \{(x, 0, 0) \mid 1 \leq x < p,\ x \in \mathbb{Z}_p\},\\
A_2 &= \{(0, y, 0) \mid 1 \leq y < q,\ y \in \mathbb{Z}_q\},\\
A_3 &= \{(0, 0, z) \mid 1 \leq z < r,\ z \in \mathbb{Z}_r\},\\
A_4 &= \{(0, y, z) \mid 1 \leq y < q,\ 1 \leq z < r,\ y \in \mathbb{Z}_q,\ z \in \mathbb{Z}_r\},\\
A_5 &= \{(x, 0, z) \mid 1 \leq x < p,\ 1 \leq z < r,\ x \in \mathbb{Z}_p,\ z \in \mathbb{Z}_r\},\\
A_6 &= \{(x, y, 0) \mid 1 \leq x < p,\ 1 \leq y < q,\ x \in \mathbb{Z}_p,\ y \in \mathbb{Z}_q\}.
\end{aligned}$$

The number of vertices of all zero-divisor sets can be calculated as $|A_1| = (p-1)$, $|A_2| = (q-1)$, $|A_3| = (r-1)$, $|A_4| = (q-1)(r-1)$, $|A_5| = (p-1)(r-1)$, and $|A_6| = (p-1)(q-1)$. Moreover, the degree of each vertex in these zero-divisor sets can be determined as

$$d_u = \begin{cases} |A_2| + |A_3| + |A_4|, & u \in A_1\\ |A_1| + |A_3| + |A_5|, & u \in A_2\\ |A_1| + |A_2| + |A_6|, & u \in A_3\\ |A_1|, & u \in A_4\\ |A_2|, & u \in A_5\\ |A_3|, & u \in A_6 \end{cases}.$$

According to these subsets, we examine edges in $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$ in six cases as follows:

Case 1:
Let $e = uv$ be an edge where for all $u \in A_1$ and $v \in A_2$. In this case, $d_u = qr-1$ and $d_v = pr-1$. Hence, the number of edges between the sets A_1 and A_2 is $|A_1||A_2|$, and we attain

$$|A_1||A_2|\sqrt{(qr-1)^2 + (pr-1)^2}.$$

Case 2:
Let $e = uv$ be an edge where for all $u \in A_1$ and $v \in A_3$. In this case, $d_u = qr-1$ and $d_v = pq-1$. Hence, the number of edges between the sets A_1 and A_3 is $|A_1||A_3|$, and we have

$$|A_1||A_3|\sqrt{(qr-1)^2 + (pq-1)^2}.$$

Case 3:
Let $e = uv$ be an edge where for all $u \in A_2$ and $v \in A_3$. In this case, $d_u = pr-1$ and $d_v = pq - 1$. Hence, the number of edges between the sets A_2 and A_3 is $|A_2||A_3|$, and we get

$$|A_2||A_3|\sqrt{(pr-1)^2 + (pq-1)^2}.$$

Case 4:
Let $e = uv$ be an edge where for all $u \in A_1$ and $v \in A_4$. In this case, $d_u = qr-1$ and $d_v = p - 1$. Hence, the number of edges between the sets A_1 and A_4 is $|A_1||A_4|$, and we have

$$|A_1||A_4|\sqrt{(qr-1)^2 + (p-1)^2}.$$

Case 5:
Let $e = uv$ be an edge where for all $u \in A_2$ and $v \in A_5$. In this case, $d_u = pr-1$ and $d_v = q - 1$. Hence, the number of edges between the sets A_5 and A_2 is $|A_2||A_5|$, and we attain

$$|A_2||A_5|\sqrt{(pr-1)^2 + (q-1)^2}.$$

Case 6:
Let $e = uv$ be an edge where for all $u \in A_3$ and $v \in A_6$. In this case, $d_u = pq-1$ and $d_v = r - 1$. Hence, the number of edges between the sets A_6 and A_3 is $|A_3||A_6|$, and we get

$$|A_3||A_6|\sqrt{(pq-1)^2 + (r-1)^2}.$$

Therefore, after combining above six cases respectively, we obtain that

$$\begin{aligned}
\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) &= |A_1||A_2|\sqrt{(qr-1)^2+(pr-1)^2} \\
&+ |A_1||A_3|\sqrt{(qr-1)^2+(pq-1)^2} \\
&+ |A_2||A_3|\sqrt{(pr-1)^2+(pq-1)^2} \\
&+ |A_1||A_4|\sqrt{(qr-1)^2+(p-1)^2} \\
&+ |A_2||A_5|\sqrt{(pr-1)^2+(q-1)^2} \\
&+ |A_3||A_6|\sqrt{(pq-1)^2+(r-1)^2} \\
&= (p-1)(q-1)\sqrt{(pr-1)^2+(qr-1)^2} \\
&+ (p-1)(r-1)\sqrt{(pq-1)^2+(qr-1)^2} \\
&+ (q-1)(r-1)\sqrt{(pq-1)^2+(pr-1)^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(qr-1)^2+(p-1)^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(pr-1)^2+(q-1)^2} \\
&+ (p-1)(q-1)(r-1)\sqrt{(pq-1)^2+(r-1)^2}.
\end{aligned}$$

From this identity, we get that Sombor index of zero-divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ is

$$\begin{aligned}
(p-1)(q-1)(r-1)\Bigg[&\sqrt{(pq-1)^2+(r-1)^2}+\sqrt{(pr-1)^2+(q-1)^2} \\
&+\sqrt{(qr-1)^2+(p-1)^2}+\frac{\sqrt{(pq-1)^2+(pr-1)^2}}{(p-1)} \\
&+\frac{\sqrt{(pq-1)^2+(qr-1)^2}}{(q-1)}+\frac{\sqrt{(pr-1)^2+(qr-1)^2}}{(r-1)}\Bigg].
\end{aligned}$$

□

Example 4.4. *For zero-divisor graph of $Z_3 \times Z_5 \times Z_7$, we have $p = 3$, $q = 5$ and $r = 7$. Then,* $\mathrm{SO}(\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)) \cong 4687.67$, *and the followings are the*

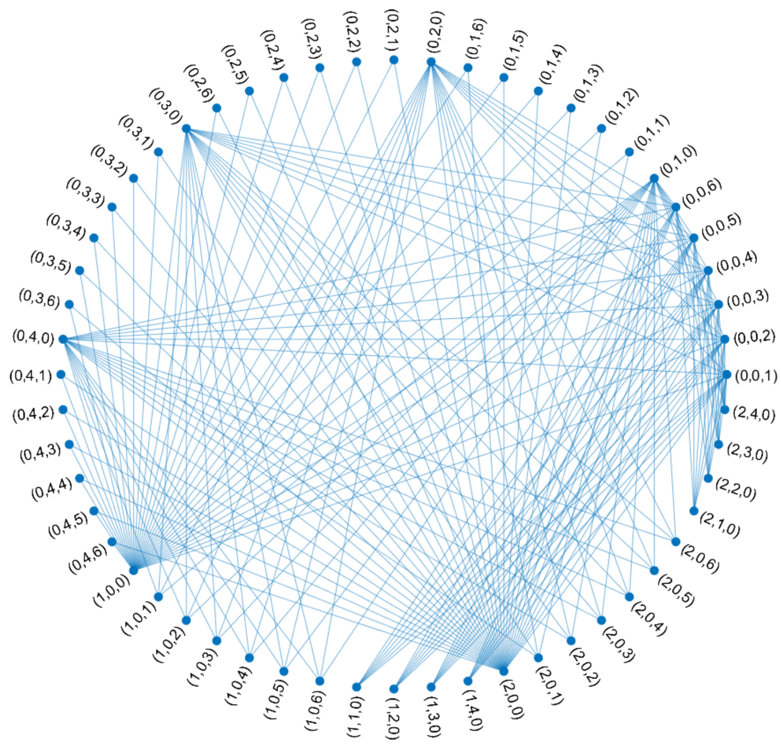

Figure 3: The graph $\Gamma(\mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7)$

set of zero-divisors of this ring.

$A_1 = \{(1,0,0),(2,0,0)\}$,
$A_2 = \{(0,1,0),(0,2,0),(0,3,0),(0,4,0)\}$,
$A_3 = \{(0,0,1),(0,0,2),(0,0,3),(0,0,4),(0,0,5),(0,0,6)\}$,
$A_4 = \{(0,1,1),(0,1,2),(0,1,3),(0,1,4),(0,1,5),(0,1,6),(0,2,1),(0,2,2),(0,2,3),$
$(0,2,4),(0,2,5),(0,2,6),(0,3,1),(0,3,2),(0,3,3),(0,3,4),(0,3,5),(0,3,6),$
$(0,4,1),(0,4,2),(0,4,3),(0,4,4),(0,4,5),(0,4,6)\}$
$A_5 = \{(1,0,1),(1,0,2),(1,0,3),(1,0,4),(1,0,5),(1,0,6),(2,0,1),(2,0,2),(2,0,3),$
$(2,0,4),(2,0,5),(2,0,6)\}$,
$A_6 = \{(1,1,0),(1,2,0),(1,3,0),(1,4,0),(2,1,0),(2,2,0),(2,3,0),(2,4,0)\}$.

The zero-divisor graph of this ring can be seen in Figure 3.

Lemma 4.5. *Assume that R_1 and R_2 be two rings. If $R_1 \cong R_2$ then $\Gamma(R_1) \cong \Gamma(R_2)$.*

Proof. Assume that $R_1 \cong R_2$ and $a, b \in R_1$ such that $ab = 0$. If ϕ is an isomorphism between R_1 and R_2 then $\phi(a)\phi(b) = 0$ in R_2. This implies that $ab \in E(\Gamma(R_1))$ and $\phi(a)\phi(b) \in E(\Gamma(R_2))$. This means that ϕ is indeed a graph isomorphism between $\Gamma(R_1)$ and $\Gamma(R_2)$. $\square$

Corollary 4.6. *$\Gamma(\mathbb{Z}_{p_1p_2\ldots p_n}) \cong \Gamma(\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_n})$ is obtained from $\mathbb{Z}_{p_1p_2\ldots p_n} \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \ldots \times \mathbb{Z}_{p_n}$ and Lemma 4.5.*

By the above arguments, we give the following corollary.

Corollary 4.7. *Let $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)$, $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)$, $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{pqr})$ be zero-divisor graphs where p, q, and r are distinct prime numbers. The followings hold:*

i) $\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q)) = \mathrm{SO}(\Gamma(\mathbb{Z}_{pq}))$

ii) $\mathrm{SO}(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = \mathrm{SO}(\Gamma(\mathbb{Z}_{pqr}))$

5 Conclusion

We compute that Sombor index of graphs $\Gamma(\mathbb{Z}_n)$ for $n \in \{p^\alpha, pq, p^2q, pqr\}$ where p, q and r are distinct prime numbers. Moreover, we introduce an algorithm which calculates the Sombor index by determining zero divisors of ring $\mathbb{Z}_n$ for given integer n. The Sombor index is a degree based topological index, so the method in this paper can be applied to other degre based topological indices. Further, one can determine the Sombor index of $\Gamma(\mathbb{Z}_n)$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$.

References

[1] M. Aijaz and S. Pirzada, Annihilating-ideal graphs of commutative rings, Asian-European Journal of Mathematics, 13(2020).

[2] S. Akbari, D. Kiani, F. Mohammadi, and S. Moradi, The total graph and regular graph of a commutative ring, Journal of Pure and Applied Algebra, 213(2009).

[3] N. Akgunes and Y. Nacaroglu, Some properties of zero divisor graph obtained by the ring z p z q z r, Asian-European Journal of Mathematics, 12(2019).

[4] N. Akgunes and M. Togan, Some graph theoretical properties over zero-divisor graphs of special finite commutative rings, Advanced Studies in Contemporary Mathematics (Kyungshang), 22(2012).

[5] A. Ali, K. C. Das, and S. Akhter, On the extremal graphs for second Zagreb index with fixed number of vertices and cyclomatic number, Miskolc Math. Notes (2019).

[6] S. Alikhani and N. Ghanbari, Sombor index of polymers, Match, 86(2021).

[7] M. An and K. C. Das, First Zagreb index, k-connectivity, beta-deficiency and k-hamiltonicity of graphs, MATCH Commun. Math. Comput. Chem, 80(2018), 141–151.

[8] D. F. Anderson and A. Badawi, The total graph of a commutative ring, Journal of Algebra, 320(2008).

[9] D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, Journal of algebra, 217(1999), 434–447.

[10] T. Asir and V. Rabikka, The Wiener index of the zero-divisor graph of zn, Discrete Applied Mathematics (2021).

[11] I. Beck, Coloring of commutative rings, Journal of Algebra, 116(1988).

[12] B. Borovicanin, K. C. Das, B. Furtula, and I. Gutman, Zagreb indices: Bounds and extremal graphs, Bounds in Chemical Graph TheoryBasics, Univ. Kragujevac, Kragujevac, 67(153)(2017).

[13] L. Buyantogtokh, B. Horoldagva, and K. C. Das, On reduced second Zagreb index, Journal of Combinatorial Optimization, 39(2020).

[14] R. Cruz, I. Gutman, and J. Rada, Sombor index of chemical graphs, Applied Mathematics and Computation, 399(2021).

[15] K. C. Das and A. Ali, On a conjecture about the second Zagreb index, Discrete Mathematics Letters, 2(2019).

[16] K. C. Das, A. S. evik, I. N. Cangul, and Y. Shang, On Sombor index, Symmetry, 13(2021).

[17] H. Deng, Z. Tang, and R. Wu, Molecular trees with extremal values of Sombor indices, International Journal of Quantum Chemistry, 121(2021).

[18] S. Fajtlowicz, On conjectures of graffiti-ii, Congr. Numer, 60(1987), 187–197.

[19] X. Fang, L. You, and H. Liu, The expected values of Sombor indices in random hexagonal chains, phenylene chains and Sombor indices of some chemical graphs, International Journal of Quantum Chemistry, 121(2021).

[20] S. Filipovski, Relations between Sombor index and some degreebased topological indices, Iranian Journal of Mathematical Chemistry, 12(2021).

[21] B. Furtula and I. Gutman, A forgotten topological index, Journal of Mathematical Chemistry, 53(2015).

[22] N. Ghanbari and S. Alikhani, Sombor index of certain graphs, Iranian Journal of Mathematical Chemistry, 12(2021).

[23] I. Gutman, Degree-based topological indices, Croatica chemica acta, 86(2013), 351–361.

[24] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, MATCH Commun. Math. Comput. Chem, 86(2021), 11–16.

[25] I. Gutman, Some basic properties of Sombor indices, Open Journal of Discrete Applied Mathematics, 4(2021), 1–3.

[26] I. Gutman and O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer Science & Business Media, 1986.

[27] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. total ϕ-electron energy of alternant hydrocarbons, Chemical Physics Letters, 17(1972).

[28] B. Horoldagva and K. C. Das, On Zagreb indices of graphs, Match, 85(2021).

[29] B. Horoldagva, K. C. Das, and T. A. Selenge, Complete characterization of graphs for direct comparing Zagreb indices, Discrete Applied Mathematics, 215(2016).

[30] B. Horoldagva and C. Xu, On Sombor index of graphs, MATCH Commun. Math. Comput. Chem, 86(2021), 703–713.

[31] V. R. Kulli and I. Gutman, Computation of Sombor indices of certain networks, International Journal of Applied Chemistry, 8(2021).

[32] Z. Lin and L. Miao, On the spectral radius, energy and Estrada index of the Sombor matrix of graphs, arXiv preprint arXiv:2102.03960(2021).

[33] I. Milovanovic, E. Milovanovic, and M. Matejic, On some mathematical properties of Sombor indices, Bull. Int. Math. Virtual Inst, 11(2021), 341–353.

[34] M. Randić, On characterization of molecular branching, Journal of the American Chemical Society, 97(1975).

[35] I. Redžepović, Chemical applicability of Sombor indices, Journal of the Serbian Chemical Society, 86(2021).

[36] T. Réti, T. Došlic, and A. Ali, On the Sombor index of graphs, Contrib. Math, 3(2021), 11–18.

[37] S. Salehifar, K. Khashyarmanesh, and M. Afkhami, On the annihilator-ideal graph of commutative rings, Ricerche di Matematica, 66(2017).

[38] Y. Shang, On the number of spanning trees, the laplacian eigenvalues, and the laplacian Estrada index of subdivided-line graphs, Open Mathematics, 14(2016).

[39] Y. Shang, Lower bounds for Gaussian Estrada index of graphs, Symmetry, 10(2018).

[40] F. Shaveisi, Some results on annihilating-ideal graphs, Canadian Mathematical Bulletin, 59(2016).

[41] P. Singh and V. K. Bhat, Adjacency matrix and Wiener index of zero divisor graph $\gamma(\mathbb{Z}_n)$, Journal of Applied Mathematics and Computing, 66(2021).

[42] D. Sinha and A. K. Rao, A note on co-maximal graphs of commutative rings, AKCE International Journal of Graphs and Combinatorics, 15(2018).

[43] A. Tehranian and H. R. Maimani, A study of the total graph, Iranian Journal of Mathematical Sciences and Informatics, 6(2011).

[44] R. Todeschini and V. Consonni, Molecular Descriptors for Chemoinformatics, volume 2, John Wiley & Sons, 2010.

[45] H. J. Wang, Graphs associated to co-maximal ideals of commutative rings, Journal of Algebra, 320(2008).

[46] Z. Wang, Y. Mao, Y. Li, and B. Furtula, On relations between Sombor and other degree-based indices, Journal of Applied Mathematics and Computing (2021).

[47] H. Wiener, Structural determination of paraffin boiling points, Journal of the American Chemical Society, 69(1947).

[48] K. Xu, F. Gao, K. C. Das, and N. Trinajstić, A formula with its applications on the difference of Zagreb indices of graphs, Journal of Mathematical Chemistry, 57(2019).

[49] M. Ye and T. Wu, Co-maximal ideal graphs of commutative rings, Journal of Algebra and its Applications, 11(2012).

[50] M. Young, Adjacency matrices of zero-divisor graphs of integers modulo n, Involve, a Journal of Mathematics, 8(2015).

[51] T. Zhou, Z. Lin, and L. Miao, The Sombor index of trees and unicyclic graphs with given matching number, arXiv preprint arXiv:2103.04645(2021).

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